3.4

$$[\varphi(x), \pi(y)] = [\varphi(x), \dot{\varphi}(y)] =$$

$$= \left[\int d\vec{k} \left(a(\vec{k}) e^{-ikx} + a^{\dagger}(\vec{k}) e^{ikx} \right), \int d\vec{p} \left(-i\omega_p a(\vec{p}) e^{-ipy} + i\omega_p a^{\dagger}(\vec{p}) e^{ipy} \right) \right] =$$

$$= \int d\vec{k} \underbrace{\int d\vec{p}}_{= \frac{d\vec{p}}{(2\pi)^3 2\omega_p}} \Big(i\omega_p e^{-ikx} e^{ipy} \underbrace{[a(\vec{k}), a^{\dagger}(\vec{p})]}_{(2\pi)^3 2\omega_k \delta(\vec{k}-\vec{p})} - i\omega_p e^{ikx} e^{-ipx} \underbrace{[a^{\dagger}(\vec{k}), a(\vec{p})]}_{-(2\pi)^3 \omega_k \delta(\vec{k}-\vec{p})} \Big) =$$

$$= \int d\vec{k} \, \frac{i}{2} \int d\vec{p} \left(2\omega_k e^{-ikx} e^{ipy} \delta(\vec{k}-\vec{p}) + 2\omega_k e^{ikx} e^{-ipy} \delta(\vec{k}-\vec{p}) \right) =$$

$$= \underbrace{\int d\vec{k}}_{= \frac{d\vec{k}}{(2\pi)^3 2\omega_k}} \frac{i}{2} 2\omega_k \int d\vec{p} \left(e^{-ikx+ipy} + e^{ikx-ipy} \right) \delta(\vec{k}-\vec{p}) =$$

$$= \frac{i}{2} \int \frac{d\vec{k}}{(2\pi)^3} \left(e^{-ik(x-y)} + e^{ik(x-y)} \right) \overset{k(x-y) = k^0(t-\bar{t}) - \vec{k}(\vec{x}-\vec{y})}{=}$$

$$= \frac{i}{2} \int \frac{d\vec{k}}{(2\pi)^3} \left(e^{i\vec{k}(\vec{y}-\vec{x})} + e^{i\vec{k}(\vec{x}-\vec{y})} \right) =$$

$$= \frac{i}{2} \left(\delta(\vec{y}-\vec{x}) + \delta(\vec{x}-\vec{y}) \right) = i\delta(\vec{x}-\vec{y})$$

3.5 $H(t) = \int d\vec{x} \left[\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \right]$

$$\varphi = \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega_k} \left[a_k e^{-ikx} + a_k^\dagger e^{ikx} \right] = \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega_k} \left[a_k e^{-i\omega_k t} e^{i\vec{k}\vec{x}} + \underbrace{a_k^\dagger e^{i\omega_k t} e^{-i\vec{k}\vec{x}}}_{\substack{\vec{k}\to-\vec{k}: \\ \omega_{-k} = \sqrt{\vec{k}^2+m^2} = \omega_k}} \right] =$$

$$= \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega_k} \left[a_k e^{-i\omega_k t} + a_{-k}^\dagger e^{i\omega_k t} \right] e^{i\vec{k}\vec{x}}$$

$$\dot{\varphi} = -i \int \frac{d\vec{k}}{(2\pi)^3\, 2} \left[a_k e^{-i\omega_k t} - a_{-k}^\dagger e^{i\omega_k t} \right] e^{i\vec{k}\vec{x}}$$

$$\vec{\nabla}\varphi = i \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega_k} \vec{k} \left[a_k e^{-i\omega_k t} + a_{-k}^\dagger e^{i\omega_k t} \right] e^{i\vec{k}\vec{x}}$$

$$\int d\vec{x}\, \frac{m^2}{2}\varphi^2 = \frac{m^2}{2} \int d\vec{x}\, \varphi^\dagger \varphi \overset{\text{théorème de Parseval}}{=}$$

$$= \frac{m^2}{8} \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{\omega_k^2} \left[a_k^\dagger e^{i\omega_k t} + a_{-k} e^{-i\omega_k t} \right] \left[a_k e^{-i\omega_k t} + a_{-k}^\dagger e^{i\omega_k t} \right]$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \frac{m^2}{\omega_k^2} \left[a_k^\dagger a_k + a_{-k} a_{-k}^\dagger + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} + a_{-k} a_k e^{-2i\omega_k t} \right]$$

$$\int d\vec{x}\, \frac{1}{2}\dot{\varphi}^2 = \frac{1}{2} \int d\vec{x}\, \dot{\varphi}^\dagger \dot{\varphi} \overset{\text{théorème de Parseval}}{=}$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \left[a_k^\dagger e^{i\omega_k t} - a_{-k} e^{-i\omega_k t} \right] \left[a_k e^{-i\omega_k t} - a_{-k}^\dagger e^{i\omega_k t} \right] =$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \left[a_k^\dagger a_k + a_{-k} a_{-k}^\dagger - a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} - a_{-k} a_k e^{-2i\omega_k t} \right]$$

$$\int d\vec{x}\, \frac{1}{2}(\vec{\nabla}\varphi)^2 = \frac{1}{2} \int d\vec{x}\, (\vec{\nabla}\varphi)^\dagger (\vec{\nabla}\varphi) \overset{\text{théorème de Parseval}}{=}$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \frac{\vec{k}^2}{\omega_k^2} \left[a_k^\dagger e^{i\omega_k t} + a_{-k} e^{-i\omega_k t} \right] \left[a_k e^{-i\omega_k t} + a_{-k}^\dagger e^{i\omega_k t} \right] =$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \frac{\vec{k}^2}{\omega_k^2} \left[a_k^\dagger a_k + a_{-k} a_{-k}^\dagger + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} + a_{-k} a_k e^{-2i\omega_k t} \right]$$

⚠ $\frac{m^2}{\omega_k^2} + \frac{\vec{k}^2}{\omega_k^2} = 1 \Rightarrow$ termes en $e^{\pm 2i\omega_k t}$ s'annullent

$$\Rightarrow H(t) = \frac{1}{4}\int \frac{d\vec{k}}{(2\pi)^3}\left[a_k^\dagger a_k + a_{-k} a_{-k}^\dagger\right] = \frac{1}{4}\int \frac{d\vec{k}}{(2\pi)^3}\left[a_k^\dagger a_k + a_k a_k^\dagger\right] =$$

$$= \frac{1}{2}\int d\tilde{k}\, \omega_k \left[a_k^\dagger a_k + a_k a_k^\dagger\right]$$

$$d\tilde{k} = \frac{d\vec{k}}{(2\pi)^3 2\omega_k}$$

$$= \int d\tilde{k}\, \omega_k \left[a_k^\dagger a_k + \text{const.}\right]$$

↳ énergie du vide diverge !

3.6 $$\left[P^\mu, a_p^\dagger\right] = \left[\int d\tilde{k}\, k^\mu a_k^\dagger a_k, a_p^\dagger\right] =$$

$$= \int d\tilde{k}\, k^\mu \left[a_k^\dagger a_k, a_p^\dagger\right] = \int d\tilde{k}\, k^\mu \left(a_k^\dagger a_k a_p^\dagger - a_p^\dagger a_k^\dagger a_k\right) =$$

$$= \int d\tilde{k}\, k^\mu a_k^\dagger \left[a_k, a_p^\dagger\right] = \int d\tilde{k}\, k^\mu a_k^\dagger\, \delta(\vec{k}-\vec{p}) \cdot (2\pi)^3 2\omega_k =$$

$$= p^\mu a_p^\dagger$$