Exercices

3.1) $\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - V(\varphi)$ avec $V(\varphi) = \frac{1}{2}m^2\varphi^2$

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - V(\varphi)$$

Euler-Lagrange : $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} - \frac{\partial \mathcal{L}}{\partial\varphi} = 0$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial\varphi} &= -m^2\varphi \\ \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)} &= \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t\varphi)} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i\varphi)} = \\ &= \partial_t(\partial_t\varphi) - \partial_i(\partial_i\varphi) = \\ &= \partial_t^2\varphi - \Delta\varphi = \Box\varphi \end{aligned}\right\} \Rightarrow \Box\varphi + m^2\varphi = 0$$

3.2) $T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}(\partial^\nu\varphi) - \eta^{\mu\nu}\mathcal{L}$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0\varphi)}(\partial^0\varphi) - \mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial\dot{\varphi}}}_{=\dot{\varphi}=\pi}\dot{\varphi} - \mathcal{L} = \pi\dot{\varphi} - \mathcal{L} = \mathcal{H}$$

3.3) $\varphi(x) = \int d\tilde{k} \left[a(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{ikx} \right]$

$$\Box\varphi + m^2\varphi = \partial_t^2\varphi - (\vec{\nabla}\varphi)^2 + m^2\varphi \overset{?}{=} 0$$

$$kx = k^0 x^0 - k^i x_i = k^0 \cdot t - k^i x_i \qquad k^0 = \sqrt{\vec{k}^2 + m^2}$$

$$\partial_t \varphi = \int d\tilde{k} \left[a(\vec{k}) e^{-ikx} (-ik^0) + a^\dagger(\vec{k}) e^{ikx} (ik^0) \right]$$

$$\partial_t^2 \varphi = -\int d\tilde{k} \left[a(\vec{k}) e^{-ikx} (k^0)^2 + a^\dagger(\vec{k}) e^{ikx} (k^0)^2 \right]$$

$$(k^0)^2 = \vec{k}^2 + m^2$$

$$\vec{\nabla}\varphi = \int d\tilde{k} \left[a(\vec{k}) e^{-ikx} (i\vec{k}) + a^\dagger(\vec{k}) e^{ikx} (-i\vec{k}) \right]$$

$$\vec{\nabla}^2\varphi = -\int d\tilde{k} \left[a(\vec{k}) e^{-ikx} \vec{k}^2 + a^\dagger(\vec{k}) e^{ikx} \vec{k}^2 \right]$$

$$\Rightarrow \Box\varphi + m^2\varphi = -(\vec{k}^2 + m^2) \int d\tilde{k} [\ldots] + \vec{k}^2 \int d\tilde{k} [\ldots] + m^2 \int d\tilde{k} [\ldots] = 0$$

3.4

$$[\varphi(x), \pi(y)] = [\varphi(x), \dot{\varphi}(y)] =$$

$$= \left[\int d\vec{k} \left(a(\vec{k}) e^{-ikx} + a^{\dagger}(\vec{k}) e^{ikx} \right), \int d\vec{p} \left(-i\omega_p a(\vec{p}) e^{-ipy} + i\omega_p a^{\dagger}(\vec{p}) e^{ipy} \right) \right] =$$

$$= \int d\vec{k} \underbrace{\int d\vec{p}}_{= \frac{d\vec{p}}{(2\pi)^3 2\omega_p}} \Big(i\omega_p e^{-ikx} e^{ipy} \underbrace{[a(\vec{k}), a^{\dagger}(\vec{p})]}_{(2\pi)^3 2\omega_k \delta(\vec{k}-\vec{p})} - i\omega_p e^{ikx} e^{-ipx} \underbrace{[a^{\dagger}(\vec{k}), a(\vec{p})]}_{-(2\pi)^3 \omega_k \delta(\vec{k}-\vec{p})} \Big) =$$

$$= \int d\vec{k} \, \frac{i}{2} \int d\vec{p} \left(2\omega_k e^{-ikx} e^{ipy} \delta(\vec{k}-\vec{p}) + 2\omega_k e^{ikx} e^{-ipy} \delta(\vec{k}-\vec{p}) \right) =$$

$$= \underbrace{\int d\vec{k}}_{= \frac{d\vec{k}}{(2\pi)^3 2\omega_k}} \frac{i}{2} 2\omega_k \int d\vec{p} \left(e^{-ikx+ipy} + e^{ikx-ipy} \right) \delta(\vec{k}-\vec{p}) =$$

$$= \frac{i}{2} \int \frac{d\vec{k}}{(2\pi)^3} \left(e^{-ik(x-y)} + e^{ik(x-y)} \right) \overset{k(x-y) = k^0(t-\bar{t}) - \vec{k}(\vec{x}-\vec{y})}{=}$$

$$= \frac{i}{2} \int \frac{d\vec{k}}{(2\pi)^3} \left(e^{i\vec{k}(\vec{y}-\vec{x})} + e^{i\vec{k}(\vec{x}-\vec{y})} \right) =$$

$$= \frac{i}{2} \left(\delta(\vec{y}-\vec{x}) + \delta(\vec{x}-\vec{y}) \right) = i\delta(\vec{x}-\vec{y})$$