Exercices

6.1) $\psi_\alpha \to \tilde{\psi}_\alpha = e^{iq\theta}\psi_\alpha \simeq (1+iq\theta)\psi_\alpha$

$$\partial_\mu\psi_\alpha \to \partial_\mu\tilde{\psi}_\alpha = e^{iq\theta}\partial_\mu\psi_\alpha = (1+iq\theta)\,\partial_\mu\psi_\alpha$$

$$\bar{\psi}_\beta \to \tilde{\bar{\psi}}_\beta = e^{-iq\theta}\,\bar{\psi}_\beta \simeq (1-iq\theta)\,\bar{\psi}_\beta$$

$$\partial_\mu\bar{\psi}_\beta \to \partial_\mu\tilde{\bar{\psi}}_\beta = e^{-iq\theta}\partial_\mu\bar{\psi}_\beta \simeq (1-iq\theta)\,\bar{\psi}_\beta$$

$\mathcal{L}$ invariant $\Rightarrow \delta\mathcal{L} = 0$

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\psi_\alpha}\right)\delta\psi_\alpha + \left(\frac{\partial\mathcal{L}}{\partial\bar{\psi}_\beta}\right)\delta\bar{\psi}_\beta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}\,\delta(\partial_\mu\psi_\alpha) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_\beta)}\,\delta(\partial_\mu\bar{\psi}_\beta) =$$

$$\overset{\text{IPP}}{=} \left(\frac{\partial\mathcal{L}}{\partial\psi_\alpha}\right)\delta\psi_\alpha + \left(\frac{\partial\mathcal{L}}{\partial\bar{\psi}_\beta}\right)\delta\bar{\psi}_\beta + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}(\delta\psi_\alpha)\right] - \left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}\right](\partial\psi_\alpha) +$$

$$+ \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_\beta)}(\partial\bar{\psi}_\beta)\right] - \left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_\beta)}\right](\partial\bar{\psi}_\beta) =$$

$$\overset{\text{Euler-Lagrange}}{=} \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha)}(\delta\psi_\alpha) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_\beta)}(\delta\bar{\psi}_\beta)\right] =$$

$$= \frac{i}{2}\,\partial_\mu\left[\bar{\psi}_\beta\,\gamma^\mu_{\beta\alpha}(iq\theta)\psi_\alpha - (-iq\theta\,\bar{\psi}_\beta)\,\gamma^\mu_{\beta\alpha}\,\psi_\alpha\right] =$$

$$= -\frac{\theta}{2}\,q\,\partial_\mu\left[\bar{\psi}_\beta\gamma^\mu_{\beta\alpha}\psi_\alpha + \bar{\psi}_\beta\gamma^\mu_{\beta\alpha}\psi_\alpha\right] = -\theta\,\partial_\mu\underbrace{\left[q\,\bar{\psi}\gamma^\mu\psi\right]}_{=:J^\mu} = 0$$

6.2)

$$P^\mu = \int d\vec{x}\, \frac{i}{2}\, \bar{\psi}\gamma^0 \overleftrightarrow{\partial^\mu} \psi = \frac{i}{2}\int d\vec{x} \left[\bar{\psi}\gamma^0(\partial^\mu\psi) - (\partial^\mu\bar{\psi})\gamma^0\psi \right]$$

$$\psi = \int d\tilde{k} \sum_{\alpha=1,2} \left[b(\vec{k},\alpha)\, u(\vec{k},\alpha)\, e^{-ikx} + d^\dagger(\vec{k},\alpha)\, v(\vec{k},\alpha)\, e^{+ikx} \right]$$

$$\partial_\mu\psi = \int d\tilde{k} \sum_{\alpha=1,2} \left[(-ik^\mu)\, b(\vec{k},\alpha)\, u(\vec{k},\alpha)\, e^{-ikx} + (ik^\mu)\, d^\dagger(\vec{k},\alpha)\, v(\vec{k},\alpha)\, e^{+ikx} \right]$$

$$\Rightarrow \frac{i}{2}\int d\vec{x}\, \bar{\psi}\gamma^0(\partial^\mu\psi) = \frac{i}{2}\int d\vec{x}\, \psi^\dagger \underbrace{\gamma^0\gamma^0}_{=1}(\partial^\mu\psi)$$

Parseval combiné avec : $\bar{u}_\alpha(\vec{k})\, u_\beta(\vec{p}) = \delta_{\alpha\beta}\, \delta(\vec{k}-\vec{p})$

$$\bar{v}_\alpha(\vec{k})\, v_\beta(\vec{p}) = -\delta_{\alpha\beta}\, \delta(\vec{k}-\vec{p})$$

$$\bar{u}_\alpha(\vec{k})\, v_\beta(\vec{p}) = \bar{v}_\alpha(\vec{k})\, u_\beta(\vec{p}) = 0$$

$$\Rightarrow \frac{i}{2}\int d\vec{x}\, \psi^\dagger(\partial^\mu\psi) = \int d\tilde{k} \sum_{\alpha=1,2} \frac{1}{2}k^\mu \left[b^\dagger(\vec{k},\alpha)\, b(\vec{k},\alpha) - d(\vec{k},\alpha)\, d^\dagger(\vec{k},\alpha) \right]$$

Analogue : $\frac{i}{2}\int d\vec{x}\, (\partial^\mu\bar{\psi})\gamma^0\psi =$

$$= -\int d\tilde{k} \sum_{\alpha=1,2} \frac{1}{2}k^\mu \left[b^\dagger(\vec{k},\alpha)\, b(\vec{k},\alpha) - d(\vec{k},\alpha)\, d^\dagger(\vec{k},\alpha) \right]$$

$$\Rightarrow P^\mu = \int d\tilde{k} \sum_{\alpha=1,2} k^\mu \left[b^\dagger(\vec{k},\alpha)\, b(\vec{k},\alpha) - d(\vec{k},\alpha)\, d^\dagger(\vec{k},\alpha) \right]$$

6.3) $\{b(\vec{k},\alpha), b^\dagger(\vec{p},\beta)\} = \{d(\vec{k},\alpha), d^\dagger(\vec{p},\beta)\} = (2\pi)^3 \frac{E}{m} \delta(\vec{k}-\vec{p})\, \delta_{\alpha\beta}$

$$\{\psi_\alpha(t,\vec{x}), \psi^\dagger_\beta(t,\vec{y})\} = \Big\{ \int d\tilde{k} \left[b(\vec{k},\alpha)u(\vec{k},\alpha)e^{-ikx} + d^\dagger(\vec{k},\alpha)v(\vec{k},\alpha)e^{+ikx}\right],$$

$$\int d\tilde{p} \left[b^\dagger(\vec{p},\beta)\, u^\dagger(\vec{p},\beta) e^{+ipy} + d(\vec{p},\beta)\, v^\dagger(\vec{p},\beta) e^{-ipy}\right] \Big\} =$$

$$= \dots =$$

$$= \int d\tilde{k} \int d\tilde{p} \left[\left\{d^\dagger v e^{ikx}, d v^\dagger e^{-ipy}\right\} + \left\{b u e^{-ikx}, b^\dagger u^\dagger e^{ipy}\right\}\right] =$$

$$= \int d\tilde{k} \int d\tilde{p} \left[\underbrace{vv^\dagger}_{=\delta_{\alpha\beta}} \underbrace{\{d^\dagger, d\}}_{=(2\pi)^3 \frac{E}{m}\delta(\vec{k}-\vec{p})\,\delta_{\alpha\beta}} e^{i(kx-py)} + \underbrace{uu^\dagger}_{=\delta_{\alpha\beta}} \underbrace{\{b, b^\dagger\}}_{=(2\pi)^3\frac{E}{m}\delta(\vec{k}-\vec{p})\delta_{\alpha\beta}} e^{i(-kx+py)} \right]$$

$$= \int d\tilde{k} \left[e^{ik(x-y)} + e^{ik(y-x)}\right]\delta_{\alpha\beta} = \delta(x-y)\, \delta_{\alpha\beta}$$

6.4) $u^\dagger(-\vec{k},\beta)\, v(\vec{k},\alpha) = \dfrac{(-k\!\!\!/+m)(k\!\!\!/+m)}{2m(m+E)} \underbrace{u^\dagger(\vec{0},\beta)\, v(\vec{0},\alpha)}_{=(a\, b\, 0\, 0)\begin{pmatrix}0\\0\\c\\d\end{pmatrix}} = 0$

Analogue: $v^\dagger(-\vec{k},\beta)u(\vec{k},\alpha) = 0$

$$Q = q \int d\vec{x} :\psi^\dagger\psi:$$

Parseval + relations démontrées:

$$\Rightarrow Q = q \int d\tilde{k} \sum_{\alpha=1,2} \left[b^\dagger(\vec{k},\alpha) b(\vec{k},\alpha) - d^\dagger(\vec{k},\alpha) d(\vec{k},\alpha)\right]$$

6.5]

$$\psi(x) = \int d\tilde{k} \sum_\alpha \left[b(\vec{k},\alpha)\, u(\vec{k},\alpha)\, e^{-ikx} + d^\dagger(\vec{k},\alpha)\, v(\vec{k},\alpha)\, e^{+ikx} \right]$$

$$\bar{\psi}(y) = \int d\tilde{p} \sum_\beta \left[b^\dagger(\vec{p},\beta)\, \bar{u}(\vec{p},\beta)\, e^{+ipy} + d(\vec{p},\beta)\, \bar{v}(\vec{p},\beta)\, e^{-ipy} \right]$$

$$\psi(x)\bar{\psi}(y) = \int d\tilde{k} \int d\tilde{p} \sum_\alpha \sum_\beta \left[b(\vec{k},\alpha) u(\vec{k},\alpha) e^{-ikx} + d^\dagger(\vec{k},\alpha) v(\vec{k},\alpha) e^{+ikx} \right] \times$$

$$\times \left[b^\dagger(\vec{p},\beta) \bar{u}(\vec{p},\beta) e^{+ipy} + d(\vec{p},\beta) \bar{v}(\vec{p},\beta) e^{-ipy} \right] =$$

$$\Rightarrow \quad \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \langle 0 | \int d\tilde{k} \int d\tilde{p} \sum_\alpha \sum_\beta b(\vec{k},\alpha)\, b^\dagger(\vec{p},\beta)\, u(\vec{k},\alpha)\, \bar{u}(\vec{p},\beta)\, e^{i(py-kx)} | 0 \rangle =$$

$$= \int d\tilde{k} \int d\tilde{p} \sum_\alpha \sum_\beta \underbrace{\langle 0 | b(\vec{k},\alpha)\, b^\dagger(\vec{p},\beta) | 0 \rangle}_{= \langle 0 | \{ b(\vec{k},\alpha),\, b^\dagger(\vec{p},\beta) \} | 0 \rangle \,=\, (2\pi)^3 \frac{E}{m} \delta(\vec{k}-\vec{p})\, \delta_{\alpha\beta}} u(\vec{k},\alpha)\, \bar{u}(\vec{p},\beta)\, e^{i(py-kx)}$$

$$\Rightarrow \quad \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \int d\tilde{k} \underbrace{\sum_\alpha u(\vec{k},\alpha)\, \bar{u}(\vec{k},\alpha)}_{= \frac{\not{k}+m}{2m}} e^{-ik(x-y)} =$$

$$\Rightarrow \quad \langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \int d\tilde{k} \left(\frac{\not{k}+m}{2m} \right) e^{-ik(x-y)} = \int d\tilde{k}\, \Lambda_+(k)\, e^{-ik(x-y)}$$

De même : $\langle 0 | \bar{\psi}(y) \psi(x) | 0 \rangle = \int d\tilde{k} \left(\frac{-\not{k}+m}{2m} \right) e^{+ik(x-y)} = \int d\tilde{k}\, \Lambda_-(k)\, e^{+ik(x-y)}$

$$\Rightarrow \quad \langle 0 | T\{ \psi(x) \bar{\psi}(y) \} | 0 \rangle =$$

$$= \int d\tilde{k} \left[\theta(x^0 - y^0)\, \Lambda_+(k)\, e^{-ik(x-y)} + \theta(y^0 - x^0)\, \Lambda_-(k)\, e^{+ik(x-y)} \right]$$

$$= i\, S_F(x-y)$$