Exercices

① $\Box A^\mu(x) = J^\mu(x) = \int d^4y\, \delta(x-y)\, J^\mu(y)$

$$A^\mu(x) = \int d^4y\, G_{ret}(x-y)\, J^\mu(y)$$

$$\Rightarrow J^\mu(x) = \int d^4y\, \Box_x G_{ret}(x-y)\, J^\mu(y) \quad \Rightarrow \quad \Box_x G_{ret}(x-y) = \delta(x-y)$$

$$G_{ret}(x-y) = \frac{1}{(2\pi)^4}\int dk\, G_{ret}(k)\, e^{-ik(x-y)}$$

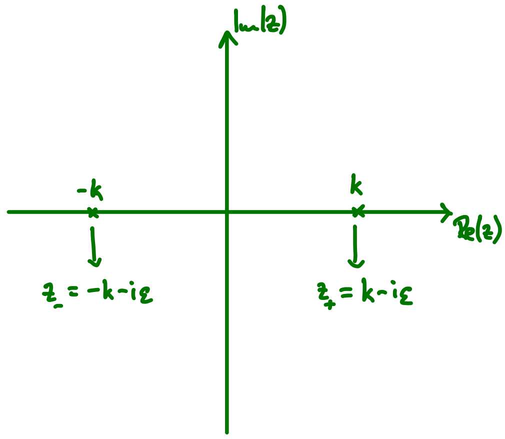
$$\Rightarrow \Box_x G_{ret}(x-y) = \frac{1}{(2\pi)^4}\int dk\, (-i)^2 k^2\, G_{ret}(k)\, e^{-ik(x-y)} = \frac{1}{(2\pi)^4}\int dk\, e^{-ik(x-y)}$$

$$\Rightarrow -k^2 G_{ret}(k^2) = 1 \quad \Rightarrow \quad G_{ret}(k) = \frac{-1}{k^2}$$

$$G_{ret}(x-y) = \frac{1}{(2\pi)^4}\int dk\, G_{ret}(k)\, e^{-ik(x-y)} = \frac{1}{(2\pi)^4}\int dk\, \frac{-1}{k^2}\, e^{-ik(x-y)}$$

$$= \int \frac{d\vec{k}}{(2\pi)^3}\, e^{+i\vec{k}(\vec{x}-\vec{y})} \underbrace{\int \frac{dk^0}{(-2\pi)}\, \frac{e^{-ik^0(x^0-y^0)}}{(k^0)^2-|\vec{k}|^2}}_{\rightsquigarrow\, I \text{ par théorème des résidus...}}$$

$$I = \int_{-\infty}^{+\infty} dz\, \frac{e^{-iz(x^0-y^0)}}{z^2-k^2} = \int_{-\infty}^{+\infty} dz\, \underbrace{\frac{e^{-iz(x^0-y^0)}}{(z-k_+)(z-k_-)}}_{F(z)}$$



$$I = -2\pi i\left[\mathrm{Res}(F, z_+) + \mathrm{Res}(F, z_-)\right]$$

$$= -2\pi i\left[\lim_{z\to z_+}(z-z_+)F(z) + \lim_{z\to z_-}(z-z_-)F(z)\right] =$$

$$= -2\pi i\left[\lim_{z\to z_+}\frac{e^{-iz(x^0-y^0)}}{z-z_-} + \lim_{z\to z_-}\frac{e^{-iz(x^0-y^0)}}{z-z_+}\right] =$$

$$= -2\pi i\left[\frac{e^{-i(k-i\varepsilon)(x^0-y^0)}}{2k} + \frac{e^{-i(-k-i\varepsilon)(x^0-y^0)}}{-2k}\right] \xrightarrow{\varepsilon\to 0} I = -2\pi i\left[\frac{e^{-ik(x^0-y^0)}}{2k} - \frac{e^{+ik(x^0-y^0)}}{2k}\right]$$

$$\Rightarrow G_{ret}(x-y) = \int \frac{d\vec{k}}{(2\pi)^3}\, e^{+i\vec{k}(\vec{x}-\vec{y})} \cdot \frac{(-2\pi i)}{(-2\pi)} \left[\frac{e^{-ik^0(x^0-y^0)}}{2k^0} - \frac{e^{+ik^0(x^0-y^0)}}{2k^0} \right] \theta(x^0-y^0) =$$

$$\vec{r} = \vec{x}-\vec{y}$$
$$r^0 = x^0-y^0$$

$$= \frac{-i}{(2\pi)^3} \int \frac{d\vec{k}}{2k^0}\, e^{+i\vec{k}\vec{r}} \left[e^{+ik^0r^0} - e^{-ik^0r^0} \right] \theta(r^0)$$

$$= \frac{-i}{(2\pi)^3} \int \frac{|\vec{k}|^2\, d|\vec{k}| - d\cos\theta\, d\phi}{2k^0}\, e^{+i|\vec{k}|\cdot|\vec{r}|\cdot\cos\theta} \left[e^{+ik^0r^0} - e^{-ik^0r^0} \right] \theta(r^0) =$$

$$= \frac{-i}{(2\pi)^3} \int \frac{d|\vec{k}|}{2} |\vec{k}| \cdot (2\pi) \left[e^{+ik^0r^0} - e^{-ik^0r^0} \right] \theta(r^0) \int_{-1}^{1} d\cos\theta\, e^{+i|\vec{k}|\cdot|\vec{r}|\cdot\cos\theta} =$$

$$= \frac{-i}{2(2\pi)^2} \int d|\vec{k}|\, |\vec{k}| \left[e^{+ik^0r^0} - e^{-ik^0r^0} \right] \theta(r^0)\, \frac{-e^{-i|\vec{k}||\vec{r}|} + e^{+i|\vec{k}||\vec{r}|}}{i|\vec{k}||\vec{r}|} =$$

$$= \frac{1}{2(2\pi)^2|\vec{r}|} \int dk^0 \left[e^{+ik^0r^0} - e^{-ik^0r^0} \right]\left[e^{+i|\vec{k}||\vec{r}|} - e^{-i|\vec{k}||\vec{r}|} \right] \theta(r^0) =$$

$$= \frac{1}{2(2\pi)^2|\vec{r}|} \int_0^{+\infty} dk^0 \left[e^{+ik^0(r^0+|\vec{r}|)} - e^{+ik^0(r^0-|\vec{r}|)} - e^{+ik^0(-r^0-|\vec{r}|)} + e^{+ik^0(-r^0+|\vec{r}|)} \right] \theta(r^0) =$$

$$= \frac{1}{2(2\pi)^2|\vec{r}|} \int_0^{+\infty} dk^0 \left[e^{+ik^0(r^0+|\vec{r}|)} + e^{+ik^0(-r^0+|\vec{r}|)} \right] \theta(r^0)$$

$$+ \frac{1}{2(2\pi)^2|\vec{r}|} \int_{-\infty}^{0} dk^0 \left[e^{+ik^0(r^0+|\vec{r}|)} + e^{+ik^0(-r^0+|\vec{r}|)} \right] \theta(r^0) =$$

$$= \frac{1}{4\pi|\vec{r}|} \Big[-\underbrace{\delta(r^0+|\vec{r}|)}_{=0} + \delta(r^0-|\vec{r}|) \Big] \theta(r^0) = \frac{1}{4\pi|\vec{r}|}\, \delta(r^0-|\vec{r}|)$$

$$\delta(x) = \frac{1}{2\pi} \int e^{ikx}\, dk$$

② $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 - J_\mu A^\mu$$

$$= -\frac{1}{4}\left(2\,\eta^{\mu\sigma}\eta^{\nu\rho}\partial_\mu A_\nu F_{\sigma\rho}\right) - \frac{\lambda}{2}\left(\eta^{\mu\nu}\partial_\mu A_\nu\right)^2 - \eta^{\mu\nu}J_\nu A_\mu$$

Équations d'Euler-Lagrange : $\displaystyle \partial_\beta \frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} = \frac{\partial\mathcal{L}}{\partial A_\alpha}$

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = -\eta^{\mu\nu}J_\nu \frac{\partial A_\mu}{\partial A_\alpha} = -\eta^{\mu\nu}J_\nu\delta^\alpha_\mu = -J^\alpha$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} = -\frac{1}{2}\eta^{\mu\sigma}\eta^{\nu\rho}\left[\underbrace{\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\beta A_\alpha)}}\cdot F_{\sigma\rho} + \partial_\mu A_\nu \underbrace{\frac{\partial F_{\sigma\rho}}{\partial(\partial_\beta A_\alpha)}}\right] - \lambda\underbrace{\eta^{\mu\nu}\partial_\mu A_\nu}_{(\partial A)}\cdot\eta^{\mu\nu}\underbrace{\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\beta A_\alpha)}}_{\delta_\mu^\beta\delta_\nu^\alpha} - \eta^{\mu\nu}J_\nu \underbrace{\frac{\partial A_\mu}{\partial(\partial_\beta A_\alpha)}}_{=0}$$

$$\Rightarrow \frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} = -\frac{1}{2}\eta^{\mu\sigma}\eta^{\nu\sigma}\left[\delta_\mu^\beta\delta_\nu^\alpha F_{\sigma\rho} + \partial_\mu A_\nu\left(\delta_\sigma^\beta\delta_\rho^\alpha - \delta_\rho^\beta\delta_\sigma^\alpha\right)\right] - \lambda(\partial\cdot A)\,\eta^{\mu\nu}\delta_\mu^\beta\delta_\nu^\alpha$$

$$= -\frac{1}{2}\left[F^{\beta\alpha} + F^{\beta\alpha}\right] - \lambda\eta^{\beta\alpha}(\partial A) = -F^{\beta\alpha} - \lambda\eta^{\beta\alpha}(\partial A)$$

$$= F^{\alpha\beta} + \lambda\,\eta^{\alpha\beta}(\partial A)$$

$$\Rightarrow \partial_\beta\frac{\partial\mathcal{L}}{\partial(\partial_\beta A_\alpha)} = -\partial_\beta F^{\beta\alpha} - \lambda\,\partial_\beta\eta^{\beta\alpha}(\partial A)$$

$$= \partial_\beta\partial^\alpha A^\beta - \partial_\beta\partial^\beta A^\alpha - \lambda\partial^\alpha(\partial A) =$$

$$= \partial^\alpha\partial_\beta A^\beta - \Box A^\alpha - \lambda\partial^\alpha(\partial A) =$$

$$= (1-\lambda)\partial^\alpha(\partial A) - \Box A^\alpha$$

$\Rightarrow$ Euler-Lagrange : $\Box A^\alpha + (\lambda-1)\partial^\alpha(\partial A) = J^\alpha$

③ $$T^{\mu\nu} = -F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4}\eta^{\mu\nu}(F\cdot F)$$

$$T^{00} = -F^{0\alpha}F^{0}{}_{\alpha} + \frac{1}{4}\underbrace{\eta^{00}}_{=1}(F\cdot F) =$$

$$= -\underbrace{F^{00}F^{0}{}_{0}}_{=0} + \underbrace{F^{0i}F^{0}{}_{i}}_{=E^iE_i} + \frac{1}{4}\underbrace{F^{\alpha\beta}F_{\alpha\beta}}_{=\underbrace{F^{00}F_{00}}_{=0} - \underbrace{F^{0i}F_{0i}}_{E^iE_i} - \underbrace{F^{i0}F_{i0}}_{E^iE_i} + \underbrace{F^{ij}F_{ij}}_{\varepsilon^{ijk}\varepsilon_{ijl}B_kB^l = 2B_kB^k}}$$

$$\Longrightarrow T^{00} = \frac{1}{2}E^iE_i + \frac{1}{2}B_kB^k = \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2$$

$$T^{0i} = -F^{0\alpha}F^{i}{}_{\alpha} + \frac{1}{4}\underbrace{\eta^{0i}}_{=0}(FF) = -\underbrace{F^{00}}_{=0}F^{i}{}_{0} + F^{0k}F^{i}{}_{k} = E_k\left(\varepsilon^{i}{}_{kl}B^{l}\right) =$$
$$= (\vec{E}\times\vec{B})^i$$

④ $$\left[A^\mu(t,\vec{x}),\ \pi^\nu(t,\vec{y})\right] = i\eta^{\mu\nu}\,\delta(\vec{x}-\vec{y})$$

$$= \left[A^\mu(t,\vec{x}),\ F^{\nu 0}(t,\vec{y}) - \lambda\eta^{\nu 0}(\partial A)\right] =$$

$$= \left[A^\mu(t,\vec{x}),\ F^{\nu 0}(t,\vec{y})\right] - \lambda\left[A^\mu(t,\vec{x}),\ \eta^{\nu 0}(\partial A)\right] \overset{\lambda=1}{=}$$

$$= \left[A^\mu(t,\vec{x}),\ \partial^\nu A^0(t,\vec{y}) - \underbrace{\partial^0 A^\nu(t,\vec{y})}_{=\dot{A}(t,\vec{y})}\right] - \left[A^\mu(t,\vec{x}),\ \underbrace{\eta^{\nu 0}(\partial^\alpha A_\alpha)}_{\to\ \partial^\nu A^0}\right]$$

$$\Longrightarrow\ -\left[A^\mu(t,\vec{x}),\ \dot{A}^\nu(t,\vec{y})\right] = i\eta^{\mu\nu}\,\delta(\vec{x}-\vec{y})$$

⑤ $$G_F(k) = \frac{-1}{k^2} = \frac{-1}{(k^0)^2 - \vec{k}^2} = \frac{-1}{(k^0)^2 - \omega_k^2} \qquad \vec{k}^2 = \omega_k^2 \ (\text{car ici } m=0)$$

$$\Rightarrow G_F(x-y) = \int \frac{dk}{(2\pi)^4} G_F(k)\, e^{-i(x-y)k} =$$

$$= \underbrace{\int \frac{dk^0}{(2\pi)} \frac{-1}{(k^0)^2 - \omega_k^2} e^{-ik^0(x^0-y^0)}}_{=I} \int \frac{d\vec{k}}{(2\pi)^3} e^{+i\vec{k}(\vec{x}-\vec{y})}$$

Analogue au cas scalaire:

(i) $x^0 > y^0$: on boucle le contour "par le bas"

$$I = \int \frac{dk^0}{(2\pi)} \frac{-1}{(k^0 - \omega_k + i\varepsilon)(k^0 + \omega_k - i\varepsilon)} e^{-ik^0(x^0-y^0)} \stackrel{\downarrow}{=}$$

théorème des résidus

$$= -\frac{2\pi i}{(2\pi)} \left[e^{-i(\omega_k - i\varepsilon)(x^0-y^0)} \frac{-1}{2\omega_k - 2i\varepsilon} \right] \xrightarrow{\varepsilon \to 0} \frac{i}{2\omega_k} e^{-i\omega_k(x^0-y^0)}$$

(ii) $x^0 < y^0$: on boucle "par le haut"

$$I = -\frac{2\pi i}{2\pi} \left[e^{-i(-\omega_k + i\varepsilon)(x^0-y^0)} \frac{1}{-2\omega_k + 2i\varepsilon} \right] \xrightarrow{\varepsilon \to 0} \frac{i}{2\omega_k} e^{+i\omega_k(x^0-y^0)}$$

$$\Rightarrow G_F(x-y) = i \int \frac{d\vec{k}}{(2\pi)^3} e^{+i\vec{k}(\vec{x}-\vec{y})} \left[\Theta(x^0-y^0) \frac{i}{2\omega_k} e^{-i\omega_k(x^0-y^0)} + \Theta(y^0-x^0) \frac{i}{2\omega_k} e^{+i\omega_k(x^0-y^0)} \right] =$$

$$= i \int d\tilde{k} \left[\Theta(x^0-y^0)\, e^{-ik(x-y)} + \Theta(y^0-x^0)\, e^{+ik(x-y)} \right]$$

$$A_\mu(x) = \int d\tilde{k} \sum_\lambda \Big[\underbrace{a(\vec{k},\lambda)\, e_\mu(\vec{k},\lambda)\, e^{-ikx}}_{\to A_\mu^+(x)} + \underbrace{a^\dagger(\vec{k},\lambda)\, e_\mu(\vec{k},\lambda)\, e^{+ikx}}_{A_\mu^-(x)} \Big]$$

$$\langle 0 | A_\mu(x) A_\nu(y) | 0 \rangle = \langle 0 | \left[A_\mu^+(x) + A_\mu^-(x) \right] \left[A_\nu^+(y) + A_\nu^-(y) \right] | 0 \rangle =$$
$$= \langle 0 | A_\mu^+(x) A_\nu^-(y) | 0 \rangle = \langle 0 | [A_\mu^+(x), A_\nu^-(y)] | 0 \rangle$$

$$[A^+_\mu(x), A^-_\nu(y)] = \int d\tilde{k} \int d\tilde{p}\; e^{-ikx} e^{-ipy} \underbrace{\sum_\lambda \sum_\sigma e_\mu(\vec{k},\lambda) e_\nu(\vec{p},\sigma)}_{\text{relation de fermeture}} \underbrace{\left[a(\vec{k},\lambda)\, a^\dagger(\vec{p},\sigma) \right]}_{= (2\pi)^3\, 2\omega_k\, \delta(\vec{k}-\vec{p})\,(-\eta^{\lambda\sigma})} =$$

$$= \int d\tilde{k}\; e^{-ik(x-y)} \left(-\eta_{\mu\nu}\right)$$

$$\Rightarrow \langle 0 | A_\mu(x) A_\nu(y) | 0 \rangle = -\eta_{\mu\nu} \int d\tilde{k}\; e^{-ik(x-y)}$$

$$\Rightarrow \langle 0 | T\{ A_\mu(x) A_\nu(y) \} | 0 \rangle = \theta(x^0 - y^0) \langle 0 | A_\mu(x) A_\nu(y) | 0 \rangle +$$

$$+ \theta(y^0 - x^0) \langle 0 | A_\nu(y) A_\mu(x) | 0 \rangle =$$

$$= -\eta_{\mu\nu} \left[\theta(x^0 - y^0) \int d\tilde{k}\; e^{-ik(x-y)} + \theta(y^0 - x^0) \int d\tilde{k}\; e^{+ik(x-y)} \right.$$

$$= i\, \eta_{\mu\nu}\, G_F(x-y)$$