Exercices

3.1) $\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}(\vec{\nabla}\varphi)^2 - V(\varphi)$ avec $V(\varphi) = \frac{1}{2}m^2\varphi^2$

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - V(\varphi)$$

Euler-Lagrange : $\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} - \frac{\partial\mathcal{L}}{\partial\varphi} = 0$

$$\left.\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\varphi} &= -m^2\varphi \\
\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} &= \partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)} - \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi)} = \\
&= \partial_t(\partial_t\varphi) - \partial_i(\partial_i\varphi) = \\
&= \partial_t^2\varphi - \Delta\varphi = \Box\varphi
\end{aligned}\right\} \Rightarrow \Box\varphi + m^2\varphi = 0$$

3.2) $T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}(\partial^\nu\varphi) - \eta^{\mu\nu}\mathcal{L}$

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}(\partial^0\varphi) - \mathcal{L} = \underbrace{\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}}_{=\dot{\varphi}=\pi}\dot{\varphi} - \mathcal{L} = \pi\dot{\varphi} - \mathcal{L} = \mathcal{H}$$

3.3)
$$\varphi(x) = \int d\tilde{k}\left[a(\vec{k})e^{-ikx} + a^\dagger(\vec{k})e^{ikx}\right]$$

$$\Box\varphi + m^2\varphi = \partial_t^2\varphi - (\vec{\nabla}\varphi)^2 + m^2\varphi \stackrel{?}{=} 0$$

$$kx = k^0x^0 - k^ix_i = k^0\cdot t - k^ix_i \qquad k^0 = \sqrt{\vec{k}^2+m^2}$$

$$\partial_t\varphi = \int d\tilde{k}\left[a(\vec{k})\,e^{-ikx}(-ik^0) + a^\dagger(\vec{k})\,e^{ikx}(ik^0)\right]$$

$$\partial_t^2\varphi = -\int d\tilde{k}\left[a(\vec{k})e^{-ikx}(k^0)^2 + a^\dagger(\vec{k})e^{ikx}(k^0)^2\right]$$

$$(k^0)^2 = \vec{k}^2 + m^2$$

$$\vec{\nabla}\varphi = \int d\tilde{k}\left[a(\vec{k})\,e^{-ikx}(i\vec{k}) + a^\dagger(\vec{k})\,e^{ikx}(-i\vec{k})\right]$$

$$\vec{\nabla}^2\varphi = -\int d\tilde{k}\left[a(\vec{k})e^{-ikx}\,\vec{k}^2 + a^\dagger(\vec{k})e^{ikx}\,\vec{k}^2\right]$$

$$\Rightarrow \Box\varphi + m^2\varphi = -(\vec{k}^2+m^2)\int d\tilde{k}\,[\ldots] + \vec{k}^2\int d\tilde{k}\,[\ldots] + m^2\int d\tilde{k}\,[\ldots] = 0$$

3.4) $[\varphi(x), \pi(y)] = [\varphi(x), \dot\varphi(y)] =$

$$= \left[\int d\tilde{k} \left(a(\vec{k}) e^{-ikx} + a^\dagger(\vec{k}) e^{ikx} \right), \int d\tilde{p} \left(-i\omega_p a(\vec{p}) e^{-ipy} + i\omega_p a^\dagger(\vec{p}) e^{ipy} \right) \right] =$$

$$= \int d\tilde{k} \underbrace{\int d\tilde{p}}_{= \frac{d\vec{p}}{(2\pi)^3 2\omega_p}} \Big(i\omega_p e^{-ikx} e^{ipy} \underbrace{[a(\vec{k}), a^\dagger(\vec{p})]}_{(2\pi)^3 2\omega_k \delta(\vec{k}-\vec{p})} - i\omega_p e^{ikx} e^{-ipx} \underbrace{[a^\dagger(\vec{k}), a(\vec{p})]}_{-(2\pi)^3 \omega_k \delta(\vec{k}-\vec{p})} \Big) =$$

$$= \int d\tilde{k} \, \frac{i}{2} \int d\vec{p} \left(2\omega_k e^{-ikx} e^{ipy} \delta(\vec{k}-\vec{p}) + 2\omega_k e^{ikx} e^{-ipy} \delta(\vec{k}-\vec{p}) \right) =$$

$$= \underbrace{\int d\tilde{k}}_{= \frac{d\vec{k}}{(2\pi)^3 2\omega_k}} \frac{i}{2} 2\omega_k \int d\vec{p} \left(e^{-ikx+ipy} + e^{ikx-ipy} \right) \delta(\vec{k}-\vec{p}) =$$

$k(x-y) =$
$k^0(t-t) - \vec{k}(\vec{x}-\vec{y})$

$$= \frac{i}{2} \int \frac{d\vec{k}}{(2\pi)^3} \left(e^{-ik(x-y)} + e^{ik(x-y)} \right) =$$

$$= \frac{i}{2} \int \frac{d\vec{k}}{(2\pi)^3} \left(e^{i\vec{k}(\vec{y}-\vec{x})} + e^{i\vec{k}(\vec{x}-\vec{y})} \right) =$$

$$= \frac{i}{2} \left(\delta(\vec{y}-\vec{x}) + \delta(\vec{x}-\vec{y}) \right) = i\delta(\vec{x}-\vec{y})$$

3.5 $H(t) = \int d\vec{x} \left[\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{m^2}{2}\varphi^2 \right]$

$$\varphi = \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega_k} \left[a_k e^{-ikx} + a_k^\dagger e^{ikx} \right] = \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega_k} \left[a_k e^{-i\omega_k t} e^{i\vec{k}\vec{x}} + \underbrace{a_k^\dagger e^{i\omega_k t} e^{-i\vec{k}\vec{x}}}_{\vec{k}\to-\vec{k}:\ \omega_{-k} = \sqrt{\vec{k}^2+m^2} = \omega_k} \right] =$$

$$= \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega_k} \left[a_k e^{-i\omega_k t} + a_{-k}^\dagger e^{i\omega_k t} \right] e^{i\vec{k}\vec{x}}$$

$$\dot{\varphi} = -i \int \frac{d\vec{k}}{(2\pi)^3\, 2} \left[a_k e^{-i\omega_k t} - a_{-k}^\dagger e^{i\omega_k t} \right] e^{i\vec{k}\vec{x}}$$

$$\vec{\nabla}\varphi = i \int \frac{d\vec{k}}{(2\pi)^3\, 2\omega_k} \vec{k} \left[a_k e^{-i\omega_k t} + a_{-k}^\dagger e^{i\omega_k t} \right] e^{i\vec{k}\vec{x}}$$

$$\int d\vec{x}\, \frac{m^2}{2}\varphi^2 = \frac{m^2}{2} \int d\vec{x}\, \varphi^\dagger \varphi \;\stackrel{\leftarrow \text{théorème de Parseval}}{=}$$

$$= \frac{m^2}{8} \int \frac{d\vec{k}}{(2\pi)^3} \frac{1}{\omega_k^2} \left[a_k^\dagger e^{i\omega_k t} + a_{-k} e^{-i\omega_k t} \right] \left[a_k e^{-i\omega_k t} + a_{-k}^\dagger e^{i\omega_k t} \right]$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \frac{m^2}{\omega_k^2} \left[a_k^\dagger a_k + a_{-k} a_{-k}^\dagger + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} + a_{-k} a_k e^{-2i\omega_k t} \right]$$

$$\int d\vec{x}\, \frac{1}{2}\dot{\varphi}^2 = \frac{1}{2} \int d\vec{x}\, \dot{\varphi}^\dagger \dot{\varphi} \;\stackrel{\leftarrow \text{théorème de Parseval}}{=}$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \left[a_k^\dagger e^{i\omega_k t} - a_{-k} e^{-i\omega_k t} \right] \left[a_k e^{-i\omega_k t} - a_{-k}^\dagger e^{i\omega_k t} \right] =$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \left[a_k^\dagger a_k + a_{-k} a_{-k}^\dagger - a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} - a_{-k} a_k e^{-2i\omega_k t} \right]$$

$$\int d\vec{x}\, \frac{1}{2}(\vec{\nabla}\varphi)^2 = \frac{1}{2} \int d\vec{x}\, (\vec{\nabla}\varphi)^\dagger (\vec{\nabla}\varphi) \;\stackrel{\leftarrow \text{théorème de Parseval}}{=}$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \frac{\vec{k}^2}{\omega_k^2} \left[a_k^\dagger e^{i\omega_k t} + a_{-k} e^{-i\omega_k t} \right] \left[a_k e^{-i\omega_k t} + a_{-k}^\dagger e^{i\omega_k t} \right] =$$

$$= \frac{1}{8} \int \frac{d\vec{k}}{(2\pi)^3} \frac{\vec{k}^2}{\omega_k^2} \left[a_k^\dagger a_k + a_{-k} a_{-k}^\dagger + a_k^\dagger a_{-k}^\dagger e^{2i\omega_k t} + a_{-k} a_k e^{-2i\omega_k t} \right]$$

⚠ $\frac{m^2}{\omega_k^2} + \frac{\vec{k}^2}{\omega_k^2} = 1 \Rightarrow$ termes en $e^{\pm 2i\omega_k t}$ s'annulent

$$\Rightarrow H(t) = \frac{1}{4}\int \frac{d\vec{k}}{(2\pi)^3}\left[a_k^\dagger a_k + a_{-k} a_{-k}^\dagger\right] = \frac{1}{4}\int \frac{d\vec{k}}{(2\pi)^3}\left[a_k^\dagger a_k + a_k a_k^\dagger\right] =$$

$$d\tilde{k} = \frac{d\vec{k}}{(2\pi)^3\, 2\omega_k}$$

$$= \frac{1}{2}\int d\tilde{k}\, \omega_k \left[a_k^\dagger a_k + a_k a_k^\dagger\right]$$

$$= \int d\tilde{k}\, \omega_k \left[a_k^\dagger a_k + \text{const.}\right]$$

↳ énergie du vide diverge!

3.6| $$[P^\mu, a_p^\dagger] = \left[\int d\tilde{k}\, k^\mu a_k^\dagger a_k\,,\, a_p^\dagger\right] =$$

$$= \int d\tilde{k}\, k^\mu \left[a_k^\dagger a_k, a_p^\dagger\right] = \int d\tilde{k}\, k^\mu \left(a_k^\dagger a_k a_p^\dagger - a_p^\dagger a_k^\dagger a_k\right) =$$

$$= \int d\tilde{k}\, k^\mu a_k^\dagger \left[a_k, a_p^\dagger\right] = \int d\tilde{k}\, k^\mu a_k^\dagger\, \delta(\vec{k}-\vec{p}) \cdot (2\pi)^3 2\omega_k =$$

$$= p^\mu a_p^\dagger$$