Exercices

2.1] $\varphi_n = \varphi_k e^{ikna}$ $\qquad\qquad$ $\ddot\varphi_n = -\omega_0^2\left(2\varphi_n - \varphi_{n-1} - \varphi_{n+1}\right)$

$$\Rightarrow \ddot\varphi_k e^{ikna} = -\omega_0^2\left(2\varphi_k e^{ikna} - \varphi_k e^{ik(n-1)a} - \varphi_k e^{ik(n+1)a}\right)$$

$$\Rightarrow \ddot\varphi_k = -\omega_0^2\varphi_k\left(2 - e^{-ika} - e^{ika}\right) = -\omega_0^2\varphi_k \underbrace{\left(2 - 2\cos(ka)\right)}_{=4\sin^2\left(\frac{ka}{2}\right)}$$

2.3] $$\ddot\varphi_n = -\omega_0^2\left[2\varphi_n - \varphi_{n-1} - \varphi_{n+1}\right] = -\omega_0^2\left[(\varphi_n - \varphi_{n-1}) + (\varphi_n - \varphi_{n+1})\right]$$

$$= \omega_0^2\left[(\varphi_{n+1} - \varphi_n) - (\varphi_n - \varphi_{n-1})\right] =$$

$$= \omega_0^2 a\left[\frac{\varphi_{n+1} - \varphi_n}{a} - \frac{\varphi_n - \varphi_{n-1}}{a}\right] \overset{*}{=} \omega_0^2 a\left[\varphi'(x,t) - \varphi'(x-a,t)\right] =$$

$$= \omega_0^2 a^2 \frac{\varphi'(x,t) - \varphi'(x-a,t)}{a} \overset{*}{=} \underbrace{\omega_0^2 a^2}\, \varphi''(x-a,t)$$

* limite continue $\qquad\qquad$ $\hookrightarrow \omega_0^2 a^2 = \frac{k}{m} a^2 = \frac{ka}{m/a} = \frac{\alpha}{\mu}$

2.4] $$\varphi(x,t) = \frac{1}{\sqrt{L_0}} \sum_k \left[A_k e^{-i(\omega_k t - kx)} + A_k^* e^{i(\omega_k t - kx)}\right]$$

$$\dot\varphi(x,t) = \frac{1}{\sqrt{L_0}} \sum_k \left[-i\omega_k A_k e^{-i(\omega_k t - kx)} + i\omega_k A_k^* e^{i(\omega_k t - kx)}\right]$$

$$= \frac{i\omega_k}{\sqrt{L_0}} \sum_k \left[-A_k e^{-i(\omega_k t - kx)} + A_k^* e^{i(\omega_k t - kx)}\right]$$

$$\hookrightarrow A_k = \frac{1}{2}\frac{1}{\sqrt{L_0}} \int_0^{L_0} \left[\varphi(x,0) e^{-ikx} - \frac{1}{i\omega_k}\dot\varphi(x,0) e^{-ikx}\right] dx$$

$$= \frac{1}{2\sqrt{L_0}} \int_0^{L_0} \left[\varphi(x,t) + \frac{i}{\omega_k}\frac{\Pi(x,0)}{\mu}\right] e^{-ikx}\, dx$$

2.5| $[\varphi_i(t), P_j(t)] = i\delta_{ij}$

Limite continue: $x = ia$, $y = ia$, $\frac{1}{a}\delta_{ij} \to \delta(x-y)$

$$\Rightarrow \quad [\varphi(x,t), \pi(y,t)] = i\,\delta(x-y)$$

$$[\varphi_i(t), \varphi_j(t)] = 0 \quad \Rightarrow \quad [\varphi(x,t), \varphi(y,t)] = 0$$

$$[P_i(t), P_j(t)] = 0 \quad \Rightarrow \quad [\pi(x,t), \pi(y,t)] = 0$$

2.6| $\mathcal{H}(t) = \int_0^{L_0} \left[\frac{\mu}{2}\dot{\varphi}^2 + \frac{\alpha}{2}\varphi'^2\right] dx$

$$\dot{\varphi}(x,t) = \sum_k \sqrt{\frac{\omega_k}{2\mu L_0}}\left[a_k e^{-i\omega_k t} - a_{-k}^+ e^{i\omega_k t}\right] e^{ikx} = \sum_k F_k e^{ikx}$$

$$\varphi'(x,t) = \sum_k \frac{k}{\sqrt{2\mu\omega_k L_0}}\left[a_k e^{-i\omega_k t} + a_{-k}^+ e^{i\omega_k t}\right] e^{ikx} = \sum_k G_k e^{ikx}$$

$$\frac{\mu}{2}\int \dot{\varphi}^2 dx = \frac{\mu}{2}\int \dot{\varphi}^+\dot{\varphi}\, dx \overset{*}{=} \frac{\mu}{2} L_0 \sum_k F_k^+ F_k =$$

$*$ Parseval

$$= \frac{\mu L_0}{2}\sum_k \frac{\omega_k}{2\mu L_0}\left[a_k^+ e^{i\omega_k t} - a_{-k} e^{-i\omega_k t}\right]\left[a_k e^{-i\omega_k t} - a_{-k}^+ e^{i\omega_k t}\right] =$$

$$= \sum_k \frac{\omega_k}{4}\left[a_k^+ a_k + a_{-k} a_{-k}^+ - a_k^+ a_{-k}^+ e^{i2\omega_k t} - a_{-k} a_k e^{-i2\omega_k t}\right]$$

$$\frac{\alpha}{2}\int \varphi'^2 dx = \frac{\alpha}{2}\int \varphi'^+ \varphi'\, dx \overset{*}{=} \frac{\alpha}{2} L_0 \sum_k G_k^+ G_k$$

$$= \frac{\alpha L_0}{2}\sum_k \frac{k^2}{2\mu\omega_k L_0}\left[a_k^+ e^{i\omega_k t} + a_{-k} e^{-i\omega_k t}\right]\left[a_k e^{-i\omega_k t} + a_{-k}^+ e^{i\omega_k t}\right] =$$

$$= \sum_k \underbrace{\frac{\alpha k^2}{4\mu\omega_k}}_{=\frac{\omega_k}{4}}\left[a_k^+ a_k + a_{-k} a_{-k}^+ + a_k^+ a_{-k}^+ e^{i2\omega_k t} + a_{-k} a_k e^{-i2\omega_k t}\right]$$

$$\Rightarrow \mathcal{H}(t) = \sum_k \frac{\omega_k}{4}\left[2 a_k^+ a_k + 2 a_{-k} a_{-k}^+\right] = \sum_k \frac{\omega_k}{2}\left[a_k^+ a_k + a_k a_k^+\right]$$