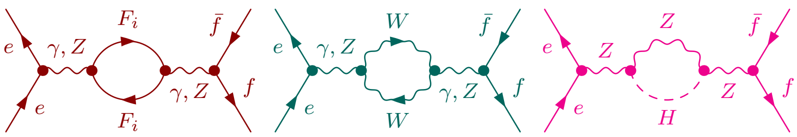
Renormalisation of the Electroweak Theory
Introduction and Precision Tests of the Standard Model



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1 Introduction: The Gauge symmetry principle

The electroweak theory combines two fundamental principles:
1.The gauge symmetry principle which must have been familiar to you from QED even if here it is used for a non-Abelian theory and
2. The concept of hidden symmetry or what is often called, the breaking of the symmetry. I prefer hidden since this is done so that gauge invariance for physical observables is maintained. Among other things, gauge invariance is crucial for the renormalisation of the theory and allows a full quantum treatment of the theory. Since one can then pursue a higher order treatment of the theory, the latter becomes very predictive and can then be compared to very precise measurements. Most precise measurements have been done at the Z-peak at LEP and SLC. These, with some other low energy data, and together with the recent discovery of the top quark and more recently a rather precise measurement of the W mass, have truly crowned the $\mathcal{SM}$ and elevate it to the status of a theory.

1.1 Electromagnetism as a prototype

Maxwell equations are the first successful attempts to unify two seemingly disparate phenomena and to bring the electric field and magnetic field together under the same entity. At heart is the *local* conservation of the electric charge as embodied in the current $j^\mu = (\rho, \overrightarrow{j})$

$$\begin{aligned} div\overrightarrow{E} &= \rho \qquad div\overrightarrow{B} = 0 \\ Curl\overrightarrow{E} + \frac{\partial\overrightarrow{B}}{\partial t} &= 0 \qquad Curl\overrightarrow{B} - \frac{\partial\overrightarrow{E}}{\partial t} = \overrightarrow{j} \end{aligned} \tag{1}$$

Gauge invariance is also at work here. One manifestation of the gauge invariance principle is the fact that in electrostatics the electric field, and hence the electrostatic force, depends only on the *difference* of potential. The gauge field idea is best known in the quantised version of Maxwell's equations. The quantum field creating and annihilating photons is the vector potential $A^\mu(x)$, which is related to the electromagnetic field strength by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. The field strength, and thereby Maxwell's equations, are invariant under gauge transformations,

$$A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x) \; . \tag{2}$$

Maxwell equations can then be written in a very compact form as

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \varepsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = 0. \tag{3}$$

Gauge invariance is also familiar from the quantum mechanics description of a charged particle interacting with an electromagnetic field. Schrodinger's equation for a free particle of mass m , $(1/2m)(-i\overrightarrow{\nabla})^2\psi = i\partial\psi/\partial t$, is invariant under a *global* phase transformation $\psi \to \exp(i\lambda)\psi$. If one insists that the equation remains valid under *local* phase transformation, in other words that the transformation can be different at different points in space-time, $\lambda \to q\Lambda(x = (t, \overrightarrow{x}))$, one is then led to introduce a *compensating* vector field which transforms exactly like Eq. 2. This prescription gives the familiar Schrodinger's equation of a particle of charge q with the electromagnetic field $A^\mu = (V, \overrightarrow{A})$,

$$(1/2m)\left(-i\overrightarrow{\nabla} + q\overrightarrow{A}\right)^2 \psi = (i\partial/\partial t + qV)\,\psi \ . \tag{4}$$

This principle is carried over to the relativistic quantum case by requiring that all derivatives ∂_μ be replaced by *covariant derivatives* $D_\mu = \partial_\mu - iqA_\mu$, in analogy with Eq. 4 which clearly displays the combination of the space-time derivatives and the vector potential components. An important consequence of this, is that all charged particles couple exactly the same way to the electromagnetic field, the *coupling is universal* . The essential point is that charged fields and covariant derivatives of charged fields have identical local transformations,

$$\psi(x) \to U(x)\psi(x) \ , \qquad D_\mu\psi(x) \to U(x)D_\mu\psi(x) \ , \qquad U(x) = e^{iq\Lambda(x)} \tag{5}$$

The group of transformations, $U(x)$, in Eq. 5, is an Abelian $U(1)$ group since it does not matter in which order we apply successive transformations of the form $U(x)$.

Electromagnetism is a long range force where the messenger of the force, the photon is massless. As is known also, electromagnetic fields have only two transverse independent polarisations (one also speaks of the photon as having only 2 helicity states), despite the fact that the photon is a spin-1 described by a vector A_μ. In fact all these properties, the derivation of Maxwell's equations that *unify* the electric and magnetic fields are embodied in the simple Lagrangian describing (free) photons

$$\mathcal{L}_{\rm em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \equiv \frac{1}{4}\left((\overrightarrow{E} + i\overrightarrow{B})^2 + (\overrightarrow{E} - i\overrightarrow{B})^2\right) \ . \tag{6}$$

$(\overrightarrow{E} \pm i\overrightarrow{B})$ displays the two helicity states of the photon, or polarisation of the field. A mass term for the photon would be represented by $m^2 A_\mu A^\mu$ which breaks gauge invariance as it is not invariant under Eq. 2. QED which is the gauge theory describing the interaction of electrons with the photons has been tremendously successful.

1.2 Weak interactions and non-Abelian gauge theories

Considering how elegant and powerful the gauge invariance principle is and how it dictates the form of the interaction, it was natural to extend it to the weak interactions. So much so, since processes as different as β-decay ($n \to p + e^- + \bar{\nu}_e$), muon decay ($\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$) or muon capture ($\mu^- + p \to n + \nu_\mu$) seemed to be of the same nature and have the same strength, pointing to universality. There are however some important differences with QED.
These processes are charged current processes representing $e^- \leftrightarrow \nu_e$ or $n \leftrightarrow p$ transitions and thus contrary to QED, they involve a change in the identity of the matter, spin-1/2 field. Moreover, the structure of the weak current suggested that only left-handed fields are involved. For example before the formulation of the electroweak theory muon decay was represented by a four-point interaction along Fermi's prescription and takes the form

$$\mathcal{L}_{\text{Fermi}} = -\frac{G_F}{\sqrt{2}} \bar{\nu}_\mu \gamma^\alpha (1-\gamma_5)\mu \; \bar{e}\gamma^\alpha(1-\gamma_5)\nu_e \tag{7}$$

G_F is the Fermi constant.

Chirality (left-handed and right-handed states) of a fermion corresponds, at high-energy, to the helicity state of the fermion. An electron field can be decomposed then as $e = e_L + e_R$. $e_L = (1-\Gamma_5)/2e$. Electromagnetism treats these two components on a equal footing, preserving symmetry under parity. Indeed the gauge coupling of electrons to photons writes $\bar{e}\gamma^\mu A_\mu e = \bar{e}_L\gamma^\mu A_\mu e_L + \bar{e}_R\gamma^\mu A_\mu e_R$. γ^μ are a set of matrices which may be thought of as a relativistic generalisation of the Pauli spin matrices. Note in passing that the gauge interaction does not mix these two states. In contrast, the charged current to which one would like to associate a charged gauge boson, $W^\pm$, is of the form, $\bar{e}_L\gamma^\mu W^+_\mu \nu_{e(L)}$. To make this resemble the electromagnetic current, one can write it as $\bar{E}_L\gamma^\mu W^+_\mu \tau^+ E_L$. The entity $E_L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}$ should be considered as a doublet (one speaks of an isospin doublet) and $\tau^\pm$ are the raising and lowering Pauli matrices.

This two-level transition is also very familiar to us from quantum mechanics as is the use of the Pauli matrices τ. The smallest group of gauge transformation acting on the doublet E_L (and n,p) generalising Eq. 5, is the non-Abelian $SU(2)_L$, which besides $\tau^\pm$ has also "a neutral" generator τ^3. There will therefore be 3 compensating *gauge fields*: $W^\pm_\mu, W^3_\mu$. $SU(2)_L$ symmetry predicts the coupling of W^3: $\bar{E}_L\gamma_\mu W^3_\mu \tau^3 E_L = \bar{\nu}_e\gamma_\mu W^3_\mu \nu_e \; - \; \bar{e}_L\gamma_\mu W^3_\mu e_L$. Unfortunately this neutral current does not correspond to the electromagnetic current. For a start it involves neutral particles. On the other hand, it does include a part (left-handed) of the electromagnetic current. Therefore we do have a partial unification or at least a unified description of the weak and electromagnetic interaction. To fully reconstruct the electromagnetic current from the neutral isospin current, one must postulate the existence of at least another neutral current. In the standard model this is introduced via a $U(1)_Y$ neutral current, associated with the hypercharge Y and a gauge field B_μ. The latter couples to both the left-handed doublet and right-handed (*e.g* e_R) isospin singlet. The photon and the Z will appear as a superposition of the fields B_μ and W^3_μ.

It is also appropriate to say, at this stage, that the neutron and the proton are made up of quarks, u and d, that form a doublet under $SU(2)$ and which are bound by the strong force. Each quark carries a set of three colours. The messenger of the colour force, strong interaction, is the gluon. The gauge group here is $SU(3)$ which means in fact that we have eight types of gluons, corresponding to the eight generators of the fundamental representation of $SU(3)$ represented by the 8 Gell-Mann matrices T^a. e, ν_e, u, d form the first generation of matter particles of the SM model. One has discovered three such families. These fermions are listed in Table. 1 which gives their respective charge under the three gauge groups $SU(2)_L, U(1)_Y, SU(3)_c$.
The gauge coupling constants of these three fundamental interactions, which help define the generalisation of the gauge transformations (Eq. 5) and the covariant derivative are,

Table 1: *Quantum numbers of the first generation of quarks and leptons, and of the Higgs doublet field, Φ, within the SM. The rows give the irreducible representations under colour SU(3) and weak SU(2), and the hyper-charges for the various multiplets. The electric charge (in units of e) is given in the last column. The assignments for the quarks and leptons of the second and third generations* ((c, s, ν_μ, μ) *and* (t, b, ν_τ, τ)) *are identical to the ones for the first generation.*

	SU(3)	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$	$Q = T_3 + Y$
$Q = (u_L, d_L)$	3	2	$\frac{1}{6}$	$(\frac{2}{3}, -\frac{1}{3})$
u_R	3	1	$\frac{2}{3}$	$\frac{2}{3}$
d_R	3	1	$-\frac{1}{3}$	$-\frac{1}{3}$
$L = (\nu_L, e_L)$	1	2	$-\frac{1}{2}$	$(0, -1)$
e_R	1	1	-1	-1
ν_R	1	1	0	0

respectively, g, g' and g_s. The gauge transformations and the covariant derivative are then

$$\begin{aligned} \psi(x) &\rightarrow U(x)\psi(x) = e^{i\theta_3^a(x)T^a} e^{i\theta_2^j(x)\frac{\tau^j}{2}} e^{i\theta_1(x)Y}\psi(x) , \\ D_\mu &= \partial_\mu - ig_s T^a A_\mu^a - ig\frac{\tau^i}{2}W_\mu^i - ig'YB_\mu . \end{aligned} \tag{8}$$

The covariant derivatives completely specify the interactions of all known fermions and gauge bosons and encode the universality of the gauge couplings via the matter Lagrangian

$$\mathcal{L}_{\text{matter}} = \sum_{j=Q,u_R,d_R,L,e_R,\nu_R} \overline{\psi}_j i\gamma^\mu D_\mu \psi_j . \tag{9}$$

An important observation concerning non-Abelian gauge theories is that the gauge fields are self-interacting. Not only the matter fields carry charge but also the gauge fields (isospin for $W^{1,2,3}$, colour for the gluons). This is imposed by the non-Abelian gauge transformations. For example the generalisation of the field strength, $F_{\mu\nu}$ for $SU(2)$ is

$$W_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\mu W_\nu^i + g\epsilon^{ijk} W_\mu^j W_\nu^k . \tag{10}$$

The first two terms enter the kinetic term as in the Abelian case, whereas the last term describes the self-interaction. The gauge field Lagrangian is simply the square of the field strength tensors,

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}A^{a,\mu\nu}A_{\mu\nu}^a - \frac{1}{4}W^{i,\mu\nu}W_{\mu\nu}^i - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} + \mathcal{L}_{\text{gauge fix.}} + \mathcal{L}_{\text{ghost}} , \tag{11}$$

apart from gauge fixing and ghost terms which are required for a proper quantization of the theory. Physical observables are of course, like in QED, independent of the choice of gauge. Although the gauge fixing explicitly breaks gauge invariance, since it freezes the fields in a particular configuration, the ghost Lagrangian is there to somehow, at the quantum level, bring back a remnant of the gauge symmetry. These issues are detailed in the Appendix but for the main course we will leave out such technicalities.

The gauge Lagrangian uniquely fixes the W^+W^-Z and $W^+W^-\gamma$ triple gauge vertices (TGV) which have been measured at LEP (, as well as the quartic couplings such as $W^+W^-Z\gamma$ in terms of a single parameter, the gauge coupling g. One important property is that the non-Abelian gauge symmetry predicts the gyromagnetic moment of the $W^\pm$ to be 2, like that of the charged elementary fermions.

2 Spontaneous symmetry breaking and mass generation

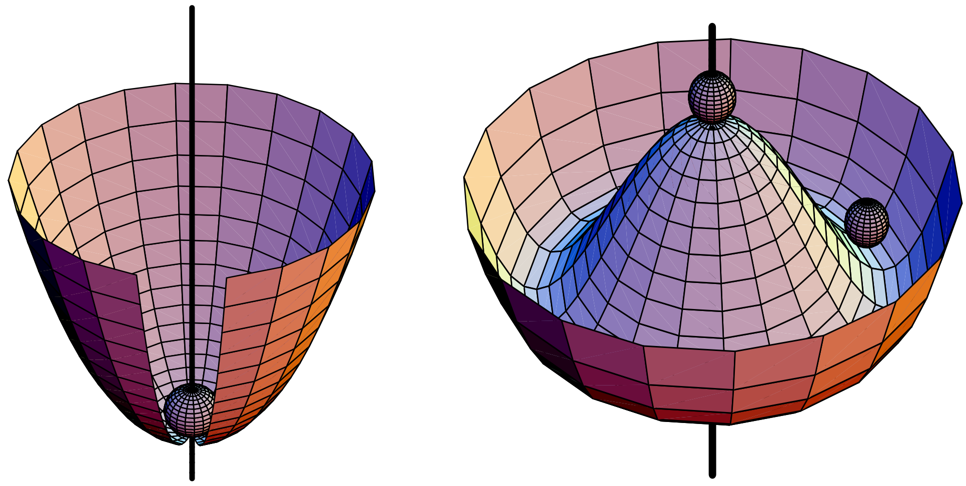
The main blow to the construct so far is that the weak interactions describe a short range force, in other words the $W^\pm$ and the Z are massive. As pointed out above for QED, a mass term introduced by hand destroys the gauge invariance of the theory. This major hurdle was solved by borrowing and adapting an idea that is encountered in many solid-state physics phenomena. In such systems, the Hamiltonian is invariant under a symmetry but the *ground state* of the system breaks this symmetry. Such is the case with a ferromagnet below the Curie temperature. In this case, rotational symmetry is broken by the ground state (all spins of the atoms aligned in the same direction) despite the fact that the dynamics (the Heisenberg spin-spin Hamiltonian) does not select any preferred direction. This spontaneous symmetry breaking is also at work in superconductivity where, with the Meissner effect, the fact that the magnetic field enters the solid over a very short range could be associated with a massive photon.

2.1 The Higgs mechanism and mass generation for the gauge bosons

One usually thinks about (and most often defines) the vacuum as a state where all fields have zero expectation value. However it may happen, like what is depicted in Fig. 1, that the state with zero energy is not the most stable. The system will choose stability or minimum energy (bottom of the well) rather than the state with maximum symmetry (top of the mountain). Such potentials are provided by scalar fields, with a potential of the form $V = \lambda(|\phi|^2 - v^2/2)^2$ $(\lambda > 0)$, where v is the vacuum expectation value (v.e.v) of the scalar field $< 0|\phi|0 >= v/\sqrt{2}$. This scale is the origin of the mass of the gauge bosons and also the fermions. It is as if some of the charged fermions and gauge bosons moving in such vacuua feel a resistance and behave like they have mass. To see how this works, it is most simple to take QED as an example to illustrate how a mass for the photon can be introduced in a *gauge invariant* way.

One needs to take a charged scalar field ϕ. The latter can be represented by a complex field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. For our purposes it is best to rewrite this in polar coordinate as $\phi = (h + v)e^{i\theta/v}/\sqrt{2}$, where both h and θ have zero v.e.v. θ characterises the rotational symmetry of the potential. The interaction of this scalar field with the electromagnetic field is described by a fully gauge invariant Lagrangian constructed through the use of

Figure 1: *Scalar potentials. The first panel shows a symmetric potential, under rotation around the vertical axis, with a stable minimum where the ball is resting. In this situation* $< 0|\phi|0 >= 0$. *The second panel, resembling a mexican hat, illustrates spontaneous symmetry breaking. The symmetric configuration at the top of the hat is unstable. The system will pick up* any *stable configuration along the brim with* $< 0|\phi|0 >\neq 0$. *The Goldstone mode therefore represents this azimuthal direction whereas the radial component is the Higgs field.*



the covariant derivative, Eq. 5. Upon expansion of this Lagrangian we find

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2\left(eA_\mu + \frac{1}{v}\partial_\mu\theta\right)^2 + \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{\lambda}{4}(h^2+2vh)^2 \\ &+ \frac{1}{2}\left(eA_\mu + \frac{1}{v}\partial_\mu\theta\right)(h^2+2vh)\ . \end{aligned} \quad (12)$$

One sees clearly that the gauge field has acquired a mass, $m_\gamma = ev$ (second term in Eq. 12: $+e^2v^2A_\mu A^\mu/2$). Because of gauge invariance we can always make a (local) phase transformation on the field $\phi = (h+v)e^{i\theta/v}/\sqrt{2}$ such that the "phase" θ/v is set to zero. This gauge where $\theta \to 0$ in Eq. 12 is the *unitary gauge* where only the *physical states* h, A_μ are left. Counting the number of degrees of freedom before and after symmetry breaking, we find the same number of course. Before, one had two scalars and one massless spin-1 which has only two (transverse) states of polarisation. After symmetry breaking, one of the scalars, θ, transmutes into the longitudinal polarisation of the "heavy photon". In fact it would be more appropriate to say that the gauge symmetry is hidden, once we choose a particular gauge, since the gauge symmetry is present in the dynamics. The θ field is a Goldstone boson, it corresponds, as seen in Eq. 12, to a massless pseudo-scalar. h is the Higgs scalar field whose mass is given by $\sqrt{2\lambda v^2}$. There is also a coupling hAA which proportional to the mass of the gauge boson.

A very similar approach is applied to the weak interaction. Since we need three massive gauge bosons, the three longitudinal states will be provided by three Goldstone bosons. This is most simply and economically provided by a Higgs doublet, Φ, with quantum numbers such that the vacuum is left invariant under electromagnetic gauge transformations, so that the photon remains massless. Thus $Y_\Phi = -1/2$,

$$\begin{aligned}
\Phi &= \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+H) \end{pmatrix} e^{i\omega^j \frac{\tau^j}{2v}} \\
\mathcal{L}_{\text{Higgs}} &= (D^\mu\Phi)^\dagger(D_\mu\Phi) - V(\Phi^\dagger\Phi)\ , \qquad \text{with} \qquad V(\Phi^\dagger\Phi) = \lambda\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2 .
\end{aligned} \tag{13}$$

where H is the physical Higgs field of the electroweak theory, $\omega^{1,2,3}$ the Goldstone bosons and v the vacuum expectation value.

2.1.1 Fermion masses

We already stressed the fact that QED treated both electron chiralities on the same footing, in particular both e_L and e_R have the same electric charge. Therefore the electron mass term $m_e(\bar{e}_R e_L + \bar{e}_L e_R)$ is gauge invariant in QED. According to Table 1 the SM has no left and right-handed multiplets with identical $SU(2)$ and $U(1)_Y$ charge, hence, a fermion mass term introduced by hand would break the symmetry of the SM. Once again, however, mass terms are possible via the Higgs mechanism. Let us consider the masses for the charged leptons. The left doublet L, right-handed singlet l_R and the Higgs field Φ combine so that they form a neutral symmetric object under $SU(2) \times U(1)$. The masses are introduced via Yukawa couplings y_l as

$$\begin{aligned}
-\mathcal{L}_m^l &= \sum_{i=e,\mu,\tau} y_l^i \left(\bar{L}_i \Phi l_{R,i} + \bar{l}_{R,i}\Phi^\dagger L_i\right) \longrightarrow \text{unitary gauge} \longrightarrow \\
&\sum_{i=e,\mu,\tau} \frac{y_l^i v}{\sqrt{2}}\left(1 + \frac{H}{v}\right)\bar{l}_i l_i \ , \quad \frac{y_l^i v}{\sqrt{2}} = m_l^i\ .
\end{aligned} \tag{14}$$

This exhibits a common important feature that applies also to quarks, namely that the couplings of the Higgs to fermions is proportional to the fermion mass. Because in the quark sector one is generating masses for both the up and down quark, one also induces mixing between the three-families in the charged current (but not in the neutral currents) . This is also the source of CP-violation in the model. It is important to stress that although masses and mixing are introduced in a gauge invariant way, one nonetheless needs to introduce a large number of ad-hoc Yukawa couplings, contrary to the gauge boson masses that are expressed through the universal gauge coupling.

3 Salient features of the model

Within the SM the scale of all particle masses is set by the Higgs v.e.v., v. For the gauge bosons, the mass term is produced by the kinetic energy term of the Higgs doublet field, $\mathcal{L}_{\text{Higgs}}$ in Eq. 13. The latter reveals several key predictions of the SM which can be tested at LEP.

1. The Higgs Lagrangian generates mass terms for the charged $W^\pm$ and for Z fields. W^3_μ and B_μ combine to give the massless photon and the Z,

$$\begin{aligned} Z_\mu &= \cos\theta_W W^3_\mu - \sin\theta_W B_\mu = \frac{1}{\sqrt{g^2+g'^2}}\left(gW^3_\mu - g'B_\mu\right) \\ A_\mu &= \sin\theta_W W^3_\mu + \cos\theta_W B_\mu \,. \end{aligned} \tag{15}$$

   This relation defines the weak mixing angle, or, more precisely, $\sin\theta_W$ and $\cos\theta_W$.

2. The W and Z masses are given by

$$M_W = \frac{gv}{2} \qquad \text{and} \qquad M_Z = \sqrt{g^2+g'^2}\,\frac{v}{2} = \frac{M_W}{\cos\theta_W}\,. \tag{16}$$

   The mass relation $M_W^2 = \rho M_Z^2 \cos^2\theta_W$ with $\rho = 1$ is a consequence of the breaking the electroweak symmetry by a scalar doublet Higgs field.

3. The mass terms for the W and Z are directly related to HWW and HZZ couplings, of strength $2M_W^2/v$ and $2M_Z^2/v$, respectively. This coupling of a single scalar to two gauge bosons requires the existence of a v.e.v. for the Higgs doublet field. Normally, gauge bosons couple to pairs of scalars only.

4. The Higgs boson mass is given by $M_H^2 = 2\lambda v^2$. Since the quartic coupling λ is a free parameter, there is no intrinsic prediction for M_H within the SM.

5. The same Higgs mechanism is responsible for the mass of the fermions. The coupling of the Higgs being proportional to the mass of the fermion, an intermediate mass Higgs, $M_H < 2M_W$, will decay predominantly to b-quarks.

It is important to stress that the model gives a very nice unified description of the weak and electromagnetic interactions. However the model does not fully unify these interactions, since we still have *two* (independent) gauge couplings g, g'. Viewed another way the model does not *predict* $\sin\theta_W$. Nonetheless the gauge principle automatically leads to the universality of the gauge coupling.

3.1 Gauge interactions of the fermions

We will be dealing mainly with the coupling of the gauge bosons to the fermions and in particular to the LEP/SLc phenomenology. It is therefore instructive to look again at the explicit form of this interaction as derived from Eq. 9,

$$\mathcal{L}_F = e\sum_i Q^i \bar{\psi}_i \gamma^\mu \psi_i A_\mu \quad (QED) \tag{17}$$

$$\frac{g}{2\sqrt{2}} \sum_i \bar{\psi}_i \gamma^\mu (1-\gamma^5)\left(T^+ W^+_\mu + T^- W^-_\mu\right)\psi_i \quad (CC) \tag{18}$$

$$\frac{g}{2\cos\theta_W} \sum_i \bar{\psi}_i \gamma^\mu (g_V^i - g_A^i \gamma^5)\psi_i Z_\mu \quad (NC) \tag{19}$$

where the vector and axial vector couplings are

$$g_V^i = T_3^i - 2Q^i \sin^2 \theta_W, \tag{20}$$
$$g_A^i = T_3^i, \tag{21}$$

Q^i is the electric charge of fermion i, T_3^i is its weak isospin and the vector and axial vector couplings are given by

$$g_V^i = T_3(i) - 2Q_i s_W^2; , \qquad g_A^i = T_3(i) \qquad \text{and} \qquad g = \frac{e}{\sin\theta_W} , \qquad g' = \frac{e}{\cos\theta_W} . \tag{22}$$

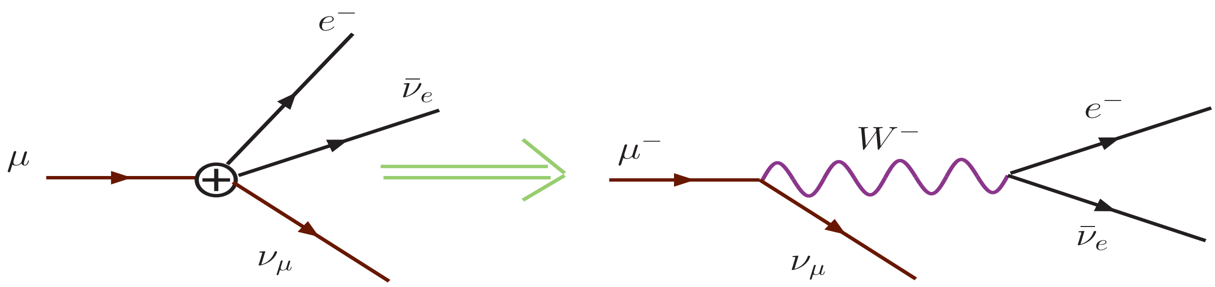
It is also very useful for a comparison with the charged current (CC) to rewrite the neutral current of the Z in terms of left and right handed currents

$$\frac{g}{2\cos\theta_W} \sum_i \bar{\psi}_i \gamma^\mu \left(g_L^i (1-\gamma^5) + g_R^i (1+\gamma^5) \right) \psi_i Z_\mu \quad (NC) \tag{23}$$

with the redefinition

$$g_V = g_L + g_R \tag{24}$$
$$g_A = g_L - g_R \tag{25}$$

The charged weak current on the other "hand is only left-handed". It is now possible to make contact between the Fermi constant and the fundamental parameters of the weak interaction as described by the SM. Muon decay in the two models is described by



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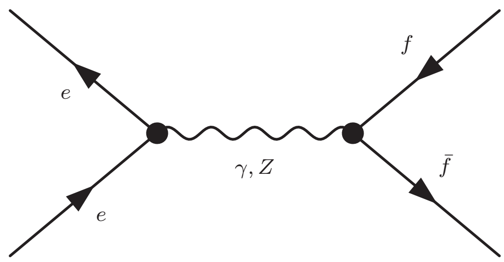
Going through the calculation with the ingredients just listed and the Fermi Lagrangian leads to the identification

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \frac{1}{2v^2} \tag{26}$$

4 Phenomenology at the Z pole

The most accurate measurements in the electroweak model come from experiment conducted at the Z pole either at LEP1 or SLC. These are e^+e^- machines with a very clean environment running at energies tuned at the Z mass. The Z is then produced in e^+e^- collisions and shortly decays into a fermion pair. The cross section for a final fermion $f \neq e$ is dominated by Z exchange



$$\frac{d\sigma_Z^f}{d\Omega} = \frac{9}{4}\frac{s\Gamma_{ee}\Gamma_{f\bar{f}}/M_Z^2}{(s-M_Z^2)^2+s^2\Gamma_Z^2/M_Z^2}\left[(1+\cos^2\theta)(1-P_eA_e) + 2\cos\theta A_f(-P_e+A_e)\right] \tag{27}$$

P_e is the beam polarisation of the electron as in the SLC experiment. $s = 4E_b^2$with E_b the beam energy. Γ_Z is the total Z width. Γ_{ee} and $\Gamma_{f\bar{f}}$ are the partial widths for $Z \to ee$ and $Z \to f\bar{f}$, correspondingly. (The partial widths are related to the Z branching fractions as $B(Z \to f\bar{f}) = \Gamma_{f\bar{f}}/\Gamma_Z$.) In (27) θ is the angle between the incident electron and the outgoing fermion.

A_f is the left-right coupling constant asymmetry:

$$A_f = \frac{2g_V^f g_A^f}{(g_V^f)^2+(g_A^f)^2} = \frac{(g_L^f)^2-(g_R^f)^2}{(g_L^f)^2+(g_R^f)^2} \tag{28}$$

Notice that $A_f \leq 1$. Since g_V and g_A only depend on the quantum numbers of the particles, it follows that A_f is the same for all charged leptons, all up-type quarks and all down-type quarks. For example, for charged leptons

$$g_V^\ell = -\frac{1}{2} - 2(-1)\sin^2\theta_W \sim -0.50 + 0.462 = -0.038 \tag{29}$$

$$g_A^\ell = -\frac{1}{2} = -0.50 \tag{30}$$

We see that because of an accidental cancellation $g_V^\ell \ll g_A^\ell$. For the asymmetry we then get (at tree level)

$$A_\ell = \frac{2(-0.038)(-0.5)}{(-0.038)^2+(-0.5)^2} \sim 0.15 \tag{31}$$

This small value of A_ℓ makes it particularly sensitive to electroweak vacuum polarization corrections (which have an impact on $\sin^2\theta_W$). In terms of $\sin^2\theta_W$, we have

$$A_\ell = \frac{2(1-4\sin^2\theta_W)}{1+(1-4\sin^2\theta_W)^2}. \tag{32}$$

Therefore small changes in $\sin^2\theta_W$ are amplified by a factor of 8 in the leptonic asymmetry A_ℓ.

4.1 Extraction of parameters

♢ Mass and Width of the Z

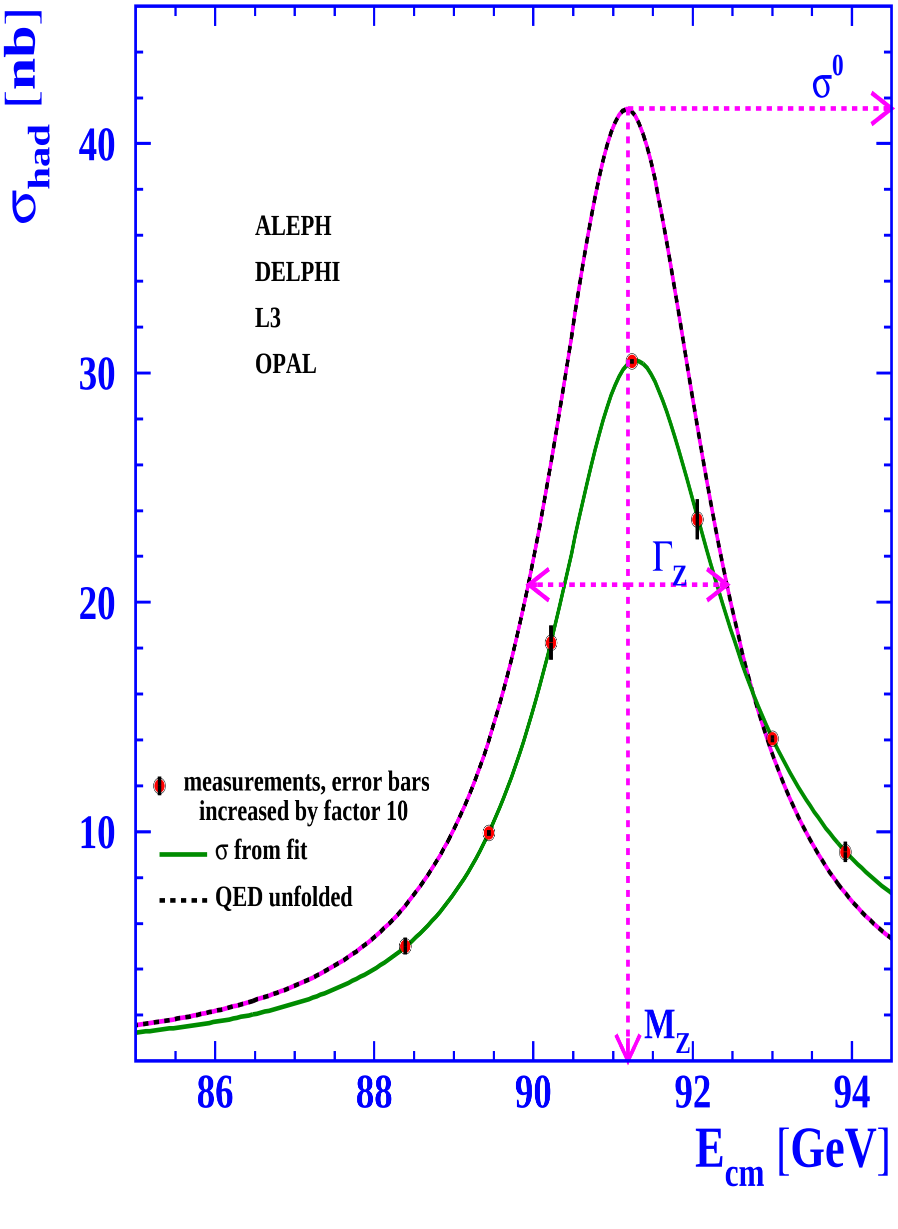


Figure 2: *The Z resonance. The effect of initial state radiation is also shown.*

A scan around the Z peak yield the mass of the Z and its width. For example from the total cross section into hadrons at LEP1, $P_e = 0$, we also get the peak cross section

$$\sigma_{had} = 12\pi \frac{\Gamma_{ee}\Gamma_{had}}{M_Z^2\Gamma_Z^2}. \tag{33}$$

This is shown in Fig. 2.

◊ Partial widths

Looking at various exclusive final states one can determine the following ratios

$$R_\ell \equiv \frac{\Gamma_{had}}{\Gamma_\ell} \sim \frac{0.7}{0.0333} \sim 21; \tag{34}$$

$$R_b \equiv \frac{\Gamma_{bb}}{\Gamma_{had}}; \tag{35}$$

$$R_c \equiv \frac{\Gamma_{cc}}{\Gamma_{had}}; \tag{36}$$

with $\ell = \{e, \mu, \tau\}$. Basically this amounts to measuring the Z branching fractions.

Having measured the total width from the peak scan and all visible partial widths, one can determine the invisible partial width (for $Z \to \nu\bar{\nu}$):

$$\Gamma_{inv} = \Gamma_Z - \Gamma_{had} - \Gamma_{ee} - \Gamma_{\mu\mu} - \Gamma_{\tau\tau} \tag{37}$$

and count the number of neutrino species N_ν coupling to the Z. Assuming that $\Gamma_{inv} = N_\nu \Gamma_\nu$, where Γ_ν is the Z partial width to pairs of a single neutrino species, we have

$$N_\nu = \frac{\Gamma_{inv}/\Gamma_\ell}{(\Gamma_\nu/\Gamma_\ell)_{SM}}. \tag{38}$$

◊A^f_{FB}

The unpolarized forward-backward asymmetry is determined by a simple counting method in $e^+e^- \to f\bar{f}$ scattering

$$A^f_{FB} \equiv \frac{\sigma^f_F - \sigma^f_B}{\sigma^f_F + \sigma^f_B} = \frac{3}{4} A_e A_f, \tag{39}$$

where σ^f_F (σ^f_B) is the total cross-section for forward (backward) scattering of f with respect to the incident e^- direction.

◊A^f_{LR}: The Left-right asymmetry

The left-right asymmetry is defined as

$$A^f_{LR} \equiv \frac{1}{P_e} \frac{\sigma^f(-|P_e|) - \sigma^f(+|P_e|)}{\sigma^f(-|P_e|) + \sigma^f(+|P_e|)} = A_e, \tag{40}$$

where $\sigma^f(P_e)$ is the total (integrated over all angles) cross-section for producing $f\bar{f}$ pairs with an electron beam of polarisation P_e.

◊$\bar{A}^f_{FB}$: The Left-right forward-backward asymmetry.

Quantity	Group(s)	Value
M_Z [GeV]	LEP	91.1876 ± 0.0021
Γ_Z [GeV]	LEP	2.4952 ± 0.0023
Γ(had) [GeV]	LEP	1.7444 ± 0.0020
Γ(inv) [MeV]	LEP	499.0 ± 1.5
$\Gamma(\ell^+\ell^-)$ [MeV]	LEP	83.984 ± 0.086
$\sigma_{\rm had}$ [nb]	LEP	41.541 ± 0.037
R_e	LEP	20.804 ± 0.050
R_μ	LEP	20.785 ± 0.033
R_τ	LEP	20.764 ± 0.045
$A_{FB}(e)$	LEP	0.0145 ± 0.0025
$A_{FB}(\mu)$	LEP	0.0169 ± 0.0013
$A_{FB}(\tau)$	LEP	0.0188 ± 0.0017
R_b	LEP + SLD	0.21664 ± 0.00065
R_c	LEP + SLD	0.1718 ± 0.0031
$R_{s,d}/R_{(d+u+s)}$	OPAL	0.371 ± 0.023
$A_{FB}(b)$	LEP	0.0995 ± 0.0017
$A_{FB}(c)$	LEP	0.0713 ± 0.0036
$A_{FB}(s)$	DELPHI,OPAL	0.0976 ± 0.0114
A_b	SLD	0.922 ± 0.020
A_c	SLD	0.670 ± 0.026
A_s	SLD	0.895 ± 0.091
A_{LR} (hadrons)	SLD	0.15138 ± 0.00216
A_{LR} (leptons)	SLD	0.1544 ± 0.0060
A_μ	SLD	0.142 ± 0.015
A_τ	SLD	0.136 ± 0.015

Table 2: Measurements of key quantities at the Z peak

The left-right forward-backward asymmetry is defined as

$$\bar{A}^f_{FB} \equiv \frac{\sigma^f_F(-|P_e|) - \sigma^f_B(-|P_e|) - \sigma^f_F(+|P_e|) + \sigma^f_B(+|P_e|)}{\sigma^f_F(-|P_e|) + \sigma^f_B(-|P_e|) + \sigma^f_F(+|P_e|) + \sigma^f_B(+|P_e|)} = \frac{3}{4} P_e A_f. \tag{41}$$

It allows a direct determination of the various A_f.

A compendium of measurements for the various observables in shown in Table 2.

5 W mass and non-abelian couplings at LEP2

LEP increased its energy primarily in order to study W pair production. The latter furnishes a rather precise determination of the mass of the W, a key parameter in the SM. This mass is extracted by studying the energy behaviour of the cross section at threshold which extremely sensitive to the mass. Another method is to reconstruct this mass from the invariant mass of the decay products of the W. $e^+e^- \to W^+W^-$ cross section is shown in Fig. 3. Especially at larger energies the energy behaviour of the cross

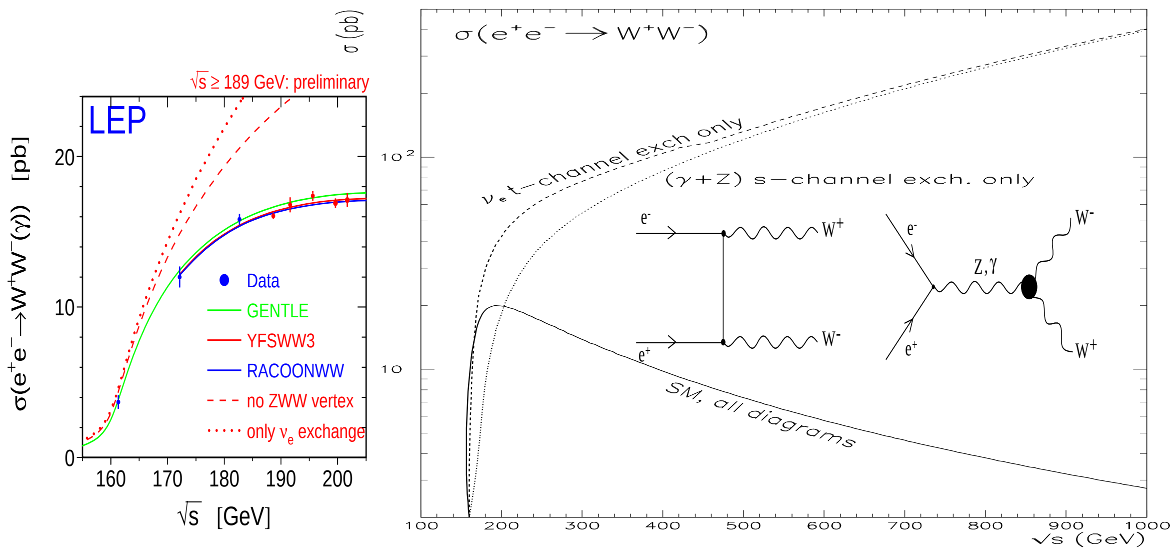


Figure 3: $e^+e^- \rightarrow W^+W^-$ *at LEP2.*

section helps check that the self-couplings of the W are as expected and are directly related to the gauge coupling of the theory. if this were not so unitarity would have been badly broken and the cross section would have risen indefinitely. The W mass is also measured at the Tevatron.

6 Top mass at the Tevatron

As we will discover the top mass is an important ingredient when it comes to testing the quantum consistency of the Standard Model. The top can not be probed directly at LEP. It is produced at the Tevatron through $q\bar{q} \rightarrow t\bar{t}$ involving a virtual s-channel gluon. The top quark decays to $t \rightarrow Wb$ almost 100% of the time. The mass is extracted from different channels according to the subsequent W decay. The status of the direct top mass measurement is shown in Fig. 4.

7 Fundamental parameters, physical parameters and the need for renormalisation

The gauge theory formulation means that the interaction of any type of known fermions and the gauge bosons can be described in a very economical way by 3 parameters (4 if one includes QCD). These are the gauge couplings of the $SU(2) \times U(1)$ group, (g, g') and the value of the v.e.v, v. Apart from these parameters, all the remaining parameters are Yukawa parameters for which one does not really have a theory. The latter are just the masses of the fermions and the Higgs, and the mixing angles of the Kobayashi-Maskawa matrix. For what I would like to describe and for the precision measurements I will be

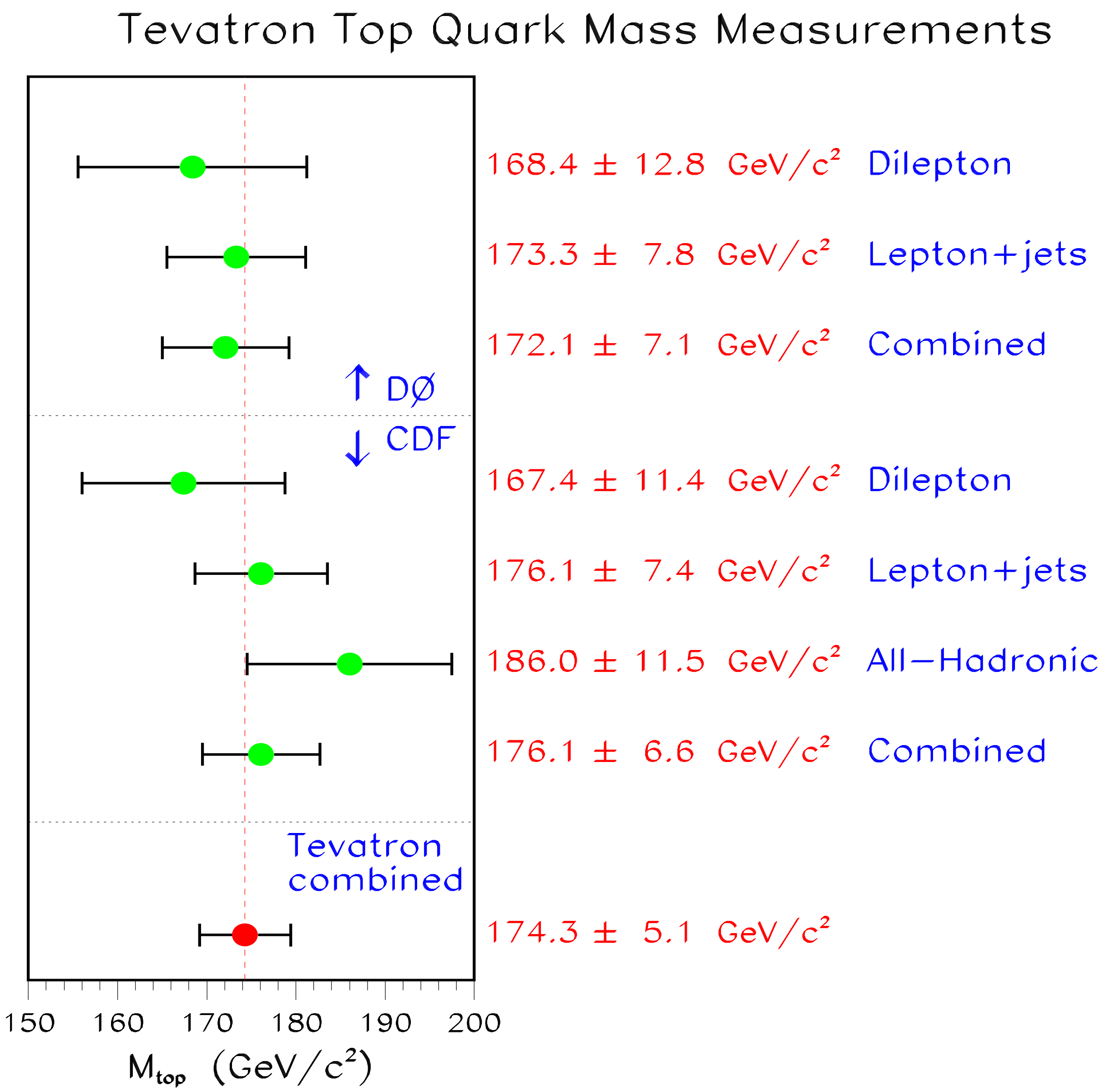


Figure 4: Run I Tevatron results for m_t and the global average.

speaking about the 3 ubiquitous parameters are enough.

The three basic parameters are then g, g', v. In principle then from these 3 parameters one can describe a full array of observables with in principle an infinite precision. This is the hallmark of the renormalisability of a theory, in that once the parameters of theory have been *defined*, one can predict any other observable. In a theory that is not renormalisable, one needs more experimental input and observables and thus these theories are not so predictive. Thus one of the first things one has to tackle is how do we define g, g', v that appear in the tree-level Lagrangian? A definition means that we need to associate them unambiguously to a set of 3 physical measurables. For example:

$$\begin{aligned} M_W &= \frac{gv}{2} \\ M_Z &= \frac{\sqrt{g^2+g'^2}v}{2} \\ e^{-2} &= g^2+g'^2 \end{aligned} \tag{42}$$

Therefore if one measures and then trade-off the set g, g', v with M_W, M_Z, e, then we should be able to express all other observables and compare them to experiment. These new three physical parameters, can be extracted almost in a straightforward manner. For example M_Z is obtained from a scan around the Z peak at LEP1 as we saw, Fig. 2.

M_W is measured by combining the measurement of the W^+W^- cross section at threshold and the reconstruction of the hadronic invariant mass from the decay of the W's at LEP2. The *LEP* (LEP1 and LEP2) results give:

$$\begin{aligned} M_Z &= 91.1875 \pm .0021\text{GeV} \qquad & \frac{\Delta M_Z}{M_Z} \sim 2.10^{-5} \\ M_W &= 80.450 \pm .039\text{GeV} \qquad & \frac{\Delta M_W}{M_W} \sim 5.10^{-4} \end{aligned} \tag{43}$$

As for e, or rather $\alpha = e^2/4\pi$, it is the old good α from QED. Well, at tree-level especially, one can only associate it to the value as defined in QED, that is in the Thomson limit. Actually the most accurate value is extracted from the electron $g-2$

$$\alpha^{-1} = 137.03599235(73) \qquad \frac{\Delta\alpha^{-1}}{\alpha^{-1}} \sim 5.10^{-9} \tag{44}$$

The precision on α is just awesome. Note also that the precision on M_W has only very recently improved after the LEP2 measurements and the Tevatron. In 1991 just after LEP/SLC started operating we had

$$\begin{aligned} M_Z &= 91.174 \pm .02\text{GeV} \qquad & \frac{\Delta M_Z}{M_Z} \sim 2.10^{-4} \\ M_W &= 79.91 \pm .39\text{GeV} \qquad & \frac{\Delta M_W}{M_W} \sim 5.10^{-3} \end{aligned} \tag{45}$$

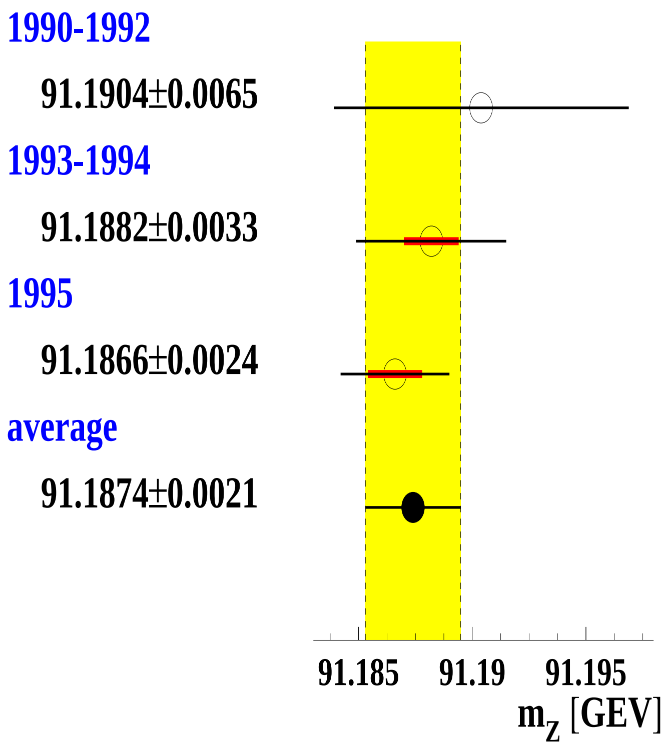


Figure 5: *Improvement in the precision on the Z mass during the LEP era.*

Note that precision on these observables are of the order $\alpha/4\pi$. Which means that radiative corrections which appear as an expansion in $\alpha/4\pi \sim 10^{-4}$ are of importance. Observe also that one has gained an order of magnitude improvement on two of the fundamental quantities needed to define the $\mathcal{SM}$.

As you see when precision measurement of the $\mathcal{SM}$ started to become crucial, the precision on the W mass was not sufficient. The problem is that if one uses as input a quantity which is poorly measured then its error will propagate to all other physical quantities which will not allow a careful and precise comparison between theory and experiment. In 1991 (and in fact much earlier than the advent of LEP) there were other electroweak quantities which were measured with a much better precision than M_W. For instance take the Fermi constant as extracted from muon decay. This was known with great precision even before the advent of the SM with the Fermi theory. In the previous lecture we encountered this quantity through the relation

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} = \frac{1}{2v^2} \tag{46}$$

Since the value used is extracted from muon decay I will use $G_\mu = G_F$. The muon decay lifetime taking into account QED corrections is shown in Fig. 6

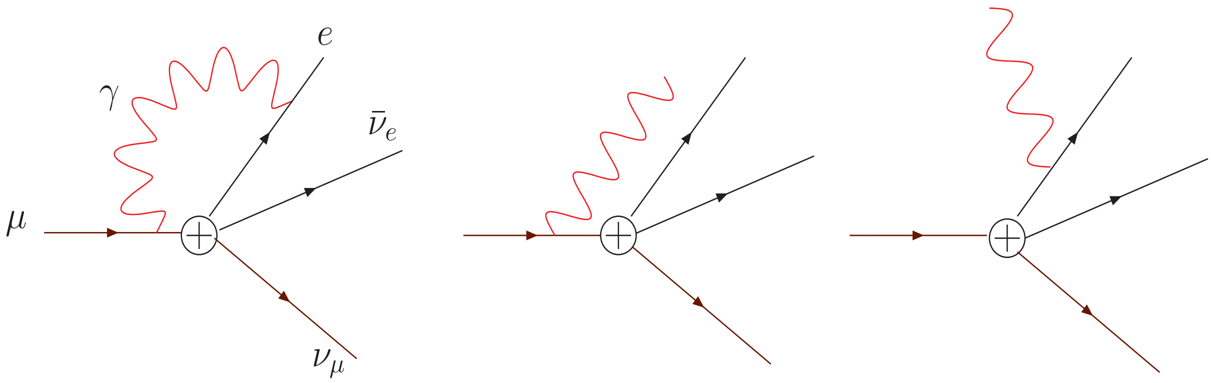


Figure 6: *Muon decay in the Fermi model including.*

It is given by

$$\tau_\mu^{-1} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} F(\frac{m_e^2}{m_\mu^2}) \left(1 + \frac{3}{5}\frac{m_\mu^2}{M_W^2}\right) \times \left[1 - 1.8098\frac{\alpha(m_\mu)}{\pi} + 6.7427\left(\frac{\alpha(m_\mu)}{\pi}\right)^2\right] \tag{47}$$

with

$$F(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x \quad \alpha^{-1}(\mu) \simeq 136 \tag{48}$$

where all QED corrections have been taken into account, Fig. 6. Note for later that the QED corrections are evaluated at the scale of the muon mass, more later about scales when we tackle RC. From this we extract

$$G_\mu = (1.16637 \pm .00001) 10^{-5} GeV^{-2} \qquad \frac{\Delta G_\mu}{G_\mu} \sim 8.6\ 10^{-6} \tag{49}$$

Another parameter that you have also seen and which pops up almost everywhere in the electroweak processes is $\sin^2\theta_W$. There are quite a few ways of introducing and defining this angle and therefore one has to be extremely careful when one mentions this parameter. Of course at tree-level all definitions are equivalent. For instance we can define it in terms of masses

$$s_W^2 = s_M^2 = 1 - M_W^2/M_Z^2 \tag{50}$$

From the latest data on the W and Z masses we get

$$s_M^2 = .22164 \pm .00079 \;\; (i.e \;\; .4\% \text{ precision}) \tag{51}$$

One can also define it in terms of the Z couplings to fermions. Recall that

$$\begin{aligned} J_Z^\mu &= \Sigma_f g_L^f \bar{f}_L \gamma^\mu f_L \; + \; g_R^f \bar{f}_R \gamma^\mu f_R = \Sigma_f g_v^f \bar{f} \gamma^\mu f \; + \; g_A^f \bar{f} \gamma^\mu \gamma_5 f \\ g_L^f &= \frac{\tau_3^f}{2}(1 - 2|Q_f| s_{\text{eff}}^2) \quad g_R^f = \frac{\tau_3^f}{2}(-2|Q_f| s_{\text{eff}}^2) \quad \tau_3^f = 2T_3^f = \pm 1. \end{aligned} \tag{52}$$

Polarised beams at the SLC allow a very precise extraction of this quantity for the electron. With $\sigma_i = \sigma(e_i^- e^+ \to f\bar{f}), i = L, R$

$$A_{LR}^e = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \frac{g_L^{f\,2} - g_R^{f\,2}}{g_L^{f\,2} + g_R^{f\,2}} = \frac{2(1 - 4s_{\text{eff}}^2)}{1 + (1 - 4s_{\text{eff}}^2)^2} \tag{53}$$

It can also be extracted from the FB asymmetries at LEP1.

$$A_{\text{FB}}^f = \frac{3}{4} A_{LR}^e A_{LR}^f \tag{54}$$

The latest measurements of s_{eff}^2 from leptonic observables gives an average value

$$s_{\text{eff}\,,\text{l}}^2 = .23159 \pm .00018 \;\; (i.e \;\; .08\% \text{ precision}) \tag{55}$$

Note only do we get a much better precision on this effective mixing angle specific to the Z observables, but more telling perhaps is that the two values are about as much as 10σ away. 10σ discrepancy can not be an experimental error. Therefore this means that there is more physics, theory, which has not been accounted for. If this discrepancy persists after including radiative corrections, then this means that some new physics is at play. Hence the importance of the RC.

Take another example from the partial widths of the Z.

$$\Gamma_1(Z \to f\bar{f}) = \frac{\sqrt{2} G_\mu M_Z^3}{3\pi}(g_V^{f\,2} + g_A^{f\,2}) \; N_c^f \stackrel{?}{=} \Gamma_2(Z \to f\bar{f}) \frac{\alpha M_Z}{3\pi} = \frac{1}{s_\theta^2 c_\theta^2}(g_V^{f\,2} + g_A^{f\,2}) \; N_c^f \tag{56}$$

Note that in the second expression I have deliberately used another s_θ^2, since we have seen that different definitions do not necessarily measure the same thing. Consider the ratio

$$\frac{\Gamma_2}{\Gamma_1} = \frac{\pi\alpha}{\sqrt{2}G_\mu M_Z^2}\frac{1}{s_\theta^2 c_\theta^2} \tag{57}$$

Using $s_\theta^2 = s_{\rm eff}^2$ we find that the two widths are off by as much as 6%! and by about 3% if s_M^2 is used! Again this is way off compared to the precision with which the leptonic widths have been measured, .1%. Radiative corrections are then again essential, this will also make it more transparent how one should express our theoretical results. Incidentally, we have seen in the expression of the muon lifetime that the QED radiative corrections have been included by using α at the scale of the muon mass. I still have not defined the notion of a running coupling constant, but for those of you who have come across this notion in their QED or QCD course, it would have been more appropriate to use for the Z partial width, the value of the electromagnetic constant at the scale M_Z, $\alpha(M_Z) \simeq 1/129$. Doing this would account for much of the previous 6% discrepancy! Therefore in a first step I would like to introduce this correction in the running of α, especially that it would follow very closely notions that you may have seen in the renormalisation of QED. Moreover this will help introduce the notion of Born-type universal corrections.

8 Infinities, Regularisation and the Running α

8.1 Infinities

We have been talking about precision measurements, discrepancies of at best a few percent depending on how we parameterise our observables and the need to go beyond tree-level (Born approximation). However as many of you know, as soon as we go beyond tree-level we start encountering loops which almost invariably are *infinite*. Take the vacuum polarisation due to a fermion of charge Q_f and mass m_f in QED, Fig.7

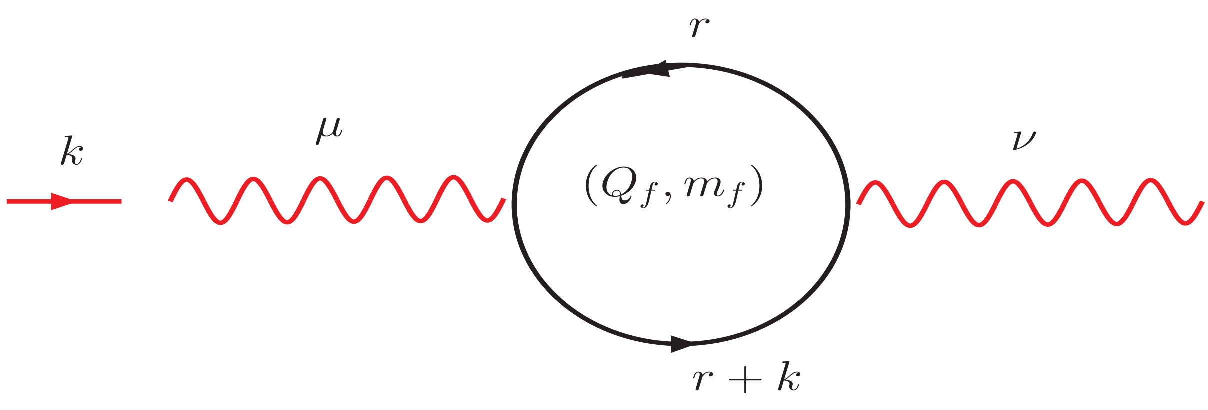


Figure 7: *Contribution of a fermion of mass m_f and charge Q_f to the photon two-point function.*

$$i\Sigma_{\mu\nu} = (-1)(-ie_0)^2(Q_f)^2 \int \frac{d^4r}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\mu \frac{i}{\not r - m}\gamma_\nu \frac{i}{\not r + \not k - m}\right) \tag{58}$$

where I have used e_0 the coupling that shows up in the tree-level (bare) Lagrangian. This is infinite since simple power counting shows that this goes like

$$\int d^4r/r^2 \to \int^\Lambda d^4r/r^2 \to \int^\Lambda r^2 \to \Lambda^2 \;\text{ with }\; \Lambda \to \infty.$$

Note that the manipulation I have just done breaks both Lorentz symmetry and gauge symmetry. This would destroy the consistency of the theory and would not be admissible. So how do we handle loops (especially divergent).
need to use a much better regularisation. Regularisation can be viewed as a formal redefinitions of infinities as limits in the mathematical sense so that one can manipulate the infinities as if they were finite. In choosing a regularisation one should be careful not to break any symmetry that is built in the original Lagrangian as conservation laws would not be valid. In QED a good regularisation is Pauli-Villars regularisation. It preserves gauge and Lorentz invariance. It is a simple replacement

$$\frac{1}{r^2-m^2} \longrightarrow \frac{1}{r^2-m^2} - \frac{1}{r^2-\Lambda^2}, \tag{59}$$

with Λ a very high cut-off. This procedure fails in non-abelian theories because Λ is a mass term which is not introduced in a gauge invariant way.

What one uses for non-abelian gauge theories, like the $\mathcal{SM}$ and QCD is dimensional regularisation.

8.2 Dimensional Regularisation

This is an analytical continuation in the number of space-time dimension. Let us go through all the steps of how this procedure is exploited.

Step 1:
In fact in the example above with fermionic loops, step-0 is to render all propagators rational:

$$(\not r - m)^{-1} \to \frac{\not r + m}{r^2 - m^2} \tag{60}$$

One could also extract all factors of e_0^2 etc..

We then continue $4 \to n$ and $r_\mu = (r_0, r_1, r_2, r_3) \to (r_0, r_1, \cdots, r_{n-1})$

The metric tensor in n dimensions has the property

$$g_\mu{}^\mu = g_{\mu\nu} g^{\nu\mu} = \mathrm{Tr}(\mathbf{1}) = n\,. \tag{61}$$

(In fact for the trace we can take any function, $f(n)$ with $f(n \to 4) \to 4$, an example is $f(n) = 2^{n/2}$.

The Dirac algebra in n dimensions

$$\{\gamma_\mu, \gamma_\nu\} \;=\; 2\,g_{\mu\nu}\,\mathbf{1} \tag{62}$$

has the consequences

$$\gamma_\mu \gamma^\mu \;=\; n\,\mathbf{1} \tag{63}$$

$$\gamma_\alpha \gamma_\mu \gamma^\alpha \;=\; (2-n)\,\gamma_\mu \tag{64}$$

$$\gamma_\alpha \gamma_\mu \gamma_\nu \gamma^\alpha \;=\; 4 g_{\mu\nu}\,\mathbf{1} \;-\; (4-n)\,\gamma_\mu \gamma_\nu \tag{65}$$

$$\gamma_\alpha \gamma_\mu \gamma_\nu \gamma_\rho \gamma^\alpha \;=\; -2\,\gamma_\rho \gamma_\nu \gamma_\mu \;+\; (4-n)\,\gamma_\mu \gamma_\nu \gamma_\rho \tag{66}$$

A consistent treatment of γ_5 in n dimensions is more subtle, but for our purposes we will not need a sophisticated prescription but just take γ_5 to anti-commute with γ_μ:

$$\{\gamma_\mu, \gamma_5\} \;=\; 0\,. \tag{67}$$

This, of course,is not needed for the $\gamma \leftrightarrow \gamma$

Then

$$\int \frac{d^4 r}{(2\pi)^4} \to \int \frac{d^n r}{(2\pi)^n} \tag{68}$$

Moving from 4 to n dimension introduces a scale, much like the "mass" term in the Pauli-Villars regularisation, however in a gauge, Lorentz invariant way. This comes about because, the Lagrangian now has mass dimension n.

From the kinetic term of a fermion and that of a boson, we conclude that in n-dim, the former has $d_f = (n-1)/2$ and the latter $d_b = (n-2)/2$. Then the gauge coupling constant (which is dimensionless in $n = 4$) has dimension $(4-n)/2$. Therefore the loop correction, e_0^2 brings in a factor μ^{4-d} for the 2 point function. Thus it is more appropriate to make the substitution

$$\int \frac{d^4 r}{(2\pi)^4} \to \mu^{4-n} \int \frac{d^n r}{(2\pi)^n} \tag{69}$$

Second Step

Move to Euclidean, or make a Wick rotation. The time component $r_0 = i r_n$ so that $r \to r_E = (r_1, r_2, \cdots, r_n)$ as a consequence $r^2 = -r_E^2$ (and $d^n r = i d^n r_E$). This applies to all momenta, after integration one can always get back to Minkowski space. This move gives more positivity to the integrands. It is legitimate due to the analyticity properties of the integrands or rather propagators $1/(r^2 - m^2) \to 1/(r^2 - m^2 + i\varepsilon)$.

Third Step

Combine all propagators into one,

$$\begin{aligned}\frac{1}{D_1 D_2 \cdots D_N} &= \Gamma(N)\int_0^1 dx_1 \cdots \int_0^1 dx_{N-1} \frac{1}{\Big(D_1 x_1 \ + \ D_2 x_2 + \cdots D_N (1 - \sum_{i=1}^{N-1} x_i)\Big)^N} \\ \Gamma(N) &= (N-1)! \end{aligned} \tag{70}$$

The new denominator will then have a structure like this

$$r^2 - 2r.P(x_i) + M^2(x_i) \tag{71}$$

where $P(x_i)$ is a combination of all momenta entering the loop and is a function of x_i M^2 combines all masses and momenta. Since by now all the integrals are regulated one can make a change of variables

$$r - P(x_i) \to r \quad \big(r^2 - 2r.P(x_i) + M^2(x_i)\big) \to r^2 + (M^2 - P^2) \equiv \big(r^2_{(E)} + \Delta_{(E)}\big) \tag{72}$$

Fourth Step

Since the integrals have been regulated we can swap the order of the integrals over the Feynman parameters and the loop variables

$$\int d^n r_E \int dx_i \to \int dx_i \int d^n r_E \tag{73}$$

The trick is to use polar co-ordinates, then

$$d^n r_E = r_E dr_E d\Omega_n \qquad d\Omega_n = d\phi(\sin\theta_1 d\theta_1)(\sin^2\theta_2 d\theta_2)\cdots(\sin^{n-2}\theta_{n-2} d\theta_{n-2}) \tag{74}$$

The integral over the angles can be performed easily with the result

$$\int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{75}$$

To show this use

$$\int_0^\pi d\theta \sin^\alpha\theta = \sqrt{\pi}\frac{\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{\alpha+2}{2})} \tag{76}$$

Then a basic integral writes

$$\int d^n r_E \frac{(r_E^2)^\alpha}{(r_E^2 + \Delta_E)^N} = \frac{\pi^{n/2}}{\Gamma(N)}\Gamma(N - n/2)\Delta_E^{-(N-n/2)} = \tilde{I}^{(N)}\Gamma(N - n/2) \tag{77}$$

to show this make a change of variables $y = \Delta^2/(r_E^2 + \Delta^2)$ and use the definition of the beta function

$$\int_0^1 dy y^{\alpha-1}(1-y)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{78}$$

Step 5

Get back to Minkowski then take the limit $n \to 4$. All divergences are now poles in

$$\varepsilon = \frac{4-n}{2} \tag{79}$$

For this one uses

$$\begin{aligned} \Gamma(-m+\varepsilon) &= \frac{(-1)^m}{m!}\left(\frac{1}{\varepsilon} + \sum_1^m \frac{1}{k} - \gamma_E + \mathcal{O}(\varepsilon)\right) \quad , \quad \gamma_E = -\Gamma'(1) \\ X^\varepsilon &= 1 + \varepsilon \ln X + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{80}$$

Usually one only needs

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \qquad \Gamma\left(\frac{\epsilon-1}{2}\right) = -\frac{2}{\epsilon} - (1-\gamma_E) + \mathcal{O}(\epsilon) \tag{81}$$

I have collected more easy-to-use formulae in the Appendix.

8.3 Vacuum polarisation in QED

Therefore let us get back to the vacuum polarisation. If you apply very carefully the 5-step recipe, here is what you should find

$$i\,\Sigma_{\mu\nu} = i\left(k_\mu k_\nu \, - \, k^2 g_{\mu\nu}\right)\Pi_{\gamma\gamma}(k^2) \tag{82}$$

Before plunging into the information contained in $\Pi_{\gamma\gamma}(k^2)$, it is important to notice that the Lorentz structure could have been guessed. The choice of a regulator which preserves gauge invariance means that $i\Sigma_{\mu\nu}$ is transverse, *i.e*

$$k^\mu \, \Sigma_{\mu\nu} = 0 \tag{83}$$

For $\Pi_{\gamma\gamma}$ we get

$$\begin{aligned} \Pi_{\gamma\gamma} &= e_0^2 \Pi_{QQ} = e_0^2 \sum_f N_c^f Q_f^2 \Pi_{\gamma\gamma}^f(k^2) \\ \Pi_{\gamma\gamma}^f(k^2) &= \frac{1}{12\pi^2}\left(\frac{1}{\varepsilon} + \ln(4\pi) - \gamma_E - 6\int_0^1 dx x(1-x)\ln(\frac{m_f^2 - k^2 x(1-x)}{\mu^2}) + \mathcal{O}(\varepsilon)\right) \\ &= \frac{1}{12\pi^2}\left(\frac{1}{\varepsilon} + \ln(4\pi) - \gamma_E + \frac{5}{3} - \ln(-k^2/\mu^2) + \cdots \qquad |k^2| \gg m_f^2\right) \\ &= \frac{1}{12\pi^2}\left(\frac{1}{\varepsilon} + \ln(4\pi) - \gamma_E - \ln(m_f^2/\mu^2) + \frac{k^2}{5m^2} + \cdots \qquad |k^2| \ll m_f^2\right) \end{aligned} \tag{84}$$

For later, note that

$$\Pi^f_{\gamma\gamma}(0) \;=\; \frac{1}{12\pi^2}\left(\frac{1}{\varepsilon} \;+\; \ln(4\pi) - \gamma_E \;-\; \ln(m_f^2/\mu^2)\right) \tag{85}$$

I will sometimes use

$$C_{UV} = \frac{1}{\varepsilon} \;+\; \ln(4\pi) - \gamma_E \tag{86}$$

We therefore see how the use of dimensional regularisation helps keep track of the infinities, as poles in $1/\varepsilon$ and also how it maintains the symmetries of the theory. Still, physical observables are finite and do not depend on spurious regulator parameters, ε. To make contact with a physical observable we need to embed our derivation of $\Pi_{\gamma\gamma}$ in the calculation of an observable. We will take a cross section. A cross section that is measured does not make a difference between what is finite at tree-level and a loop correction that could be infinite!

8.4 Charge Renormalisation

The "infinite" correction we have just evaluated enters the propagator. For this particular case one needs to resum the full set of the bubble diagrams or 1PI (one-particle irreducible), to define the fully dressed propagator at one-loop. Let us deviate slightly from the case of the photon and prepare the ground to the W and Z. At tree-level the propagator of a gauge particle in a gauge ξ is

$$iD^0_{\mu\nu} = -i\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_0^2}\right)\frac{1}{k^2 - M_0^2} \tag{87}$$

At one loop we need to add the vacuum polarisation that we have just calculated. For more general couplings than in QED, the latter has the general form

$$i\Sigma_{\mu\nu} = -i\left(g_{\mu\nu}G(k^2) - k_\mu k_\nu L(k^2)\right) \;\;,\;\; G(k^2) = G(0) + k^2\Pi(k^2) \equiv G + k^2\Pi(k^2) \tag{88}$$

The transverse part will always be denoted by $G(k^2)$ and will be split into its $k^2 = 0$ part $G(0)$ and its $\Pi(k^2)$ part. In most cases of interest, couplings to (almost) massless fermions, the longitudinal part would be of no concern to us. Note however that from our expression for $\Pi_{\gamma\gamma}$ $G = 0$ (massless gauge particle). The dressed propagator is then the geometrical sum, Fig. 8 which gives

$$iD_{\mu\nu} = -ig_{\mu\nu}\frac{1}{k^2 - M_0^2 + (G(0) + k^2\Pi(k^2))} \tag{89}$$

Note that for a massive particle, the imaginary part of $\Pi(k^2)$ which is finite defines a running width, which at the pole of the particle relates directly to the usual width.

Let us now get back to the photon and consider the cross section with an initial state with a pair of fermion-antifermion of charge Q to one with fermions of charge Q', $Q\bar{Q} \to Q'\bar{Q}'$. I'll denote the electromagnetic current for a charge Q as J^Q_μ. At tree-level we write

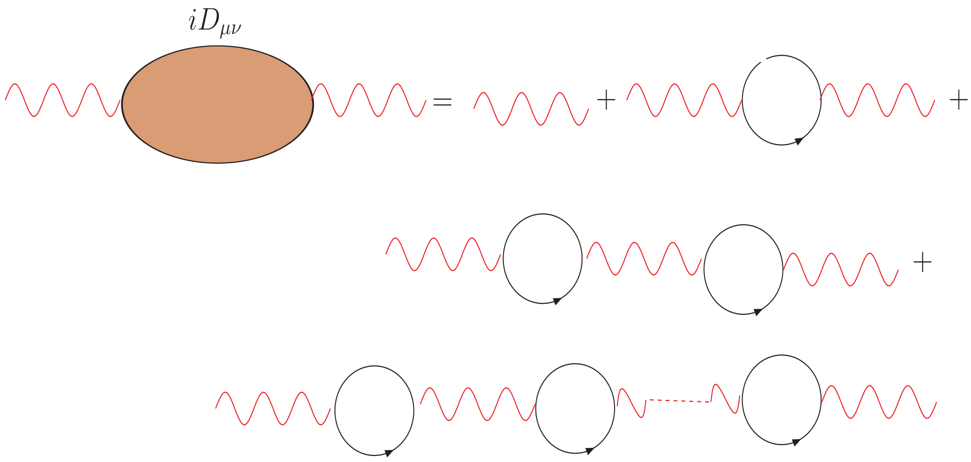
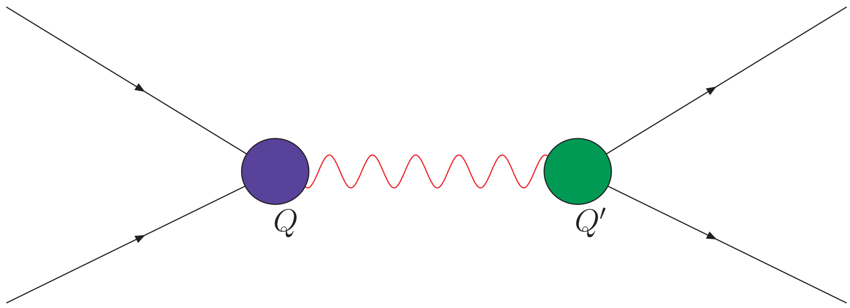


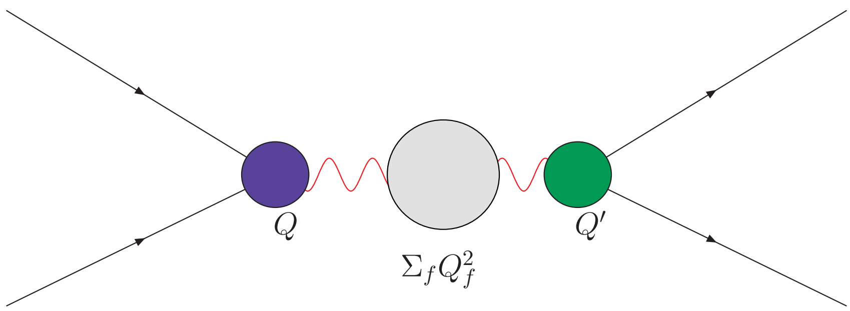
Figure 8: *Contribution of a fermion of mass m_f and charge Q_f to the photon two-point function.*



$$\mathcal{M}_0 = e_0 J^Q_\mu D_0^{\mu\nu} e_0 J^{Q'}_\mu \to e_0^2 Q \frac{1}{k^2} Q' \tag{90}$$

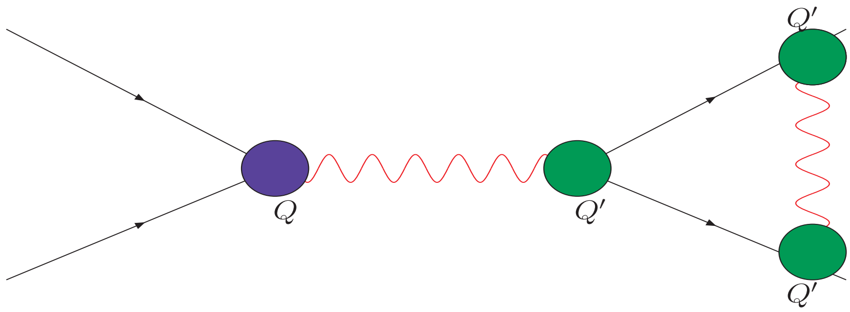
At the one-loop order we have 3 types of corrections.

♣1. The first is the correction to the self-energy of the photon.



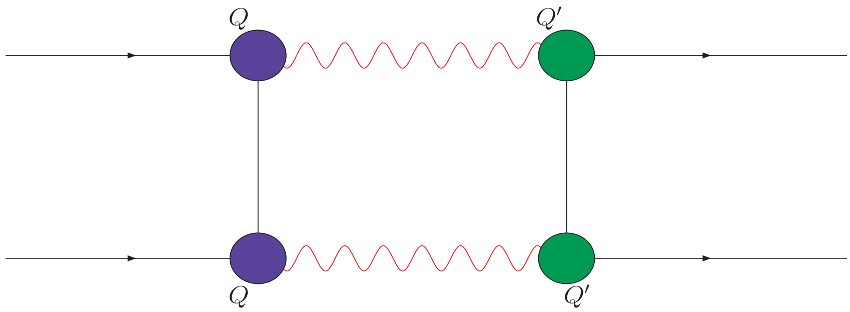
This correction has the same structure as the Born contribution. Moreover it is universal, that is it appears as the same building block for all processes. Moreover it is sensitive to all charged particles that couple to the photon, therefore it can really probe new particles. The charge structure of the two-point function, universal correction, is $Q(\sum_f Q_f^2)Q'$

♣2. The second type are vertex (and fermion self-energy) corrections.



These are dependent on the initial or final state, so that the corrections write as $(Q(Q^2))Q'$ or $Q(Q')^2Q'$.

♣3. The last class are four-point function diagrams, boxes.



These are finite by themselves. Their charge structure is $Q(QQ')Q'$.

Because the universal two-point corrections are the most interesting (for most cases) especially when we will specialise to the electroweak theory at LEP/SLC and also because of a lack of time I will concentrate essentially on the universal corrections, apart from an important vertex correction . Moreover they are gauge invariant by themselves and constitute the bulk of the corrections in most cases. With these self-energy corrections, the improved Born approximation for our process writes:

$$\mathcal{M} = \frac{1}{k^2}\; e_0^2 Q \frac{1}{1+e_0^2 \Pi_{QQ}} Q' \equiv \frac{1}{k^2}\; e_\star^2(k^2) QQ' \tag{91}$$

This has exactly the same form as the Born expression, save for the replacement $e_0^2 \to e_\star^2(k^2)$. $e_\star^2(k^2)$ is a running, scale (energy) dependent, charge:

$$e_\star^{-2}(k^2) = e_0^{-2} \; + \; \Pi_{QQ}(k^2) \tag{92}$$

It is now easy to see that once we have defined the charge at some input point, energy, then for all other energies and observables we will get a *finite* answer:

$$e_\star^{-2}(k^2) = \left(e_0^{-2} \; + \; \Pi_{QQ}(k_R^2)\right) \; + \; \left(\Pi_{QQ}(k^2) \; - \; \Pi_{QQ}(k_R^2)\right) \equiv \underbrace{e^{-2}(k_R^2)}_{\text{input}} \; + \; \underbrace{\left(\Pi_{QQ}(k^2) \; - \; \Pi_{QQ}(k_R^2)\right)}_{finite} \tag{93}$$

k_R is the renormalisation point where the charge $e^{-2}(k_R^2)$ is identified and defined. The choice of the input parameter can be based on some theoretical considerations or be directly related to an experimental set-up. We could have, for example, chosen to extract only the $1/\varepsilon$ or C_{UV} which defines some variant of the MS-scheme. However in QED, and we will do so also in the electroweak theory, the choice of the OS (On-shell scheme) is the most useful. We thus choose $k_R^2 = 0$, that is the pole or on-shell mass of the photon to define the theory (and subtract the infinities), hence the name of the scheme. In this limit,

$$\lim_{k^2 \to 0} \left(k^2 \mathcal{A}_{all\ contributions} \right) = \lim_{k^2 \to 0} k^2 \mathcal{M} = e^2 = e_\star^2(0) \tag{94}$$

which helps define

$$\begin{aligned} e^{-2} &= e_\star^{-2}(0) = e_0^{-2} + \Pi_{QQ}(0) \\ e_\star^{-2}(k^2) &= e^{-2} + (\Pi_{QQ}(k^2) - \Pi_{QQ}(0)) \equiv e^{-2} + (\Delta\Pi_{QQ}(k^2)) \\ e_\star^{-2}(k^2) & \quad \frac{e^2}{1 + (\Pi_{QQ}(k^2) - \Pi_{QQ}(0))} \end{aligned} \tag{95}$$

It is crucial to note that $e_\star^2(k^2)$ is defined in terms of the physical parameter e, extracted from a cross section, and no longer in terms of the bare parameter, e_0. It depends now on a finite difference of Π functions.

Formally α is defined as the classical charge in the Thomson limit which can be considered as the non-relativistic limit of Compton scattering $e\gamma \to e\gamma, E_\gamma \to 0$.

$$\sigma_{\text{Compton}} \to \sigma_{\text{Thomson}} = \frac{8\pi\alpha^2}{3m_e^2} \quad , \quad \alpha = \frac{e^2}{4\pi} = \frac{e_\star^2(0)}{4\pi} \tag{96}$$

The previous discussion makes clear once more than the parameters appearing in the Lagrangian get their full meaning only once they are identified with some physical observables. Thus, especially when conducting loop calculations, the parameters appearing in the Lagrangian can in principle even be infinite. These *bare parameters* however will not show up explicitly in the expression for physical observables, only the renormalised ones will. The latter are finite and defined by a set of renormalisation conditions.

To summarise,

a) Starting with a Lagrangian defined in terms of the bare parameter

$$e_0 = Z_3 e = e + \delta e, \tag{97}$$

we express the cross section in terms of this *bare* parameter. The on-shell renormalisation condition imposes

$$\frac{\delta e^2}{e^2} = \frac{\delta\alpha}{\alpha} = e^2 \Pi_{QQ}(0) = \Pi_{\gamma\gamma}(0) \tag{98}$$

b) The matrix element at one-loop will write

$$\mathcal{M}(k^2) = \frac{1}{k^2}\, e_0^2 Q \frac{1}{1+e_0^2\Pi_{QQ}} Q' \simeq \frac{QQ'}{k^2}\, \left(1 - e_0^2\Pi_{QQ}\right) e_0^2 \qquad (99)$$

c) We re-express everything in terms of the physical parameter, e^2, by rewriting e_0:

$$\mathcal{M}(k^2) \;=\; \frac{QQ'}{k^2}\left(1 - e_0^2\Pi_{QQ} + \frac{\delta\alpha}{\alpha}\right) e^2 \sim \frac{QQ'}{k^2}\left(1 - e^2\Pi_{QQ} + \frac{\delta\alpha}{\alpha}\right) \equiv \frac{Q e_\star^2(k^2) Q'}{k^2} \qquad (100)$$

We end up with a finite result, that corrects the tree-level results by terms of order $\alpha/4\pi$ in terms of a running coupling.

It is appropriate at this point to have a discussion about an important quantity which you will see in all presentations of electroweak precision data and fits. This the quantity $\Delta\alpha$. We have just seen the notion of an effective coupling and improved Born approximation. $e_\star^2(k^2)$ as defined above could develop an imaginary part, which though finite does not make the appellation of an effective coupling sensible. This combined with the fact that the imaginary part contribution is very small, we in fact define the improved Born approximation with all the effective couplings taken as real. We will do the same in the electroweak analysis and consider the effective coupling $e_\star^2(k^2)$ real. Then

$$\alpha_\star(k^2) = \frac{\alpha}{1-\Delta\alpha(k^2)} \quad , \quad \Delta\alpha(k^2) = \Pi_{\gamma\gamma}(0) - \Re e[\Pi_{\gamma\gamma}(k^2)] \qquad (101)$$

9 $\Delta\alpha(M_Z^2)$

For energies around the Z and the electroweak observables in general, one needs the running α at M_Z and therefore we will need a very good knowledge of $\Delta\alpha(M_Z^2)$. This includes effects from all those particles below the Z peak and also particles far above the Z. With a good approximation, the fermionic contribution is

$$\frac{1}{\alpha_\star(k^2)} = \frac{1}{\alpha} - \frac{1}{3\pi}\sum_{k^2 \gg m_f^2} N_c Q_f^2 \left(\ln(|k^2|/m_f^2) - \frac{5}{3}\right) + \frac{1}{3\pi}\sum_{m_f^2 \gg k^2} N_c Q_f^2 \frac{k^2}{m_f^2} \qquad (102)$$

To take into account effects from yet undiscovered particles or in general all those particles with mass above M_Z, we split $\Delta\alpha(M_Z^2)$ as

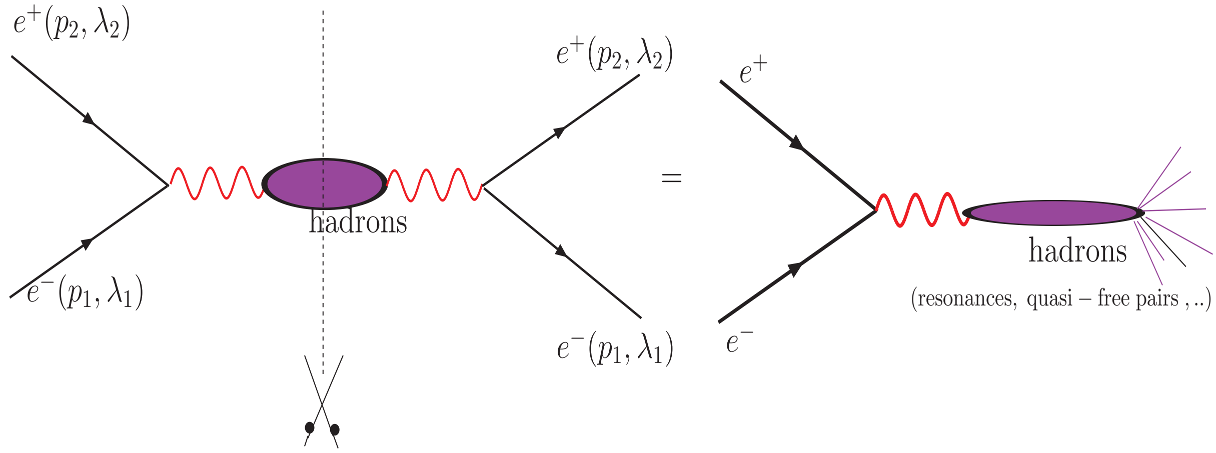
$$\begin{aligned} \Delta\alpha(M_Z^2) &= \Delta\alpha_{e,\mu,\tau}(M_Z^2) + \Delta\alpha^{(5)}_{\rm had}(M_Z^2) + \Delta\alpha_{\rm NP}(M_Z^2) \quad \text{and} \\ \alpha_Z &= \frac{\alpha}{1 - \Delta\alpha_{e,\mu,\tau}(M_Z^2) - \Delta\alpha^{(5)}_{\rm had}(M_Z^2)} \end{aligned} \qquad (103)$$

Although the top is now discovered I will still include its contribution in $\Delta\alpha_{\rm NP}(M_Z^2)$, there is also a residual contribution from W-loops which I will not include*. The top contributes a

*Within the usual linear gauge-fixing condition, the self-energy contribution is not gauge invariant. People have reverted to pinch technique methods to extract this contribution. I prefer the non-linear gauge approach.

little, we find for $m_t = 175$GeV $\Delta\alpha_{\rm top}(M_Z^2) = -.78 \times 10^{-4}$. (Use the full formula for Π_{QQ} to get this number). In principle the other contributions are then known. This is certainly the case for the leptonic contribution. You can quite happily use the above formula for (e, μ, τ) with $m_e = .5MeV, m_\mu = 106MeV, m_\tau = 1.784GeV$ to quite precisely compute this contribution. Actually higher order perturbative corrections are known up to three loops, they change the leading order very slightly. We get $\Delta\alpha_{e,\mu,\tau}(M_Z^2) = 314.98\ 10^{-4}$. One could attempt to do the same for the 5 light quarks, however especially for $u, d, s, (c)$ we are really into a region which is clearly non-perturbative and which can not be treated using naive QCD corrections. For these quarks one is really dealing with hadrons. To circumvent the problem the hadronic contribution is extracted, through a dispersion relation, to hadron production in e^+e^- .

The idea is the following, by cutting through the hadronic bubble of the photon polarisation vector, we get $\gamma^* \rightarrow hadrons$ or more exactly $e^+e^- \rightarrow hadron$ which is measured experimentally.



That this is possible is just due to the conservation of probability which in field theory is encoded in the unitarity of the S-matrix, $\mathbf{S} = \mathbf{1} + i\mathbf{T}$, where T is the familiar transition matrix. Usually one works with the reduced transition matrix where energy-momentum conservation has been factored out.

$$< f|T|i >= (2\pi)^4 \delta^4(P_f - P_i)\mathcal{T}_{fi} \tag{104}$$

Unitarity, $\mathbf{S}\mathbf{S}^\dagger = \mathbf{1}$, implies

$$\mathcal{T}_{fi} - \mathcal{T}^\star_{if} = i \sum_{n=all\ inter.\ states} (2\pi)^4 \delta^4(P_n - P_i)\mathcal{T}^\star_{nf}\mathcal{T}_{ni}. \tag{105}$$

When the final state is the same as the initial state (same momentum, polarisation, ..), we have

$$\Im\mathcal{T}_{ii} = \lambda(s, m_i^2)\sigma_{\rm tot}(i \rightarrow \sum_n) \tag{106}$$

we can take i to be an e^+e^- state then

$$
\begin{aligned}
i\mathcal{T}_{ii} &= [\bar{u}(p_1,\lambda_1)(-ie\gamma_\mu)v(p_2,\lambda_2)]\left(\frac{-ig^{\mu\nu}}{k^2+G_{\gamma\gamma}(k^2)}\right)[\bar{v}(p_2,\lambda_2)(-ie\gamma_\nu)u(p_1,\lambda_1)] \\
&= i\frac{e^2}{4}\sum_{\lambda_1,\lambda_2}[\bar{u}(p_1,\lambda_1)\gamma_\mu v(p_2,\lambda_2)\bar{v}(p_2,\lambda_2)\gamma^\mu u(p_1,\lambda_1)]\,\frac{1}{k^2+G_{\gamma\gamma}(k^2)} \\
&= -i4\pi\alpha s\frac{1}{s+G_{\gamma\gamma}(s)} \qquad\qquad s=k^2 \qquad (107)
\end{aligned}
$$

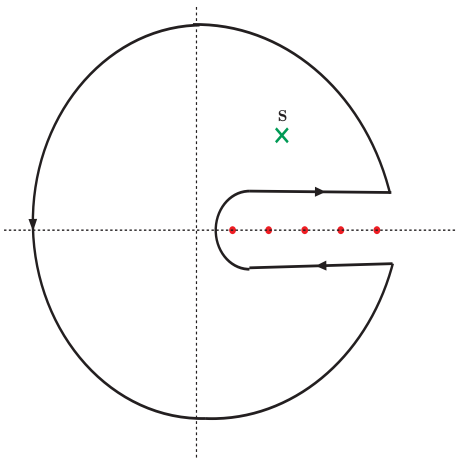
Then (with $G_{\gamma\gamma}(0)=0$)

$$
\Im\mathcal{T}^h_{ii} = +4\pi\alpha\Im\Pi^h_{\gamma\gamma}(s) = s\sigma(e^+e^- \to \text{hadrons}) \qquad (108)
$$

(we have been neglecting m_e). Normalising the hadronic cross section to the point cross section $e^+e^- \to \gamma \to \mu^+\mu^-$, we have the first important ingredient

$$
\Im\Pi^h_{\gamma\gamma}(s) = \frac{\alpha}{3\pi}R_h(s) \;\;,\;\; R_h = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \gamma \to \mu^+\mu^-)} = \frac{3\,s}{4\pi\alpha^2}\sigma(\text{hadrons}) \qquad (109)
$$

To get to $\Re e\Pi^h_{\gamma\gamma}(s)$ one appeals to Cauchy's theorem on analytical functions. $\Pi_{\gamma\gamma}(s)$ is analytical real since it is real in some region of the real axis. We thus have $\Pi_{\gamma\gamma}(s^\star) = \Pi^\star_{\gamma\gamma}(s)$, then



$$
\Pi_{\gamma\gamma}(s) = \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{ds'}{s'-s}\Pi_{\gamma\gamma}(s') \qquad (110)
$$

where we choose the contour $\mathcal{C}$ so that it encloses s and the poles are avoided. Then

$$
\begin{aligned}
\Re e\Pi^h_{\gamma\gamma}(s) &= \frac{1}{\pi}P\int_0^\infty \frac{ds'}{s'-s}\Im\Pi_{\gamma\gamma}(s') \;\; \textit{and} \\
\Re e[\Pi^h_{\gamma\gamma}(s)-\Pi^h_{\gamma\gamma}(0)] &= \frac{s}{\pi}P\int_0^\infty \frac{ds'}{(s'-s)s'}\Im\Pi_{\gamma\gamma}(s') = +\frac{\alpha}{3\pi}\,s\,P\int_0^\infty\frac{ds'}{(s'-s)s'}R_h(s') \qquad (111)
\end{aligned}
$$

So far so good, but as you can see, Fig. 9 the data for $R_h(s)$ is not continuous but available only for some scattered points.

What one usually does is to interpolate between the data sets and also extrapolate beyond what is available. Some in fact just connect the points by a straight line, some make a guess

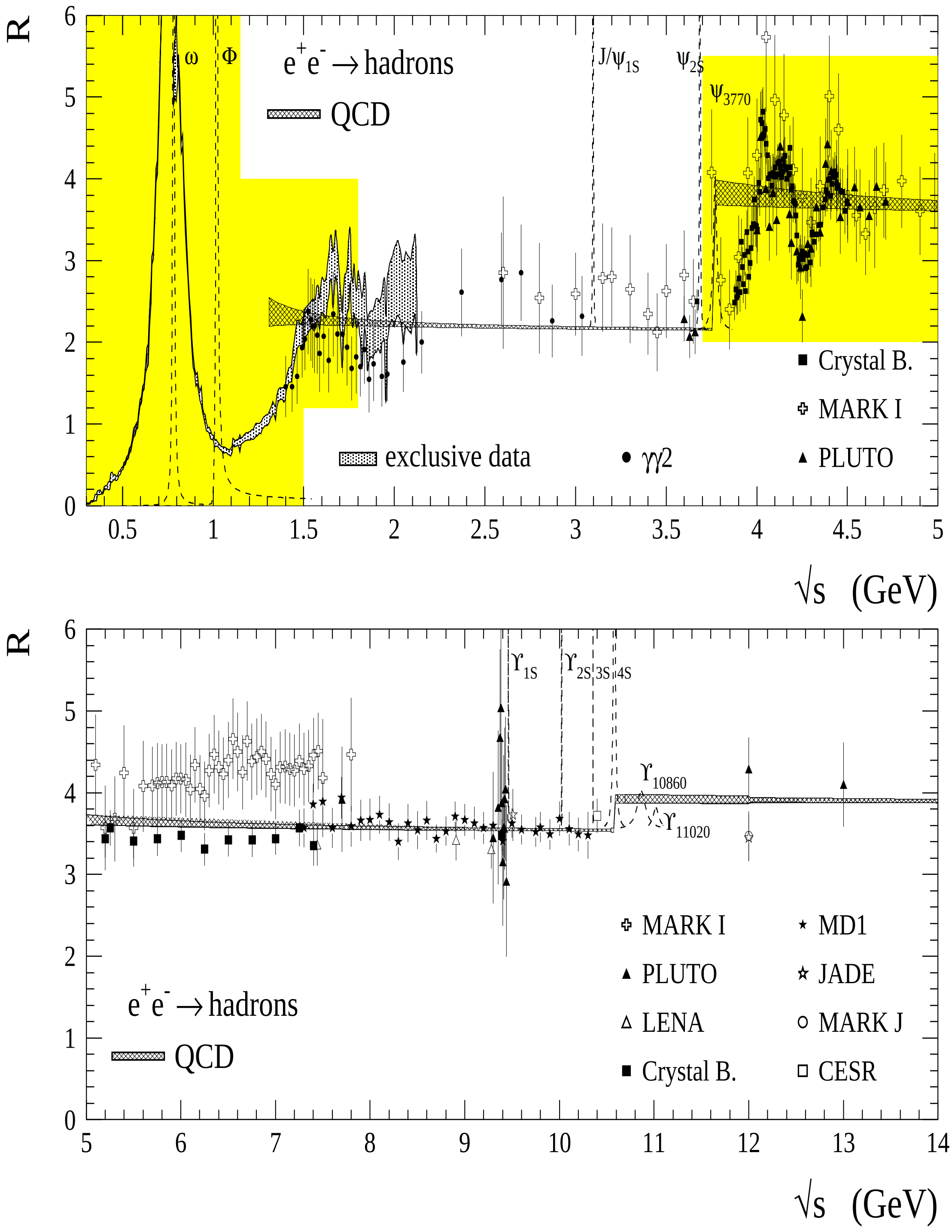


Figure 9: $R_h(s)$, *the hadronic data below* $\sqrt{s} = 15\,GeV$.

to R_h and fit the data. As you see in some regions the errors are not so small, and we need to tackle them rather properly especially in the interpolation regions. At the same time, beyond say $\sqrt{s} = 12\text{GeV}$, perturbative QCD (pQCD) can be safely applied. In this range one uses the QCD theoretical prediction for R_h where the errors are much smaller. Some have pushed down this limit quite a bit in some sophisticated perturbative QCD approach, especially before the recent BES-II data from Beijing in the region $2 - 5\text{GeV}$ was made available, Fig. 10.

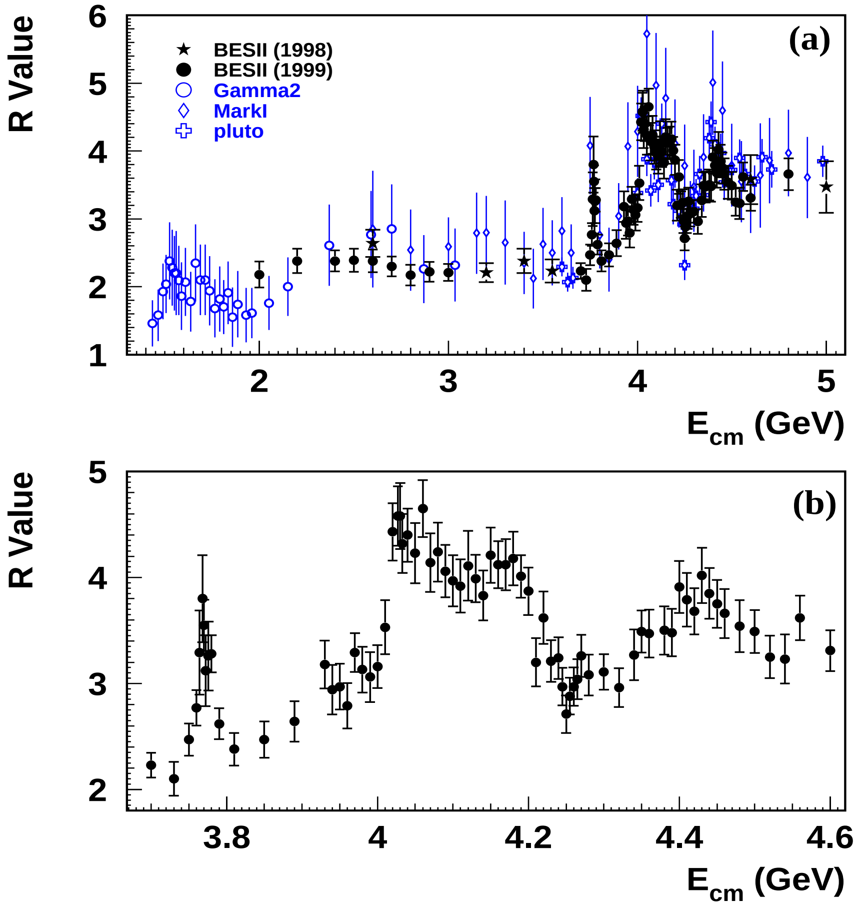


Figure 10: *New data on the hadronic cross section below* $5\,GeV$.

Conservatively the LEP fits, include the experiment based approach, with pQCD $\sqrt{s} > 12\text{GeV}$. A compilation of results based purely on experimental data and theory driven ones is shown in Fig. 11.

The latest LEP data (experiment driven) quote

$$\begin{array}{llll}
(Exp.) & 1989(\text{rescaled}) & 10^4 \times \Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = 286 \pm 9 & \alpha_Z^{-1} = 128.81 \pm .12 \\
(Exp.) & 1991 - 1995 & 10^4 \times \Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = 282 \pm 9 & \alpha_Z^{-1} = 128.87 \pm .12 \\
(Exp.) & 1995 - 2000 & 10^4 \times \Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = 280.4 \pm 6.5 & \alpha_Z^{-1} = 128.88 \pm .089 \\
(Exp.) & 2000- & 10^4 \times \Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = 273.8 \pm 2.0 & \alpha_Z^{-1} = 128.979 \pm 0.027 \\
(Th.) & 1998 & 10^4 \times \Delta\alpha^{(5)}_{\text{had}}(M_Z^2) = 276.3 \pm 1.6 & \alpha_Z^{-1} = 128.944 \pm 0.022
\end{array} \qquad (112)$$

You see that the $\Delta\alpha(M_Z^2)$ correction alone represents as much as 6%. Moreover the improvement in the precision has been steady. Just before LEP started one had a precision on α_Z

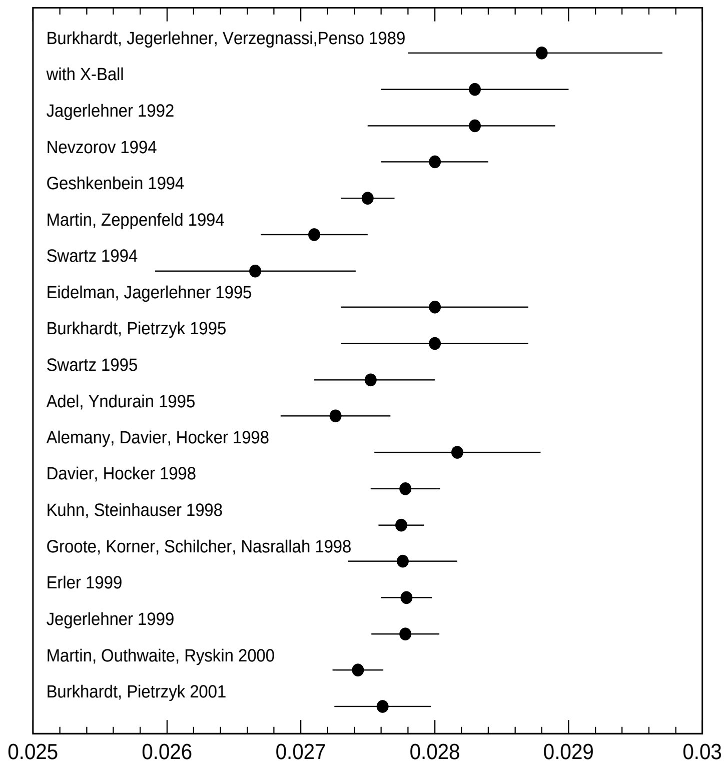


Figure 11: *How the precision on the value of* $\delta\alpha^{\mathrm{had}}(M_Z^2)$ *has evolved.*

of only about 10^{-3}, whereas now it has reached $2\ 10^{-4}$. This is a far cry from the precision on α, and still an order of magnitude worse than M_Z and G_μ, but only slightly better than M_W.

10 Radiative corrections in the EW theory

10.1 The 3 basic and fundamental parameters and how to adjust them

We have seen in QED how starting from the bare Lagrangian in terms of the classical gauge parameter e_0 we were able, after a proper redefinition of the single parameter e_0 in terms of an observable, to arrive at a finite predictive answer that took into account radiative corrections. We will try to do the same for the electroweak theory. Here one has two gauge couplings that can be used for the perturbation series, but since both are of the same order as the electromagnetic coupling and since QED will be a subset, we will use e to perform the perturbation series. As we said in the introduction and as you have seen in the course on the electroweak theory and symmetry breaking, the basic parameters are g, g', v, all others are Yukawa couplings (fermion masses and Higgs masses). We would still like to use a scheme similar to that of QED, especially that we want to recover the latter as a subset. This is the on-shell scheme which means that beside e we use M_W, M_Z as physical parameters. Therefore, starting from the bare Lagrangian with g, g', v we need the adjustments to these parameters $\delta g, \delta g', \delta v$ in order to obtain finite results once we include corrections to the propagators. Since we will be using $M_{W,Z}, e$ instead of g, g', v, we can use the inter-relations defined in Eq. 42 to move to this set. The adjustments will mean

$$
\begin{aligned}
\frac{\delta g'^{\,2}}{g'^{\,2}} &= \frac{\delta e^2}{e^2} + \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \\
\frac{\delta g^2}{g^2} &= \frac{\delta e^2}{e^2} + \frac{c_W^2}{s_W^2}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right) \\
\frac{\delta v^2}{v^2} &= -\frac{\delta e^2}{e^2} + \frac{c_W^2}{s_W^2}\frac{\delta M_Z^2}{M_Z^2} + \frac{s_W^2 - c_W^2}{s_W^2}\frac{\delta M_W^2}{M_W^2}
\end{aligned}
\qquad (113)
$$

Here we used s_W as a book-keeping device. Since we are using an on-shell scheme it is quite natural to define the mixing angle as

$$
s_M^2 = s_W^2 = 1 - \frac{M_W^2}{M_Z^2} \qquad (114)
$$

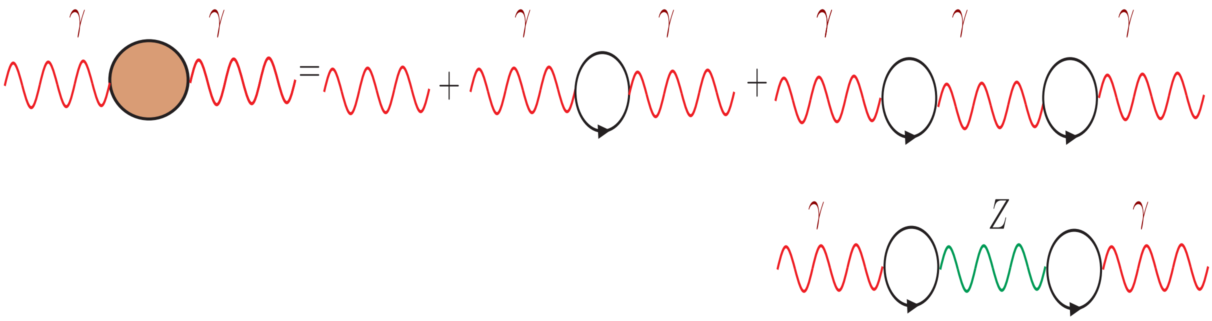
to *all orders* of perturbation theory. I need to stress this very often, this is not a fundamental parameter once we have taken the set e, M_W, M_Z as fundamental parameters. It is simply a very handy book-keeping device. Its corresponding adjustment is

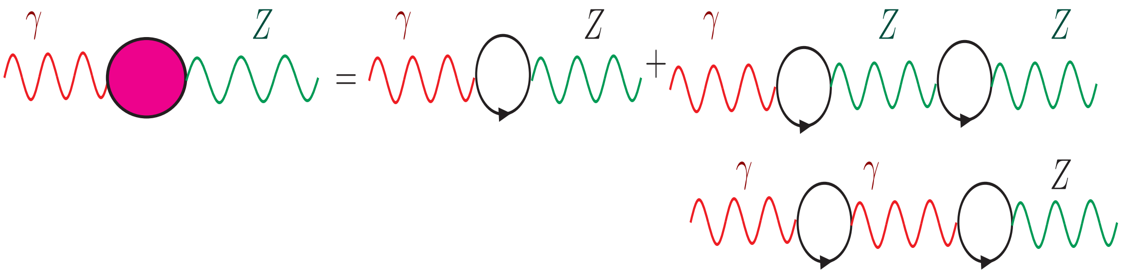
$$
\frac{\delta s_M^2}{s_M^2} = \frac{\delta s^2}{s^2} = -\frac{c_W^2}{s_W^2}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right) \quad , \quad \delta c^2 = -\delta s^2 \qquad (115)
$$

10.2 The propagators and mixing in the neutral sector

We will be dealing essentially with 4-fermion processes at LEP energies, therefore we do not have to consider the longitudinal parts of the propagators or equivalently the Goldstone bosons. Even so there is a complication when we consider the neutral sector. Indeed, at the loop level we induce a $Z-\gamma$ transition, in other words a $Z\gamma$ vacuum polarisation.

The Dyson resummation is more complex as one can see from the series of the bubble diagrams





Doing the series gives

$$
\begin{aligned}
D^{\gamma\gamma}_{\mu\nu} &= -\,ig_{\mu\nu}\,\frac{1}{k^2+G_{\gamma\gamma}(k^2)-\frac{G^2_{Z\gamma}(k^2)}{k^2-M^2_{Z,0}+G_{ZZ}(k^2)}}\\
D^{ZZ}_{\mu\nu} &= -\,ig_{\mu\nu}\,\frac{1}{k^2-M^2_{Z,0}+G_{ZZ}(k^2)-\frac{G^2_{Z\gamma}(k^2)}{k^2+G_{\gamma\gamma}(k^2)}}\\
D^{Z\gamma}_{\mu\nu} &= +\,ig_{\mu\nu}\,\frac{G_{Z\gamma}(k^2)}{k^2(k^2-M^2_{Z,0})\left\{\left(1+\frac{G_{\gamma\gamma}}{k^2}\right)\left(1+\frac{G_{ZZ}}{k^2-M^2_{Z,0}}\right)-\frac{G^2_{Z\gamma}(k^2)}{k^2(k^2-M^2_{Z,0})}\right\}}
\end{aligned}
\qquad (116)
$$

The same result can be arrived at by considering the inverse propagator matrix

$$
D^{ij}_{\mu\nu} = \begin{pmatrix} k^2+G_{\gamma\gamma}(k^2) & G_{Z\gamma}(k^2) \\ G_{Z\gamma}(k^2) & k^2-M^2_{Z,0}+G_{ZZ}(k^2) \end{pmatrix} \qquad (117)
$$

We have seen that with a good regulator $G_{\gamma\gamma}(0) = 0$. We can also show that the fermion contribution to $G_{Z\gamma} = 0$, this would be true also for the weak boson contribution if a proper gauge fixing-term is used. This can be done by taking the a gauge-fixing term for the W where the usual derivative is replaced by a $U(1)$ covariant derivative (See Appendix). Note that this has to be, otherwise one can induce a pole for the photon. Indeed

$$k^2 + G_{\gamma\gamma}(k^2) - \frac{G^2_{Z\gamma}(k^2)}{k^2 - M^2_{Z,0} + G_{ZZ}(k^2)} \neq 0 \quad \text{if} \quad G_{Z\gamma} \neq 0 \quad \text{when} \quad k^2 \to 0 \tag{118}$$

In the rest of the lecture I will therefore assume that $G_{Z\gamma}(0) = 0$, otherwise one needs to impose it. Then

$$\begin{aligned}
D^{\gamma\gamma}_{\mu\nu} &= -\,ig_{\mu\nu}\frac{1}{k^2}\frac{1}{1+\Pi_{\gamma\gamma}(k^2) - \frac{k^2\Pi^2_{Z\gamma}(k^2)}{k^2-M^2_{Z,0}+G_{ZZ}(k^2)}} \\
D^{ZZ}_{\mu\nu} &= -\,ig_{\mu\nu}\frac{1}{k^2 - M^2_{Z,0} + G_{ZZ}(k^2) - \frac{k^2\Pi^2_{Z\gamma}(k^2)}{1+\Pi_{\gamma\gamma}(k^2)}} \\
D^{Z\gamma}_{\mu\nu} &= +\,ig_{\mu\nu}\frac{\Pi_{Z\gamma}(k^2)}{(k^2 - M^2_{Z,0})\left\{(1+\Pi_{\gamma\gamma}(k^2))\left(1+\frac{G_{ZZ}}{k^2-M^2_{Z,0}}\right) - \frac{k^2\Pi^2_{Z\gamma}(k^2)}{k^2-M^2_{Z,0}}\right\}}
\end{aligned} \tag{119}$$

Of course we will only limit ourselves to corrections of order α, therefore we can see that some of the terms present in the fully dressed propagators will not contribute at this order. Therefore for our purposes it suffices to work with

$$\begin{aligned}
D^{\gamma\gamma}_{\mu\nu} &= -\,ig_{\mu\nu}\frac{1}{k^2}\frac{1}{1+\Pi_{\gamma\gamma}(k^2)} \\
D^{ZZ}_{\mu\nu} &= -\,ig_{\mu\nu}\frac{1}{k^2 - M^2_{Z,0} + G_{ZZ}(k^2)} \\
D^{Z\gamma}_{\mu\nu} &= +\,ig_{\mu\nu}\frac{\Pi_{Z\gamma}(k^2)}{k^2 - M^2_{Z,0}}
\end{aligned} \tag{120}$$

Likewise for the charged W we can take

$$D^{WW}_{\mu\nu} = -\,ig_{\mu\nu}\frac{1}{k^2 - M^2_{W,0} + G_{WW}(k^2)} \tag{121}$$

The physical Z, W masses are defined from the zero of the real part of the inverse propagators. The imaginary part(which as we will see is finite at one-loop) corresponds to the definition of the width. At one loop, this gauge invariant definition of the poles is equivalent to an on-shell definition which corresponds to the zero of the real part of the inverse propagator. For the electric charge we will keep the same definition as in pure QED.

Re-expressing the bare masses $M_{Z,0}$ and $M_{W,0}$ in terms of the physical masses, defined from the zeros of the inverse propagators, one defines for example the pole (physical) Z mass as

$$M^2_Z - M^2_Z\left(1 + \frac{\delta M^2_Z}{M^2_Z}\right) + Re\left[G_{ZZ}(M^2_Z)\right] = 0 \tag{122}$$

or

$$\delta M_Z^2 = Re\left[G_{ZZ}(M_Z^2)\right] = G_{ZZ}(0) + M_Z^2 Re\left[\Pi_{ZZ}(M_Z^2)\right] \tag{123}$$

and likewise for the W.

With these definitions, by combining the adjustments of the basic parameters with the "corrections" to the propagators one should get finite results for all observables. Since we will be dealing with the polarisation vectors it is important that we calculate them.

11 Chiral Polarisation vectors

We will only be studying the effects of the virtual fermionic corrections. This turn out to be the most important and yet are the easiest to calculate. Having grasped the idea you could try to include the full electroweak corrections with the W loops and so on. In an Appendix I have listed the contributions of all particles.

We have seen that for QED, the polarisation vector is induced by a charge Q_f writes

$$i\Sigma_{\mu\nu} = (-1)(-ie_0)^2(Q_f)^2 \int \frac{d^4r}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\mu \frac{i}{\not r - m}\gamma_\nu \frac{i}{\not r + \not k - m}\right) \tag{124}$$

In the electroweak theory, the fermionic contribution can be generalised through the couplings g_L (left-handed) and g_R right-handed (absent for the charged $W^\pm$) current. For the latter the two masses in the loop are not equal. Thus we can generalise the previous QED polarisation tensor as:

$$\begin{aligned}
i\Sigma_{\mu\nu} &= (-1)(-i)^2 \int \frac{d^4r}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\mu(g_L\gamma_L + g_R\gamma_R)\frac{i}{\not r - m_1}\gamma_\nu(g_L\gamma_L + g_R\gamma_R)\frac{i}{\not r + \not k - m_2}\right) \\
gamma_{L,R} &= \frac{1 \pm \gamma_5}{2} \\
&\to (-1)\frac{(g_L^2 + g_R^2)}{2}\mu^{4-n} \int \frac{d^nr}{(2\pi)^n} \mathrm{Tr}\left(\frac{\gamma_\mu \not r \gamma_\nu(\not r + \not k)}{(r^2 - m_1^2)((r+k)^2 - m_2^2)}\right) \\
&\to (-1)(g_Lg_R)m_1m_2\mu^{4-n} \int \frac{d^nr}{(2\pi)^n} \mathrm{Tr}\left(\frac{\gamma_\mu\gamma_\nu}{(r^2 - m_1^2)((r+k)^2 - m_2^2)}\right) \\
&\equiv g_L^2 i\Sigma_{\mu\nu}^{LL} + g_R^2 i\Sigma_{\mu\nu}^{RR} + g_Lg_R(i\Sigma_{\mu\nu}^{LR} + i\Sigma_{\mu\nu}^{RL})
\end{aligned} \tag{125}$$

Up to now we have just used the fact that traces of an odd number of $\gamma's$ is zero and that a trace with an odd number of $\gamma's$ could not contribute. Moreover a completely anti-commuting γ_5 has been used. We have also moved to n-dimension taking care to introduce the scale factor,μ, to maintain dimensionless couplings.

Following our five-step recipe, we now Feynman parameterise the denominator

$$\frac{1}{(r^2 - m_1^2)((r+k)^2 - m^2)} = \int_0^1 dx \frac{1}{r^2 - \Delta} \quad , \quad \Delta = m_1^2(1-x) \; + \; m_2^2 x - x(1-x)k^2 \tag{126}$$

Then

$$
\begin{aligned}
i\Sigma^{LL}_{\mu\nu} &= i\Sigma^{RR}_{\mu\nu} = \frac{4i}{(4\pi)^{n/2}}\Gamma(2-n/2)\int_0^1 dx \frac{1}{(\Delta/\mu^2)^{2-n/2}} \left\{ g_{\mu\nu}\left(\frac{m_1^2(1-x) \;+\; m_2^2 x}{2} - x(1-x)k^2\right)\right. \\
&\qquad\qquad \left. + k_\mu k_\nu x(1-x) \right\} \\
i\Sigma^{LR}_{\mu\nu} &= i\Sigma^{RL}_{\mu\nu} = -g_{\mu\nu} m_1 m_2 \,\frac{2i}{(4\pi)^{n/2}}\Gamma(2-n/2)\int_0^1 dx \frac{1}{(\Delta/\mu^2)^{2-n/2}} \qquad (127)
\end{aligned}
$$

Notice that in the limit where $m_{1,2} \to 0$ the $\Sigma^{LL,RR}_{\mu\nu}$ tensors are transverse. It is also easy to recover the QED results.

For the vacuum polarisation, we in fact need only the $g_{\mu\nu}$ parts for which we define and extract,

$$
\begin{aligned}
G_{LL} &= G_{RR} = -\frac{4}{(4\pi)^{n/2}}\Gamma(2-n/2)\int_0^1 dx \frac{1}{(\Delta/\mu^2)^{2-n/2}} \left\{ \left(\frac{m_1^2(1-x) \;+\; m_2^2 x}{2} - x(1-x)k^2\right)\right\} \\
G_{LR} &= G_{RL} = m_1 m_2 \,\frac{2}{(4\pi)^{n/2}}\Gamma(2-n/2)\int_0^1 dx \frac{1}{(\Delta/\mu^2)^{2-n/2}} \qquad (128)
\end{aligned}
$$

We can now expand around $n = 4$. Collecting the divergencies in

$$
C_{UV} = \frac{1}{\epsilon} - \gamma_E + \ln(4\pi) \qquad (129)
$$

And defining the finite pieces in terms of the B functions

$$
\begin{aligned}
B_0(m_1^2, m_2^2, k^2) &= B_0(m_2^2, m_2^2, k^2) = \int_0^1 dx \ln(\Delta/\mu^2) \\
B_1(m_1^2, m_2^2, k^2) &= \int_0^1 dx x \ln(\Delta/\mu^2) \\
B_2(m_1^2, m_2^2, k^2) &= \int_0^1 x(1-x) dx \ln(\Delta/\mu^2) \qquad (130)
\end{aligned}
$$

For equal masses, because of $x \leftrightarrow (1-x)$ symmetry, $B_1 = B_0/2$.

Collecting all of this one obtains

$$
\begin{aligned}
G_{LL} &= -\frac{4}{(4\pi)^2}\left\{ \frac{C_{UV}}{4}(m_1^2 + m_2^2) \;+\; k^2\left(B_2(m_1^2, m_2^2, k^2) - \frac{C_{UV}}{6}\right)\right. \\
&\qquad \left. -\frac{1}{2}\left(m_2^2 B_1(m_1^2, m_2^2, k^2) + m_1^2 B_1(m_2^2, m_1^2, k^2)\right)\right\} \\
G_{LR} &= \frac{2}{(4\pi)^2} m_1 m_2 \left\{ C_{UV} - B_0(m_1^2, m_2^2, k^2)\right\} \qquad (131)
\end{aligned}
$$

It is important to realise that the C_{UV}, or infinite, part of the Π functions does not depend on the masses and it is all contained in G_{LL}. It is tempting to make a few remarks here. Note that G_{LR} breaks chiral symmetry explicitly, so that it vanishes if any one of the masses is zero. In the chiral limit, the LL, RR currents are conserved (anomaly set aside), therefore no wonder that $G_{LL}(m_1 = m_2 = 0) \propto k^2(\cdots)$ as with QED. Note that all the divergences are real, thus all the imaginary parts that may develop from the Log's are finite.

These functions can now be exploited for the calculations of all the weak bosons self-energies. It is very instructive to write all the two-point functions in terms of the weak charge and electric assignments. These are specified by the covariant derivative

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ W_\mu^- T^-) - i\frac{g}{c_W} Z_\mu (T_3 - s_W Q) - ieA_\mu Q \tag{132}$$

We can then factorise the overall strength of the different two point functions. Note that the third component of the isospin T_3 is to be associated with L whereas the charge Q is a vector that we may see as $Q \to V = L + R$.

$$\begin{aligned}
\Pi_{\gamma\gamma} &= e^2 \Pi_{QQ} \\
\Pi_{\gamma Z} &= \left(\frac{e^2}{s_W c_W}\right)\left(\Pi_{3Q} - s_W^2 \Pi_{QQ}\right) \\
\Pi_{ZZ} &= \left(\frac{e^2}{s_W^2 c_W^2}\right)\left(\Pi_{33} - 2s_W^2 \Pi_{3Q} + s_W^4 \Pi_{QQ}\right) \\
\Pi_{WW} &= \left(\frac{e^2}{s_W^2}\right)\Pi_{11} = g^2 \Pi_{11} \\
G_{WW} &= g^2 G_{11} \\
G_{ZZ} &= \frac{g^2}{c_W^2} G_{33}
\end{aligned} \tag{133}$$

In G_{ZZ} we used the fact that we have $G_{\gamma\gamma} = G_{\gamma Z} = 0$.

From here we have that

$$G_{11}(k^2) = \frac{N_C}{2} G_{LL}(k^2, m_u, m_d)$$
$$G_{33}(k^2) = \frac{N_C}{4}\left(G_{LL}(k^2, m_u, m_u) + G_{LL}(k^2, m_d, m_d)\right) \tag{134}$$

where N_C is the number of colours. To convince yourselves about the factors of 2 in G_{11} and 4 in G_{33} you can either resort to the fermion rules, or just look at the covariant derivative[†].

Note that $G_{33}(k^2) = G_{11}(k^2)$ in the isospin limit $m_u = m_d$.

For Π_{3Q} and Π_{QQ} use the fact that

$$G_{LV} = G_{LL} + G_{LR} \quad , \quad G_{VV} = G_{LL} + G_{LR} + \Pi_{RR} + \Pi_{RL} \tag{135}$$

With all these ingredients you will notice that *ALL* the Π_{ij} , $i, j = 11, 33, 3Q, QQ$ have the same divergent term, same coefficient coefficient of $1/\epsilon$. The same holds for the k^2 independent

[†]For example the $W_\alpha e \nu_e$ vertex writes in terms of Feynman rules $-i\frac{g}{\sqrt{2}}\gamma_\alpha\gamma_L$ while the $Z_\alpha \nu \bar{\nu}_e$ is $-i\frac{g}{2c_W}\gamma_\alpha\gamma_L$

(quadratic divergence in fact) in $G_{11}(0) = G_{33}(0)$. This was to be expected, since it stems from the fact that at asymptotic scales, the $SU(2)$ and $U(1)_Y$ are not mixed, and we recover exact $SU(2)$, so that all the $SU(2)$ directions are equivalent. With these observations, it is now easy to see that any combination of the form

$$(\Pi_{ij}(k^2) - \Pi_{i',j'}(k'^2)$$

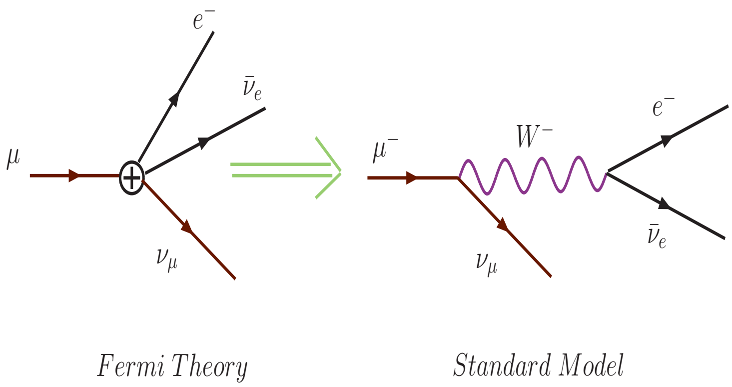
or

$$G_{33}(0) - G_{11}(0)$$

is finite . If one can therefore group all corrections in terms of these differences of self-energies, finiteness is assured.

12 First application: G_μ, M_W and Δr

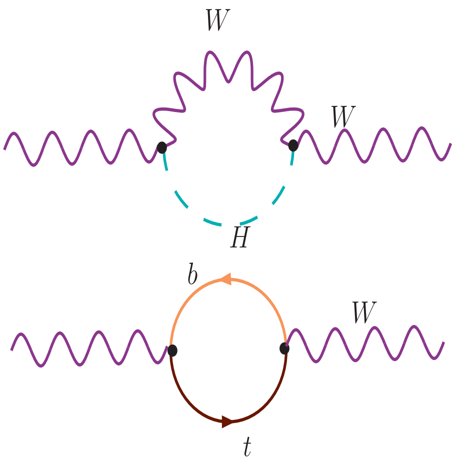
Let us go back to muon decay which we only looked at previously as a 4-point effective amplitude. We will now explicitly exhibit the W propagator and fully reveal the gauge structure.



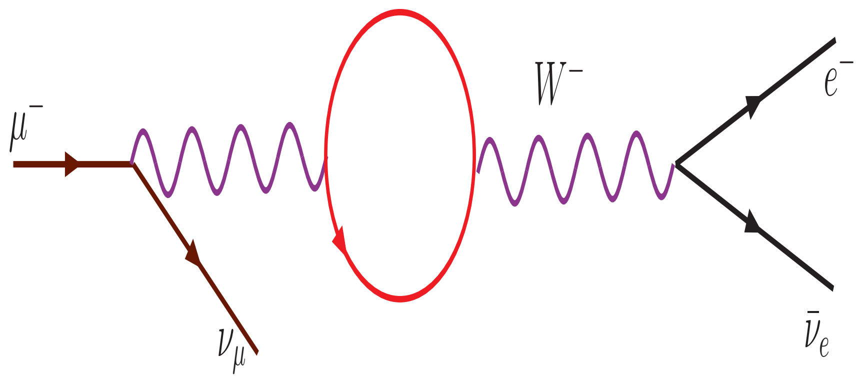
The amplitude at tree-level can be written, leaving aside the external fermion spinors, as

$$\mathcal{M}_0 = \frac{g_0^2}{k^2 - M_{W,0}^2} \quad \text{with} \quad k^2 \to 0 \tag{136}$$

The W propagator at one-loop incorporates some interesting features as it probes both the top and the Higgs



Including the loop contribution to the W propagator and redefining the $SU(2)$ coupling in terms of the physical coupling and mass, we get after specialising to $k^2 \to 0$



$$\mathcal{M}(k^2 \to 0) = \frac{g_0^2}{k^2_{\hookrightarrow 0} - M^2_{W,0} + G_{WW}(k^2_{\hookrightarrow 0})} = -\frac{g^2}{M_W^2} \frac{1 + \frac{\delta g^2}{g^2}}{1 + \frac{\delta M_W^2}{M_W^2} - \frac{G_{WW}(0)}{M_W^2}}$$
$$= -\frac{g^2}{M_W^2}\left(1 + \frac{\delta g^2}{g^2}\right)\left(1 + \Re e[\Pi_{WW}(M_W^2)]\right)^{-1} \qquad (137)$$
$$\simeq -\frac{g^2}{M_W^2}\left(1 + \frac{\delta g^2}{g^2} - \Re e[\Pi_{WW}(M_W^2)] + \delta r_{V+B}\right) \equiv -\frac{g^2}{M_W^2}(1+\Delta r)$$

Note that, in order to be complete I have added the small electroweak vertex and box corrections δr_{V+B}[‡] . This correction compared to others is quite small, as we will see. One finds

$$\delta r_{V+B} = \frac{\alpha}{4\pi s_W^2}\left(6 + \frac{7 - 4s_W^2}{2s_W^2}\ln(c_W^2)\right) \sim 6.43\ 10^{-3} \qquad (138)$$

The parameter Δr is an important ingredient in the electroweak theory and you will come across it in almost all discussions on precision measurements. It tells us also that the tree-level relation between G_μ and M_W is changed. It also *defines* how M_W is traded for G_μ:

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8M_W^2}(1+\Delta r) \quad \Delta r = \Delta\hat{r} + \delta r_{V+B} \qquad (139)$$

This means that the overall coupling as <u>measured</u> in muon decay embodies <u>all</u> the electroweak parameters g^2, M_W^2 <u>and</u> a radiative correction Δr. Of course the experiment does not make a difference between what is classical at tree-level and quantum at the loop level. Remember however that we have chosen to extract G_μ after including the purely QED radiative corrections.

Of course, the self-energy contribution to Δr must be finite. To see this we expand $\frac{\delta g^2}{g^2}$:

[‡]it is interesting to note that in a scheme where $G_{Z\gamma} \neq 0$, δr_{V+B} would be infinite.

$$
\begin{aligned}
\Delta\hat{r} &= \frac{\delta g^2}{g^2} - \Re e[\Pi_{WW}(M_W^2)] = \frac{\delta e^2}{e^2} + \frac{c_W^2}{s_W^2}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right) - \Re e[\Pi_{WW}(M_W^2)] \\
&= \Pi_{\gamma\gamma}(0) + \frac{c_W^2}{s_W^2}\left\{\underbrace{\left(\frac{G_{WW}(0)}{M_W^2} - \frac{G_{ZZ}(0)}{M_Z^2}\right)}_{\text{finite}} + \left(\Re e\Pi_{WW}(M_W^2) - \Re e\Pi_{ZZ}(M_Z^2)\right)\right\} - \Re e[\Pi_{WW}(M_W^2)]
\end{aligned}
\tag{140}
$$

To see that these combinations lead to a finite result it is very instructive to write to write all the two-point functions in terms of the weak charge and electric assignments.

We can then reorganise $\Delta\hat{r}$, as

$$
\begin{aligned}
\Delta\hat{r} &= \left(\Pi_{\gamma\gamma}(0) - \Re e\Pi_{\gamma\gamma}(M_Z^2)\right) \;+\; g^2\frac{c_W^2}{s_W^2}\left(\frac{G_{11}-G_{33}}{M_W^2}\right) \\
&+ \frac{g^2}{s_W^2}\left\{-\Re e\Pi_{33}(M_Z^2) + 2s_W^2\Re e\Pi_{3Q}(M_Z^2) + (c_W^2 - s_W^2)\Re e\Pi_{11}(M_W^2)\right\} \\
&= \left(\Pi_{\gamma\gamma}(0) - \Re e\Pi_{\gamma\gamma}(M_Z^2)\right) \;+\; g^2\frac{c_W^2}{s_W^2}\left(\frac{G_{11}-G_{33}}{M_W^2}\right) \\
&+ g^2\left\{2\Re e[\Pi_{3Q}(M_Z^2) - \Pi_{33}(M_Z^2)] \;+\; \frac{c_W^2 - s_W^2}{s_W^2}\Re e[\Pi_{11}(M_W^2) - \Pi_{33}(M_Z^2)]\right\}
\end{aligned}
\tag{141}
$$

the first part

$$
\Delta\alpha(M_Z^2) = \left(\Pi_{\gamma\gamma}(0) - \Re e\Pi_{\gamma\gamma}(M_Z^2)\right) \equiv e^2\left(\Pi_{QQ}(0) - \Re e\Pi_{QQ}(M_Z^2)\right) \tag{142}
$$

is finite as we have seen in QED. The second one is directly related to important parameter in the electroweak theory, $\Delta\rho$, nowadays also parameterised as ε_1 or αT.

$$
\Delta\rho = \varepsilon_1 = \alpha T = g^2\left(\frac{G_{33}-G_{11}}{M_W^2}\right) \tag{143}
$$

in the limit of exact $SU(2)$, global, all three components of the triplet of W would be same and hence this quantity would be zero. $SU(2)$ symmetry is broken, for instance the masses of the same fermion doublets are not equal. The most important effect is the very large masse difference between m_t and m_b. Since the G_{ii} have a dimension of a quadratic mass, we expect that the leading contribution will be proportional to $m_t^2 - m_b^2$. This is indeed what you will find if you compute this contribution from the expression of the G_{ij} that we have written in the limit $m_u^2 = m_t^2 \gg m_d^2 = m_b^2$.

As for the last two contributions, it is now easy to see that they are indeed finite. We will define

$$\varepsilon_2 = -\frac{\alpha}{4s_W^2}U = g^2\left(\Re e[\Pi_{11}(M_W^2)] - \Re e[\Pi_{33}(M_Z^2)]\right)$$
$$\varepsilon_3 = \frac{\alpha}{4s_W^2}S = g^2\left(\Re e[\Pi_{3Q}(M_Z^2)] - \Re e[\Pi_{33}(M_Z^2)]\right) = -\frac{g^2}{2}\Re e[\Pi_{3Y}(M_Z^2)] \quad (144)$$

This helps write

$$\Delta\hat{r} = \Delta\alpha(M_Z^2) \;-\; \frac{c_W^2}{s_W^2}\varepsilon_1 \;+\; \frac{c_W^2 - s_W^2}{s_W^2}\varepsilon_2 \;+\; 2\varepsilon_3 \quad (145)$$

The S, T, U ($\varepsilon_{1,2,3}$) variables are extremely sensitive to the top mass and to a lesser extent the Higgs mass, they encode effects of new particles that do not decouple, in contrast to what is encoded in $\Delta\alpha(M_Z^2)$ where all heavy particles have a negligible contribution. We will have more to say about this later.

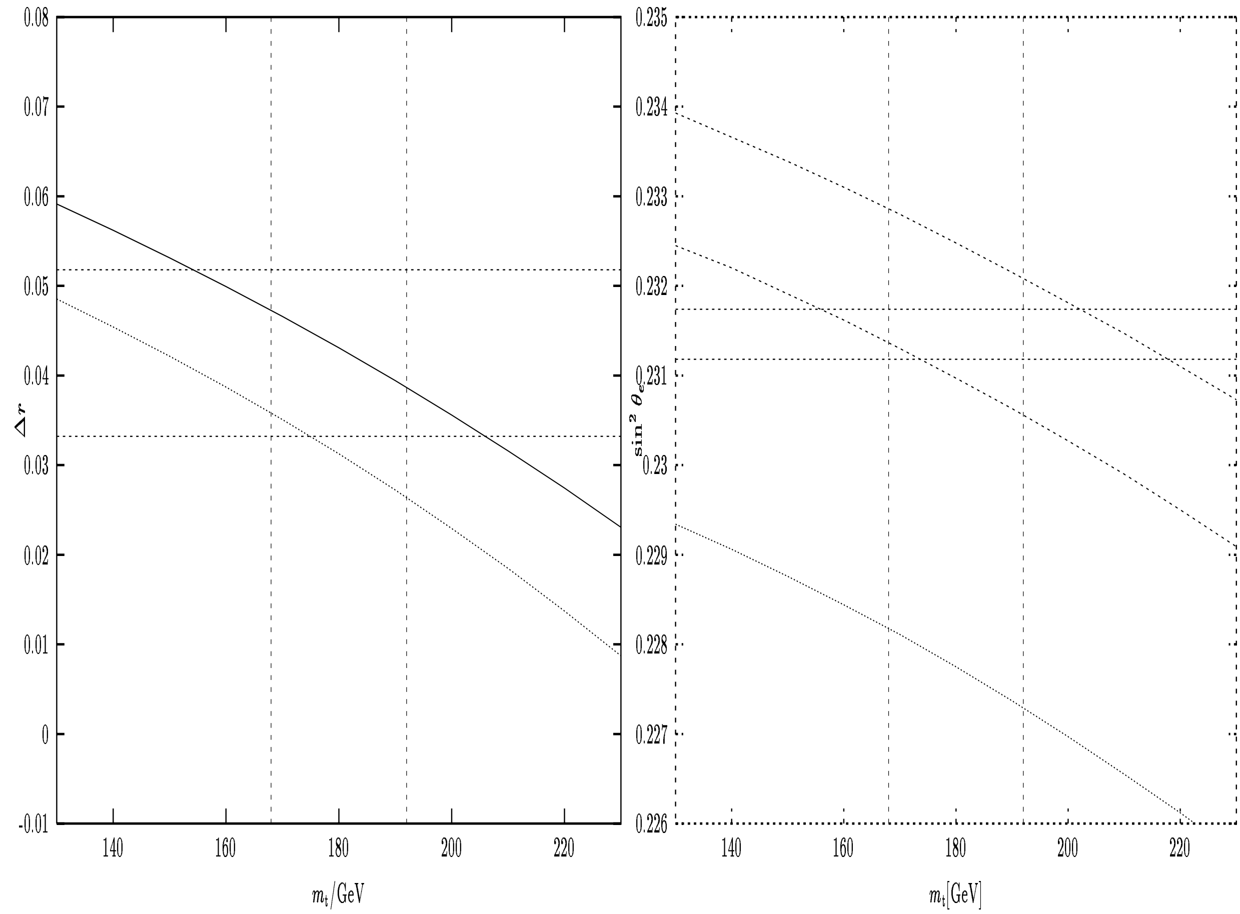


Figure 12: Δr *and* $\sin^2\theta_{eff}$ *as a function of* m_t.

One of the reasons we started by muon decay was to trade the not so precisely measured M_W (especially before *LEP*) by the very precisely measured value of G_μ. Writing now $g^2 = e^2/s_W^2$

and using the $G_\mu - M_W^2$ corrected relation we can predict

$$\frac{M_W^2}{M_Z^2} = \frac{1}{2}\left(1 + \sqrt{1 - \frac{4\pi\alpha(1+\Delta r)}{\sqrt{2}G_\mu M_Z^2}}\right) \quad ; \quad s_M^2 = s_W^2 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\pi\alpha(1+\Delta r)}{\sqrt{2}G_\mu M_Z^2}}\right) \tag{146}$$

Δr contains non-decoupled elements and could be extremely sensitive to New Physics especially that related to e-w symmetry breaking. Therefore it can be a good probe to New Physics. However since M_W is not so precisely measured one needs to wait or use some other more precisely measured observables, as we will see shortly. By the same token if we do not have all the parameters entering in Δr one can not give a very good prediction on s_W. Nonetheless, the expression of Δr suggests to define an improved s_θ^2 that should be as close to s_W^2 as possible and yet include the radiative corrections effects only from known particles. We will call this quantity s_Z^2. This we obtain by taking in Δr the $\Delta\alpha(M_Z^2)$ part that contains no new Physics, *i.e.* $\Delta\alpha(M_Z^2) \to \Delta\alpha_Z = \Delta\alpha_{e,\mu,\tau}(M_Z^2) + \Delta\alpha^{(5)}_{\rm had}(M_Z^2)$. This quantity is precisely known in terms of known parameters below the Z scale. This constitutes QED-only improved mixing angle

$$s_Z^2 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\pi\alpha_Z}{\sqrt{2}G_\mu M_Z^2}}\right) \quad \alpha_Z = \frac{\alpha}{1-\Delta\alpha_Z} \tag{147}$$

In fact we have resummed the $\Delta\alpha(M_Z^2)$ effect as we would do by using the running α. Numerically we find

$$\left\{\begin{array}{lll} s_Z^2 = .212154 & \text{for} & \alpha_Z = \alpha(0) \\ s_Z^2 = .231086 & \text{for} & \alpha_Z = 128.907 \\ s_Z^2 = .231027 & \text{for} & \alpha_Z = 128.930 \\ s_Z^2 = .230931 & \text{for} & \alpha_Z = 128.979 \\ s_Z^2 = .231019 & \text{for} & \alpha_Z = 128.944 \end{array}\right. \tag{148}$$

We see, by defining the mixing angle in terms of parameters at the Z scale, that the agreement with the measured value $s^2_{\rm eff}$ from asymmetries at the Z peak improves considerably compared to what we obtain by using the tree-level relation. In fact $s_Z^2 = .231086$ is "only" 4σ away from from the latest extracted value of $s^2_{\rm eff}$, while $s_Z^2 = .231027$ is about 7σ. Clearly this is a big improvement but we still need some New Physics. But at the same time just notice how important a very good knowledge of α_Z is! When I say "only" 4σ this is because for a long time (till about 1995) the measured effective angle at the Z peak could be accounted for by only using the running of α. Indeed in 1991 for example $s^2_{\rm eff}|_{1991} = .2329 \pm^{.0034}_{.0029}$, thus completely compatible with any of the above values with a running α, but not with $\alpha(0) = \alpha$.

This also provides an example of having as a precise value for $\Delta\alpha(M_Z^2)$ as possible. Indeed, the error on s_Z^2 from its defining parameters is

$$\delta s_Z^2 \simeq .33\left(\frac{\Delta\alpha_Z^{-1}}{\alpha_Z^{-1}} + 2\frac{\Delta M_Z}{M_Z} + \frac{\Delta G_\mu}{G_\mu}\right) \sim 3.51 \times 10^{-5}\ \left(\delta\Delta\alpha^{(5)}_{\rm had}(M_Z^2) \times 10^{-4}\right) \tag{149}$$

With the 1991 uncertainty this gives an error which is twice as big as the actual experimental precision on s_Z^2 which would not be good news if the error on $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ has not improved.

With today's uncertainty, the error is less than half the experimental precision on s_Z^2. This is still a little bit too close for comfort, especially that we have other theoretical errors. Also if a Giga-Z experiment is conducted one day, an improved $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ value would be needed. This said $s_{\rm eff}^2$ is measured from the Z-peak, that is from neutral current observables whereas we have been talking about s_Z^2. How are the two related? Let us therefore check what effective mixing angle appears when we go to 1-loop in the neutral current to which we now turn.

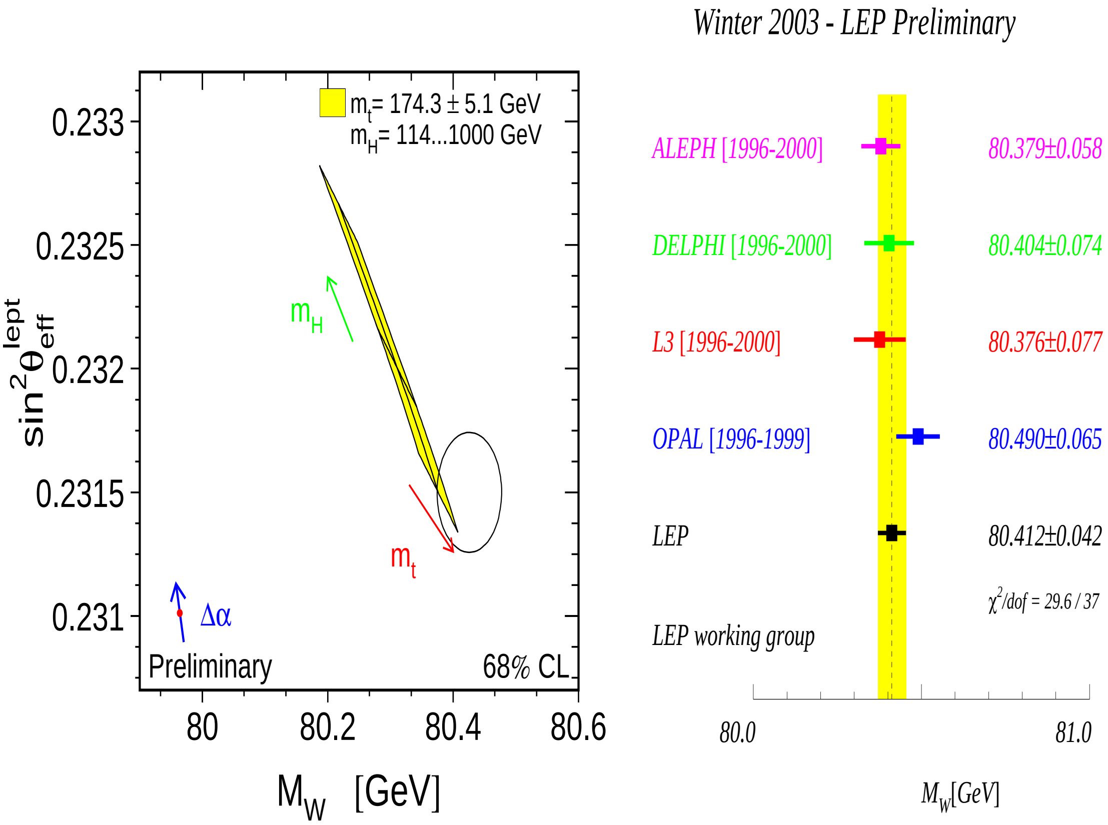


Figure 13: *Fitting M_W from precision measurements. The actual value prefers a higher top mass but a lower Higgs mass. The uncertainty from $\Delta\alpha(M_Z^2)$ is also given. The second plot shows the direct measurement of the W mass at LEP.*

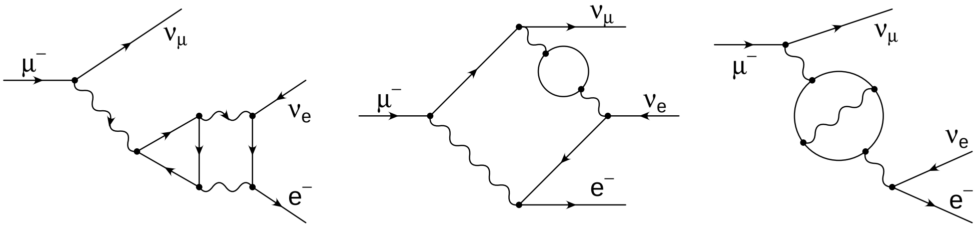


Figure 14: Examples for types of fermionic two-loop diagrams contributing to muon decay.

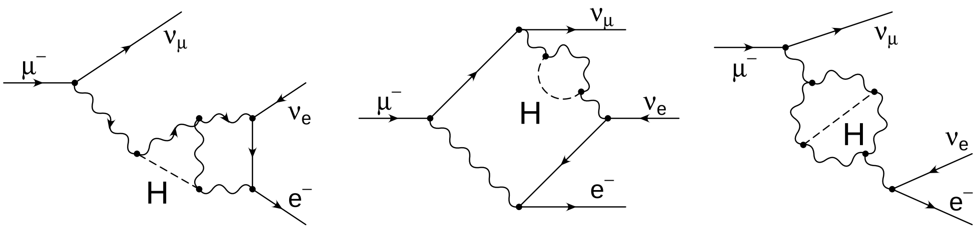


Figure 15: Examples for types of bosonic two-loop diagrams with internal Higgs bosons contributing to muon decay.

13 Latest theoretical calculations

Recently the full two-loop calculations to Δ_r have been computed. Some of diagrams that need to be calculated are shown in Figs 14, 15.

The authors have been able to parameterise the formula for the W mass quite simply:

$$\begin{aligned} M_W &= M_W^0 - c_1\,\mathrm{dH} - c_2\,\mathrm{dH}^2 + c_3\,\mathrm{dH}^4 + c_4(\mathrm{dh}-1) - c_5\,\mathrm{d}\alpha + c_6\,\mathrm{dt} - c_7\,\mathrm{dt}^2 \\ &\quad - c_8\,\mathrm{dH}\,\mathrm{dt} + c_9\,\mathrm{dh}\,\mathrm{dt} - c_{10}\,\mathrm{d}\alpha_s + c_{11}\,\mathrm{dZ}, \end{aligned} \tag{150}$$

where

$$\begin{aligned} \mathrm{dH} &= \ln\left(\frac{M_H}{100\ \mathrm{GeV}}\right), \quad \mathrm{dh} = \left(\frac{M_H}{100\ \mathrm{GeV}}\right)^2, \quad \mathrm{dt} = \left(\frac{m_t}{174.3\ \mathrm{GeV}}\right)^2 - 1, \\ \mathrm{dZ} &= \frac{M_Z}{91.1875\ \mathrm{GeV}} - 1, \quad \mathrm{d}\alpha = \frac{\Delta\alpha(M_Z^2)}{0.05907} - 1, \quad \mathrm{d}\alpha_s = \frac{\alpha_s(M_Z^2)}{0.119} - 1, \end{aligned} \tag{151}$$

with

$$\begin{aligned} M_W^0 &= 80.3800\ \mathrm{GeV}, & c_1 &= 0.05253\ \mathrm{GeV}, & c_2 &= 0.010345\ \mathrm{GeV}, \\ c_3 &= 0.001021\ \mathrm{GeV}, & c_4 &= -0.000070\ \mathrm{GeV}, & c_5 &= 1.077\ \mathrm{GeV}, \\ c_6 &= 0.5270\ \mathrm{GeV}, & c_7 &= 0.0698\ \mathrm{GeV}, & c_8 &= 0.004055\ \mathrm{GeV}, \\ c_9 &= 0.000110\ \mathrm{GeV}, & c_{10} &= 0.0716\ \mathrm{GeV}, & c_{11} &= 115.0\ \mathrm{GeV} \end{aligned} \tag{152}$$

The full result for M_W is approximated to better than 0.2 MeV over the range of 100 GeV $\leq M_H \leq$ 1TeV if all other experimental input values vary within their combined 2σ region around their central values given in

14 Corrections to the neutral current observables

In the QED warm-up we have seen how by taking into account the leading self-energy corrections we were able to obtain an improved Born-approximation. We also introduced the notion of a running coupling constant. We shall do the same now for the neutral current especially that these observables are the most precisely measured as we have seen in the introduction. We go from the same concept and consider the scattering from an initial sate defined by the quantum numbers Q, T_3 to a final state with Q', T_3'. The matrix element, including Z and γ exchange, at tree-level can be written as,

$$\mathcal{M}_0 = \underbrace{-e^2 Q \frac{1}{k^2} Q'}_{\gamma \text{ exch.}} \underbrace{- \frac{e^2}{s_W^2 c_W^2}(T_3 - s_W^2 Q)\frac{1}{k^2 - M_Z^2}(T_3' - s_W^2 Q')}_{Z \text{ exch.}} \tag{153}$$

where we have assumed the fermions to be massless, so that we can neglect the contribution of the longitudinal mode of the Z.

At one-loop we include the self-energy corrections. This includes as we have seen, $Z\gamma$ two-point function. At the same time, we re-express the bare parameters in terms of the finite parameters. We thus have,

$$\begin{aligned}
\mathcal{M} &= -e^2 \underbrace{\left(1 + \frac{\delta e^2}{e^2}\right)}_{counterterm} Q \underbrace{\frac{1}{k^2\left(1 + \Pi_{\gamma\gamma}(k^2)\right)}}_{\gamma\gamma\ 2-pt} Q' \\
&- \frac{e^2}{s_W^2 c_W^2} \underbrace{\frac{\left(1 + \frac{\delta e^2}{e^2}\right)}{\left(1 + \frac{\delta s^2}{s^2} - \frac{s_W^2}{c_W^2}\frac{\delta s^2}{s^2}\right)}}_{counterterm} \underbrace{\frac{\left(T_3 - s_W^2(1 + \frac{\delta s^2}{s^2})Q\right) \times \left(T_3' - s_W^2(1 + \frac{\delta s^2}{s^2})Q'\right)}{k^2 - M_Z^2\left(1 + \frac{\delta M_Z^2}{M_Z^2}\right) + G_{ZZ}(k^2)}}_{ZZ\ 2-pt} \\
&+ \frac{e^2}{s_W c_W}(T_3 - s_W^2 Q) \underbrace{\frac{\Pi_{\gamma Z}(k^2)}{k^2 - M_Z^2}}_{Z\gamma\ 2-pt} (Q') \\
&+ \frac{e^2}{s_W c_W}(Q) \underbrace{\frac{\Pi_{\gamma Z}(k^2)}{k^2 - M_Z^2}}_{Z\gamma\ 2-pt} (T_3' - s_W^2 Q')
\end{aligned} \tag{154}$$

Since we are working to first order in $\delta\wp, \Pi_{ij}$,($\delta\wp$ is the set of adjustments) and since $\Pi_{\gamma Z}$ is already of that order, it is very convenient to express the $Z\gamma$ contribution in a format similar to ZZ

$$
\begin{aligned}
&+ \quad \frac{e^2}{s_W c_W}(T_3 - s_W^2 Q)\frac{\Pi_{\gamma Z}(k^2)}{k^2 - M_Z^2}(Q') \\
&= \quad -\frac{e^2}{s_W^2 c_W^2}\left(T_3 - s_W^2 Q\right)\left[\frac{c_W}{s_W}\Pi_{\gamma Z}(k^2)\right]\left(-s_W^2 Q'\right) \quad \rightarrow
\end{aligned} \tag{155}
$$

$$
- \quad \frac{e^2}{s_W^2 c_W^2}\frac{\left(1+\frac{\delta e^2}{e^2}\right)}{\left(1+\frac{\delta s^2}{s^2} - \frac{s_W^2}{c_W^2}\frac{\delta s^2}{s^2}\right)}\frac{\left(T_3 - s_W^2(1+\frac{\delta s^2}{s^2})Q\right)\times\left(\frac{c_W}{s_W}\Pi_{\gamma Z}(k^2)\right)\times\left(-s_W^2(1+\frac{\delta s^2}{s^2})Q'\right)}{k^2 - M_Z^2\left(1+\frac{\delta M_Z^2}{M_Z^2}\right) + G_{ZZ}(k^2)} \tag{156}
$$

we can now regroup the $Z\gamma$ contribution with the ZZ so that

$$
\begin{aligned}
\mathcal{M} \quad &= \quad -e^2\left(1+\frac{\delta e^2}{e^2}\right)Q\frac{1}{k^2\left(1+\Pi_{\gamma\gamma}(k^2)\right)}Q' \\
&- \quad \frac{e^2}{s_W^2 c_W^2}\frac{\left(1+\frac{\delta e^2}{e^2}\right)}{\left(1+\frac{\delta s^2}{s^2} - \frac{s_W^2}{c_W^2}\frac{\delta s^2}{s^2}\right)}\frac{\left(T_3 - s_W^2(1+\frac{\delta s^2}{s^2}+\frac{c_W}{s_W}\Pi_{\gamma Z}(k^2))Q\right)\left(T_3' - s_W^2(1+\frac{\delta s^2}{s^2}\frac{c_W}{s_w}\Pi_{\gamma Z}(k^2))Q'\right)}{k^2 - M_Z^2\left(1+\frac{\delta M_Z^2}{M_Z^2}\right) + G_{ZZ}(0) + k^2\Pi_{ZZ}(k^2)}
\end{aligned} \tag{157}
$$

We first of all see that the photon exchange has kept the same structure as in QED, whereas the Z exchange can again be described in a form akin to the tree-level expression. Indeed we can make the identification

$$
\mathcal{M}^Z = -\frac{e_\star^2}{s_\star^2 c_\star^2}(T_3 - s_\star^2 Q)\frac{1}{k^2 - M_{Z\star}^2}(T_3' - s_\star^2 Q') \tag{158}
$$

where all the $\star$ quantities are k^2 dependent running parameters. For $e_\star$ we keep the same definition as in QED. but you can see that the effective s_θ^2 which appears in the neutral current is rather different from the one defined from M_W/M_Z. First of all we should make sure that it is finite. Indeed we have

$$
\begin{aligned}
\frac{\delta s^2}{s^2} + \frac{c_W}{s_W}\Pi_{\gamma Z}(k^2) \quad &= \quad -\frac{c_W^2}{s_W^2}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right) + \frac{c_W}{s_W}\Pi_{\gamma Z}(k^2) \\
&= \quad -\frac{c_W^2}{s_W^2}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right) - \Pi_{\gamma\gamma}(k^2) + \frac{e^2}{s_W^2}\Pi_{3Q}(k^2) \\
&= \quad -\left(\frac{c_W^2}{s_W^2}\left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2}\right) + \Pi_{\gamma\gamma}(0) - \Re e\Pi_{WW}(M_W^2)\right) \\
&- \quad \left(\Pi_{\gamma\gamma}(k^2) - \Pi_{\gamma\gamma}(0)\right) \; + \; \frac{e^2}{s_W^2}\left(\Pi_{3Q}(k^2) - \Re e\Pi_{11}(M_W^2)\right) \\
&\equiv \quad -\Delta\hat{r} + \Delta\alpha(k^2) \; + \; \frac{e^2}{s_W^2}\left(\Pi_{3Q}(k^2) - \Re e\Pi_{11}(M_W^2)\right)
\end{aligned} \tag{159}
$$

This is indeed finite. It is then very easy to see that

$$\begin{aligned} s_\star^2(k^2) &= s_Z^2 \Big\{ 1 + (\Delta\alpha(k^2) - \Delta\alpha(M_Z^2)) \;+\; \frac{c_W^2}{c_W^2 - s_W^2} (\Delta\alpha_{NP}(M_Z^2) - \Delta\rho + \delta r_{V+B}) \\ &+ \frac{e^2}{s_W^2}\frac{1}{c_W^2 - s_W^2}\left(\Re e\Pi_{3Q}(M_Z^2) - \Re e\Pi_{33}(M_Z^2)\right) + \frac{e^2}{s_W^2}\left(\Pi_{3Q}(k^2) - \Re e\Pi_{3Q}(M_Z^2)\right) \Big\} \end{aligned} \tag{160}$$

At the Z peak this expression simplifies greatly write

$$s_\star^2(M_Z^2) \;=\; s_Z^2 \left\{ 1 - \frac{c_W^2}{c_W^2 - s_W^2}\left(\Delta\rho \;-\; \frac{\varepsilon_3}{c_W^2} - \Delta\alpha_{NP}(M_Z^2) - \delta r_{V+B}\right)\right\} \tag{161}$$

One often sees the parameter $\Delta\kappa'$ which defines the effective mixing angle at the Z-peak

$$s_\star^2(M_Z^2) = s_Z^2(1 + \Delta\kappa') \tag{162}$$

This expression shows that indeed the large $\Delta\alpha(M_Z^2)$ correction does indeed appear in $s_\star^2(M_Z^2)$. However, $\Delta\kappa'$ encodes all the Higgs and top corrections. With $\Delta\alpha^{(5)}_{\rm had}(M_Z^2) = 273.8$ as the latest standard, I find

$$\begin{aligned} s_\star^2(M_Z^2) &= .23122 \quad m_t = 174.3\text{GeV} \quad M_H = 300\text{GeV} \\ s_\star^2(M_Z^2) &= .23018 \quad m_t = 200\text{GeV} \quad M_H = 300\text{GeV} \qquad (163) \\ s_\star^2(M_Z^2) &= .23457 \quad m_t = 200\text{GeV} \quad M_H = 300\text{GeV} \qquad (164) \end{aligned}$$

This little exercise alone shows that both $m_t = 200$GeV and $m_t = 50$GeV are excluded based on the experimental measurement of $s_\star^2(M_Z^2)$ for any value of M_H. LEP indirectly measures m_t and agrees with the direct measurement from the Tevatron.

we also give the sensitivity of $s_\star^2(M_Z^2)$ to the Higgs mass, together with the errors on the other parameters entering $s_\star^2(M_Z^2)$:

$$\delta s_\star^2(M_Z^2) = \delta s_Z^2 \;+\; 5.8510^{-4}\ln(M_H/M_Z) + 3.9310^{-5}\Delta m_t \tag{165}$$

This again shows that in order to extract M_H one needs a good measurement of $\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$ and also Δm_t which is now measured only within 5GeV.

Let us go back at the Z exchange and look at the overall strength:

$$\begin{aligned} Z(k^2) &= \frac{e^2}{s_W^2 c_W^2}\frac{\left(1 + \frac{\delta e^2}{e^2}\right)}{\left(1 + \frac{\delta s^2}{s^2} - \frac{s_W^2}{c_W^2}\frac{\delta s^2}{s^2}\right)} \times \frac{1}{k^2 - M_Z^2\left(1 + \frac{\delta M_Z^2}{M_Z^2}\right) + G_{ZZ}(0) + k^2\Pi_{ZZ}(k^2)} \\ &= \frac{e_\star^2(k^2)}{s_\star^2(k^2)c_\star^2(k^2)}\left(1 + \frac{c_W^2 - s_W^2}{s_W c_W}\Pi_{\gamma Z}(k^2) + \Pi_{\gamma\gamma}(k^2)\right)\frac{1}{k^2 - M_Z^2 + k^2\Pi_{ZZ}(k^2) - M_Z^2\Pi_{ZZ}(M_Z^2)} \end{aligned} \tag{166}$$

Note that the pre-factor alone is divergent. One needs to combine it with the propagator, which we write as

$$k^2 - M_Z^2 + k^2\Pi_{ZZ}(k^2) - M_Z^2\Pi_{ZZ}(M_Z^2) = k^2 - M_Z^{\star\,2}(k^2) \; + \; (k^2 - M_Z^2)\Pi_{ZZ}(M_Z^2)$$
$$+M_Z^2\left(\Pi_{ZZ}(k^2) - \Pi_{ZZ}(M_Z^2)\right) \quad \text{with} \quad M_Z^{\star\,2}(k^2) = \left\{M_Z^2 - (k^2 - M_Z^2)(\Pi_{ZZ}(k^2) - \Pi_{ZZ}(M_Z^2))\right\} \tag{167}$$

$M_Z^{\star\,2}$ is the running Z mass. Obviously it is finite and equals the physical Z mass at the Z pole, *i.e*,

$$M_Z^{\star\,2}(M_Z^2) = M_Z^2 \quad \text{and} \quad \frac{M_Z^{\star\,2}(M_Z^2)}{dk^2} = 0 \;\text{ at }\; k^2 = M_Z^2. \tag{168}$$

we can thus write

$$\begin{aligned} Z(k^2) &= \frac{e_\star^2(k^2)}{s_\star^2(k^2)c_\star^2(k^2)} Z_\star(k^2) \frac{1}{k^2 - M_Z^\star(k^2)^2} \\ Z_\star(k^2) &= 1 + \frac{c_W^2 - s_W^2}{s_W c_W}\Pi_{\gamma Z}(k^2) + \Pi_{\gamma\gamma}(k^2) - \Pi_{ZZ}(k^2) + (k^2 - 2M_Z^2)\tilde{\Pi}'_{ZZ}(k^2) \\ \text{with} \quad & \Pi_{ZZ}(k^2) = \Pi_{ZZ}(M_Z^2) + (k^2 - M_Z^2)\tilde{\Pi}'_{ZZ}(k^2) \end{aligned} \tag{169}$$

It is clear that $\tilde{\Pi}'_{ZZ}(k^2)$ is finite, whereas

$$\frac{c_W^2 - s_W^2}{s_W c_W}\Pi_{\gamma Z}(k^2) + \Pi_{\gamma\gamma}(k^2) - \Pi_{ZZ}(k^2) = \frac{e^2}{s^2c^2}\left(\Pi_{3Q}(k^2) - \Pi_{33}(k^2)\right) \tag{170}$$

It is worth noticing that the combination $\Pi_{3Q}(k^2) - \Pi_{33}(k^2)$ also appear in $s_\star^2(k^2)$. This implies that the effective strength of the Z to fermions at the Z peak is accounted for mostly by $\Delta\rho$

$$\begin{aligned} \frac{4\pi\alpha_\star(M_Z^2)}{s_\star^2(M_Z^2)c_\star^2(M_Z^2)} Z_\star(M_Z^2) &= 4\sqrt{2}G_\mu M_Z^2\left(1 + \Delta\rho - M_Z^2\tilde{\Pi}'_{ZZ}(M_Z^2)\right) = 4\sqrt{2}G_\mu M_Z^2\left(1 + \Delta\rho(M_Z^2)\right) \\ &\simeq 4\sqrt{2}G_\mu M_Z^2\left(1 + \Delta\rho(0)\right) \equiv 4\sqrt{2}G_\mu M_Z^2\left(1 + \Delta\rho\right) \end{aligned} \tag{171}$$

So that at the Z-peak one can effectively write

$$g_A^{\text{lepton}} = -\frac{1}{2}\sqrt{1 + \Delta\rho} \quad , \quad g_V^{\text{lepton}} = g_A^{\text{lepton}}(1 - 4s^2_{\text{eff ,lepton}}) \tag{172}$$

Exercise: Show explicitly by neglecting the small δr_{V+B}, that the neutral current strength at low energy, $Z(0)$ is simply given by

$$Z(0) = 4G_\mu\sqrt{2}\,(1 + \Delta\rho) \tag{173}$$

15 Charged current processes

Muon decay is in fact a charged current process, albeit at low energies $k^2 \to 0$. Here we consider the more general case of arbitrary momentum transfer. In this case the matrix element only involves the isospin current

$$\mathcal{M}_{cc}^{(0)} = -i\frac{e^2}{2s_W^2}J^+\frac{1}{k^2 - M_W^2}J^- \tag{174}$$

Including the self-energy corrections and the counter-terms we get

$$\mathcal{M}_{cc} = \frac{-i}{2}J^+\frac{e^2}{s_W^2}\left(1+\frac{\delta e^2}{e^2}-\frac{\delta s^2}{s^2}\right)\frac{1}{k^2 - M_W^2 + k^2\Pi_{WW}(k^2) - M_W^2\Pi_{WW}(M_W^2)}J^- \tag{175}$$

Following the same steps as with the Z current, we may write

$$\mathcal{M}_{cc} = \frac{-i}{2}J^+\frac{Z_\star^W(k^2)}{k^2 - M_W^2 + k^2\Pi_{WW}(k^2) - M_W^\star(k^2)^2}J^- \tag{176}$$

$M_W^\star(k^2)^2$ can be obtained from $M_Z^\star(k^2)^2$ by allowing all ${}_{Z,ZZ} \to {}_{W,WW}$ and its finiteness is trivial while

$$Z_\star^W(k^2) = 1 + \Pi_{\gamma\gamma}(k^2) + \frac{c_W}{s_W}\Pi_{\gamma Z}(k^2) + \Pi_{WW}(k^2)(k^2 - 2M_W^2)\left(\frac{\Re e\Pi_{WW}(k^2) - \Re e\Pi_{WW}(M_W^2)}{k^2 - M_W^2}\right) \tag{177}$$

the last term is finite and

$$\Pi_{\gamma\gamma}(k^2) + \frac{c_W}{s_W}\Pi_{\gamma Z}(k^2) - \Pi_{WW}(k^2) = \frac{e^2}{s_W^2}\left(\Pi_{3Q}(k^2) - \Pi_{11}(k^2)\right) \tag{178}$$

which is a combination we have seen already and which is finite.

Exercise: What would you obtain for $\mathcal{M}_{cc}(k^2 \to 0)$? You should be able to give the answer without any calculation!

16 An important vertex correction

The parameterisation of the RC in terms of universal quantities that emerge from the common two point functions of the gauge bosons was possible because all the fermions could be treated as almost massless and hence share basically the same correction, weighted only by their quantum number. In that respect it is only the top that is special, in fact its effects appear already in the two-point functions. Since the top and the bottom form an $SU(2)$ doublet, it is no surprise that the $Zb\bar{b}$ vertex can get an important contribution from the top. This $SU(2)$ contribution

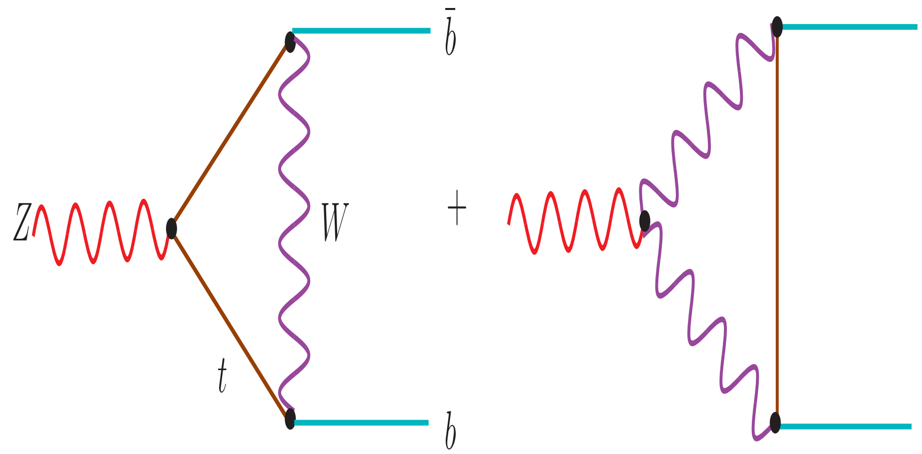


Figure 16: m_t^2 *enhanced vertex contribution to* $Z \to b\bar{b}$ *.*

only affects the left-handed part of the $Zb\bar{b}$. The contributing diagrams are shown in Fig. 16 and the matrix element which is affected

$$\mathcal{M}_{Zb_L\bar{b}_R} = -i\frac{e_\star^2}{c_\star^2 s_\star^2} Z_\mu \bar{b}\gamma_\mu b_L \left((\frac{1}{2} - \frac{1}{3}s_\star^2) - \delta v_b \right) \tag{179}$$

δv_b is the vertex correction. Since g_R is unaffected (by the vertex correction) and since $(g_L/g_R)^2 \sim 1/25$), the effect of the vertex correction on the asymmetries is quite small (for instance $\delta A_{LR}^b/A_{LR}^b \sim 4/25$ and thus the effect on the FB asymmetry is marginal). The effect on the width, on the other hand is important since $\delta\Gamma_{Zb\bar{b}}/\Gamma_{Zb\bar{b}} \sim 2\delta g_L/g_L$. This vertex correction almost "kills" the universal m_t^2 contained in $\Delta\rho$. We can write

$$\begin{aligned} \Gamma_{Zb\bar{b}} &= \Gamma^0_{Zb\bar{b}}(G_\mu, M_Z, \alpha(M_Z^2)) \left(1 + \frac{19}{13}(\Delta\rho + \delta v_b)\right) \\ \Delta\rho_t &= \frac{\alpha}{\pi}\frac{m_t^2}{M_Z^2} \quad \text{while} \quad \delta v_b = -\frac{20\alpha}{19\pi}\left(\frac{m_t^2}{M_Z^2} + \frac{13}{6}\ln(m_t^2/M_Z^2)\right) \end{aligned} \tag{180}$$

Note the numerically important logarithmic left-over from δv_b For the Z peak observables, in fact one considers the ratio R_b, in order to get rid of the final state QCD radiative corrections which I have not said much about. This ratio has made the headlines a few years ago when a discrepancy between the $\mathcal{SM}$ values and measurements was found

$$R_b = \frac{\Gamma(Z \to b\bar{b})}{\Gamma(Z \to \text{hadrons})} \simeq \left(1 - \frac{19\alpha}{13\pi}\left(\frac{m_t^2}{M_Z^2} + \frac{13}{6}\ln(\frac{m_t^2}{M_Z^2})\right)\right) R_b^0 \tag{181}$$

17 Most important combinations of two-point functions

To summarise we have seen that most observables involve a combination of two-point functions where the effects of particles at scales far above the Z peak and present energies (notably M_H, and until recently m_t) do not decouple. These are excellent probes of New Physics, especially that related to symmetry breaking since heavy physics that contributes to conserved currents like in QED do decouple. There are 3 such combinations:

$$
\begin{aligned}
\varepsilon_1 &= \alpha T = \Delta\rho = \frac{g^2}{M_W^2}\left(G_{33}(0) - G_{11}(0)\right) \\
\varepsilon_2 &= -\frac{\alpha}{4s_W^2}U = g^2\left(\Re e[\Pi_{11}(M_W^2)] - \Re e[\Pi_{33}(M_Z^2)]\right) \\
\varepsilon_3 &= \frac{\alpha}{4s_W^2}S = g^2\left(\Re e[\Pi_{3Q}(M_Z^2)] - \Re e[\Pi_{33}(M_Z^2)]\right) = -\frac{g^2}{2}\Re e[\Pi_{3Y}(M_Z^2)]
\end{aligned}
\tag{182}
$$

$\Delta\rho$ is the most important correction in the $\mathcal{SM}$, it represents the breaking of the global $SU(2)$ symmetry, for example it is badly broken by the splitting between the top and and the bottom mass. Moreover because it involves the G_{ij} which have mass dimension, its leading contribution is quadratic and therefore we expect for example the $m_t - m_b$ splitting to contribute to $\Delta\rho$ as m_t^2. This constitutes a large contribution which can be indirectly "seen" in precision measurements. Because it is also a $k^2 = 0$ effect it can be "seen" even for low-energy experiments.
noi U also measures isospin breaking albeit in a subleading way at $k^2 \sim M_Z^2$.
S is also an important parameter, it describe the mixing between the weak and hypercharge current and represent chiral symmetry breaking. For example even all heavy doublets even degenerate contribute.

The standard model contribution to $\varepsilon_{1,2,3}$, is easily calculable, one finds in leading order

$$
\begin{aligned}
\varepsilon_1 &= \frac{3G_\mu M_Z^2}{8\pi^2\sqrt{2}}\left\{\left(\frac{m_t^2}{M_Z^2} - 1\right) - 2s_M^2\ln(M_H/M_Z)\right\} + \varepsilon_1^{\rm NP} \\
\varepsilon_2 &= -\frac{G_\mu M_W^2}{2\pi^2\sqrt{2}}\ln(m_t/M_Z) + \varepsilon_2^{\rm NP} \\
\varepsilon_3 &= \frac{G_\mu M_W^2}{12\pi^2\sqrt{2}}\left\{\ln(M_H/M_Z) - \ln(m_t^2/M_Z^2)\right\} + \varepsilon_3^{\rm NP}
\end{aligned}
\tag{183}
$$

As pointed out above these variables are very sensitive to the presence of particles of high mass that nevertheless do not decouple. Most important is also the fact that they encode effects which are directly related to the breaking of the electroweak symmetry. Note for example that the effect of the Higgs, though screened through logarithms, indicates that a properly defined theory without a Higgs is almost renormalisable in the sense of that the Higgs mass would be parameterised by a cut-off scale. One needs then to see whether one can put a limit on such a scale. On the order the m_t^2 dependence indicates that without a top mass, the theory is ill defined. Indeed without a top, the theory would not be anomaly free.

These variables have been instrumental in disfavouring a host of models, especially technicolour models at least in their naive (though most attractive) implementation. Though these models have a custodial symmetry that makes them survive the $\Delta\rho$ constraint, S, ε_3 is a killer because it counts the number of mass degenerate fermions which are numerous in these models

$$
\varepsilon_3 = \frac{G_\mu M_W^2}{12\pi^2\sqrt{2}} N_c^f = 4.5\ 10^{-4} N^f_{\text{deg. fermions}} > 0
\tag{184}
$$

Whereas precision data, prefers $S > 0$. Not that the many sfermions of SUSY do not contribute to S, and they do very little to $\Delta\rho$, apart perhaps from light stops with large mixing.

We have encountered other combinations, but these are sub-sub-leading, in the sense that one expects the effects of New Physics (particle masses or an effective scale Λ) to contribute as $1/\Lambda^2$, that is the effect decouples. For example, all Π'_{ij} have a much better convergence behaviour. Objects of these sorts that we have seen are:

$$\Delta\alpha_{\rm NP} \;\; , \;\; e_5 = \Delta_Z = M_Z^2 \Pi'_{ZZ}(M_Z^2) \;\; , \;\; \left(\Pi_{3Q}(0) - \Pi_{3Q}(M_Z^2)\right) \tag{185}$$

e_5 would contribute to the strength of the Z coupling, but only in a sub-leading way.

$$\Gamma_{\rm inv}(3\nu) = \frac{\sqrt{2} G_\mu M_Z^3}{8\pi} \rho_{\rm eff} \;\; \text{,with} \;\; \rho_{\rm eff} = \left(1 + \Delta\rho - M_Z^2 \Pi'_{ZZ}(M_Z^2)\right) \tag{186}$$

remember how the leading contributions enter.

$$\begin{aligned} s^2_{\rm eff,l} &= s^2_\star(M_Z^2) = s_Z^2(1 + \Delta\kappa') \\ \Delta\kappa' &= \frac{1}{c_W^2 - s_W^2} \left(\varepsilon_3 - c_W^2(\Delta\rho - \Delta\alpha_{\rm NP}(M_Z^2) - \delta r_{VB})\right) \simeq \frac{1}{c_W^2 - s_W^2} \left(\varepsilon_3 - c_W^2 \varepsilon_1\right) \\ \Delta\hat{r} &= \Delta\alpha(M_Z^2) \; - \; \frac{c_W^2}{s_W^2}\varepsilon_1 \; + \; \frac{c_W^2 - s_W^2}{s_W^2}\varepsilon_2 \; + \; 2\varepsilon_3 \end{aligned} \tag{187}$$

18 Results and Fits to data

What follows is what the latest precision data teach us about the fundamental parameters of the weak, but also QCD, interaction. A selection of the latest fits from LEP/SLC/Tevatron follows.

Time allowing I will discuss (through figures mainly) future prospects, especially the physics at the Giga-Z project.

18.1 The Pull from all data: Overall fit

Once we have the best fit values for the floating parameters, we can predict the theoretical central values $\mathcal{O}^{th}$. We can then compare theory ($\mathcal{O}^{th}$) to experiment ($\mathcal{O}^{exp}$). The results are usually presented as "pulls", defined as

$$\text{pull} = \frac{\mathcal{O}^{th} - \mathcal{O}^{exp}}{\sigma^{th}}, \tag{188}$$

where σ^{th} is the error on $\mathcal{O}^{th}$.

All in all, the fit is successful, as evidenced from Fig. ??. One is tempted to conclude that the SM works pretty well, having been tested at the level of 1% and less.

19 How far we've come and how far will go

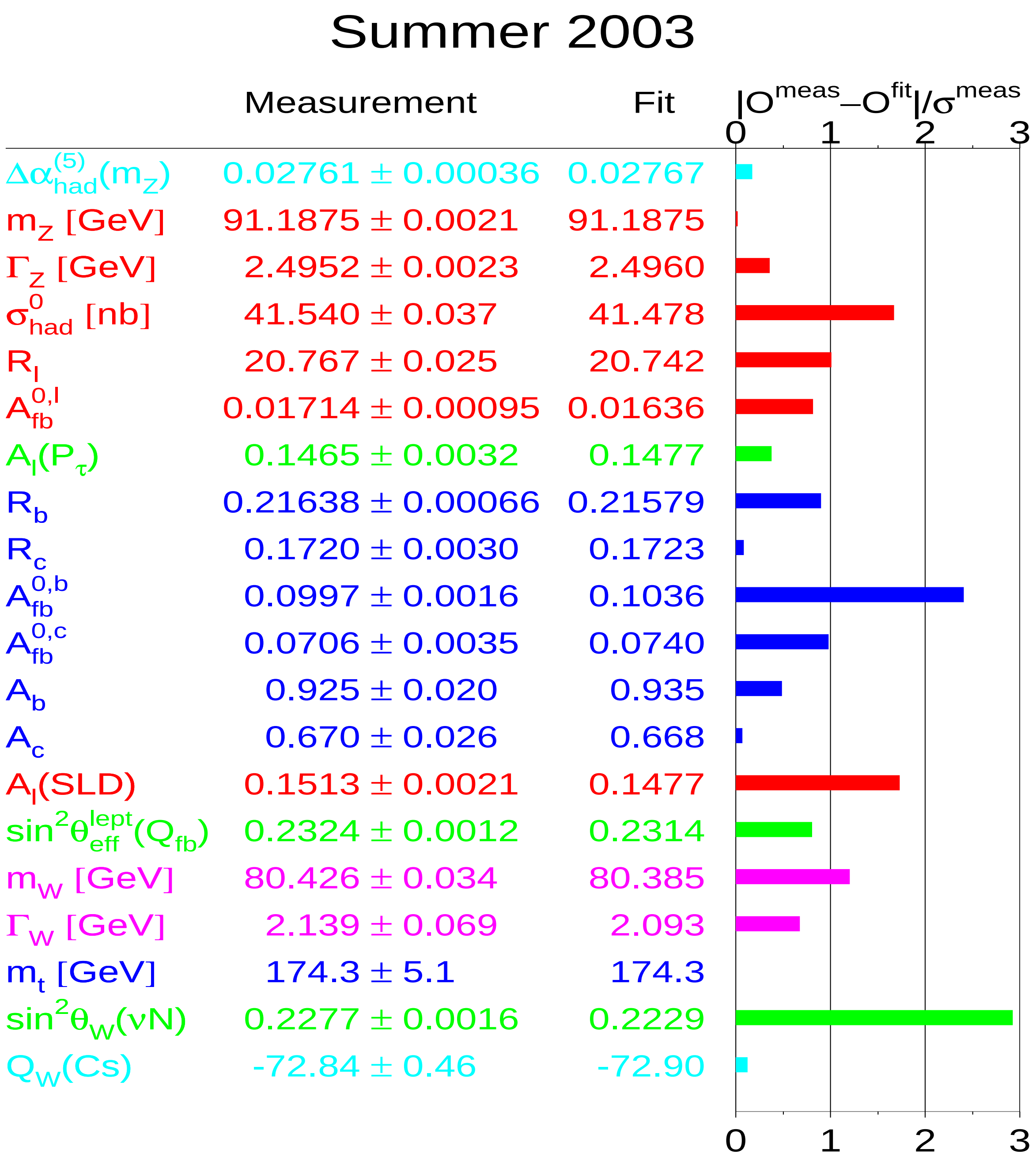


Figure 17: *The pull for all data measured at LEP. This can be seen as a consistency check between the experimental measurement and theory since the fit relies on theory. One notes that* $\sin^2\theta_W(\nu N)$ *as extracted from NuTeV is the worst. This due to a misinterpretation of the theoretical calculation. Apart from this anomaly, we see that only* A^b_{FB} *and the polarised* A_{LR} *are near* 2σ*.*

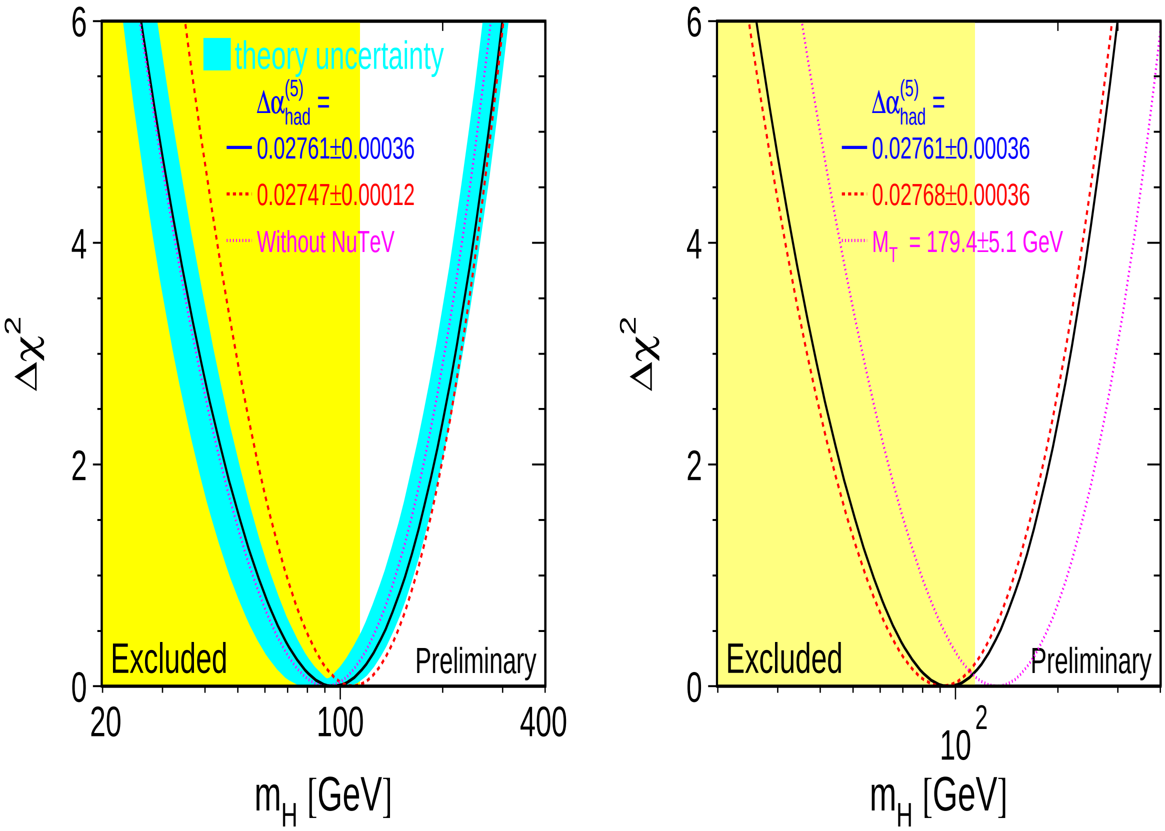


Figure 18: *The famous blue band. This gives the limit on the Higgs mass from all precision data and also after excluding the anomalous NuTeV result. The area shaded in yellow is the mass excluded from a direct search. The blue band refers to the estimated theoretical uncertainty from missing higher order terms. The* χ^2 *is given for 2 values of* $\Delta\alpha(M_Z^2)$*. From LEP electroweak group. The second plot shows what happens if one increases the average value of the top mass within* 1σ*.*

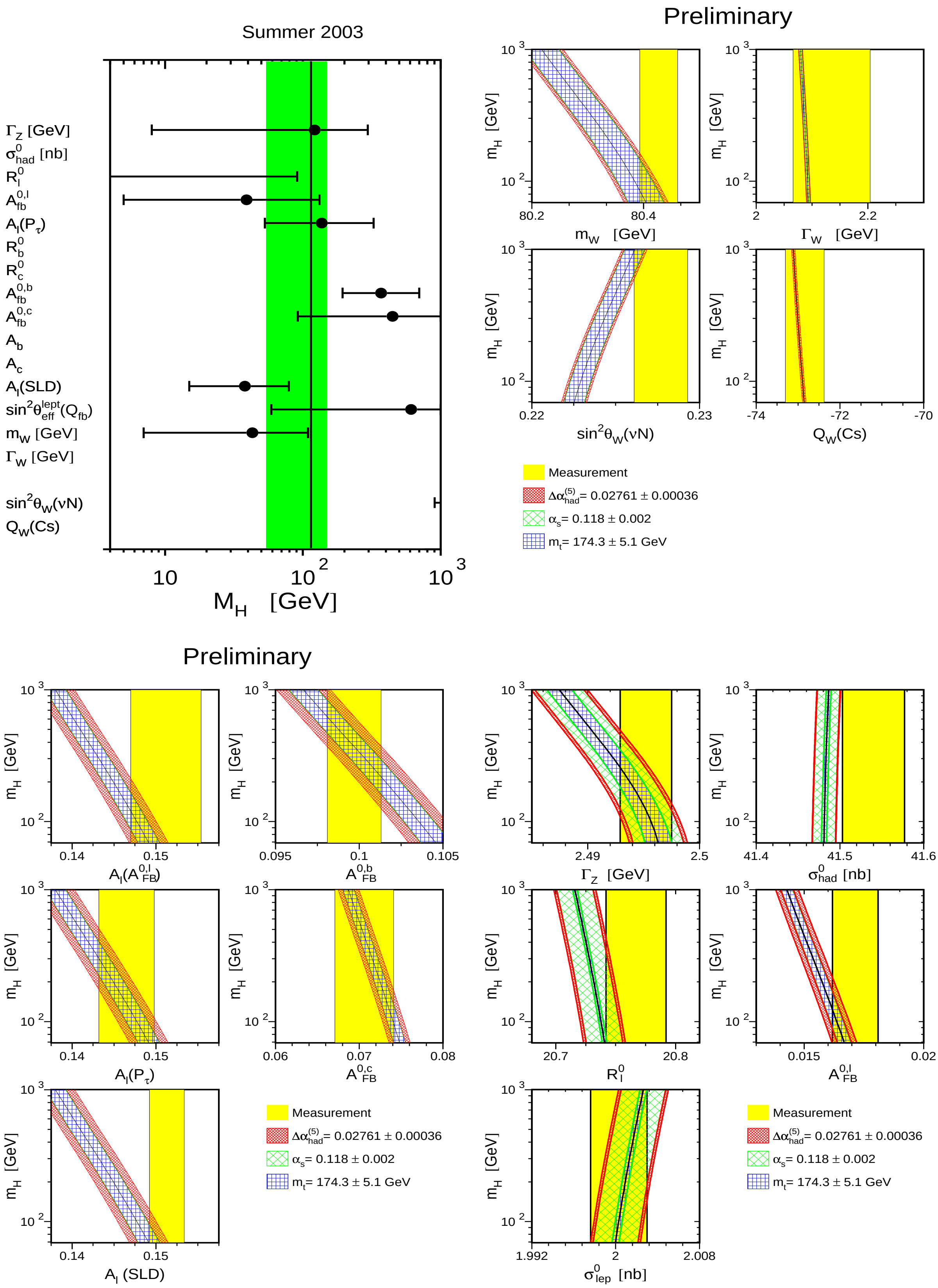


Figure 19: *M_H extraction from different observables. Not all point to a low value of M_H. Some point to values much smaller than the direct limit.*

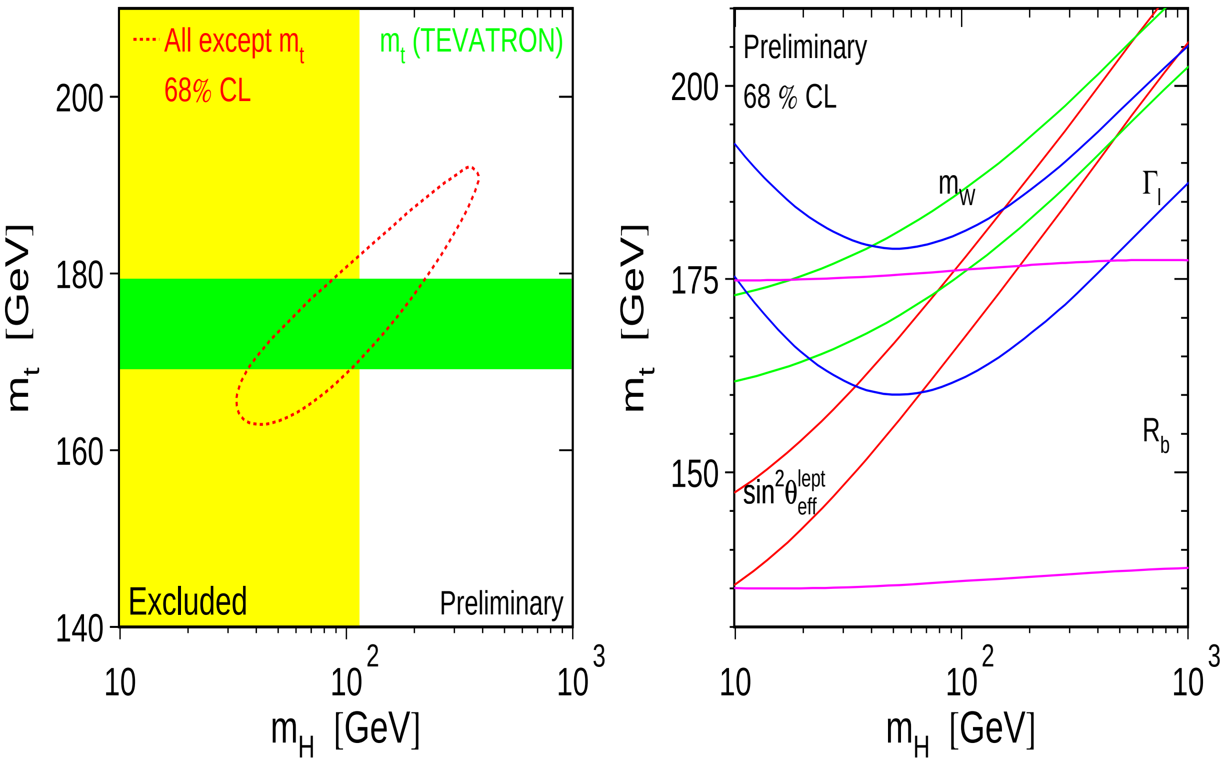


Figure 20: *Prediction of the top mass from the precision measurement and how it compares with the direct limit on the top mass from the* TEVATRON*. The plot on the left shows how the different precision observables constraint in the M_H vs m_t plan.*

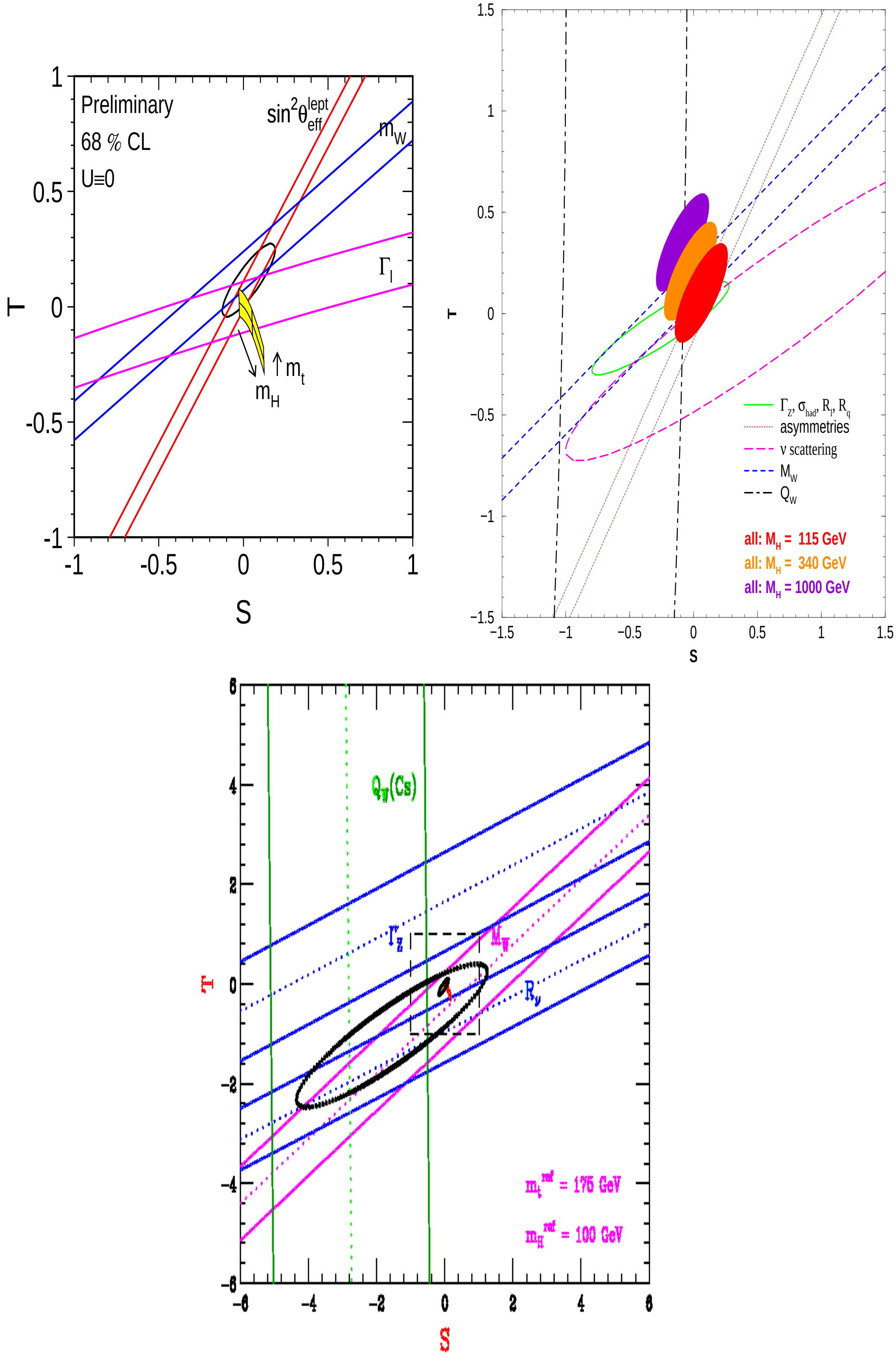


Figure 21: *Constraint in on the S and T variables. The variation from m_t within its 1σ value and M_H is shown. The second graph, which fits for different values of M_H, shows that a high value of M_H is more comfortable with $S < 0$ but $T > 0$. The last plot shows the situation in 1989 compared to the improvement made in 1999.*

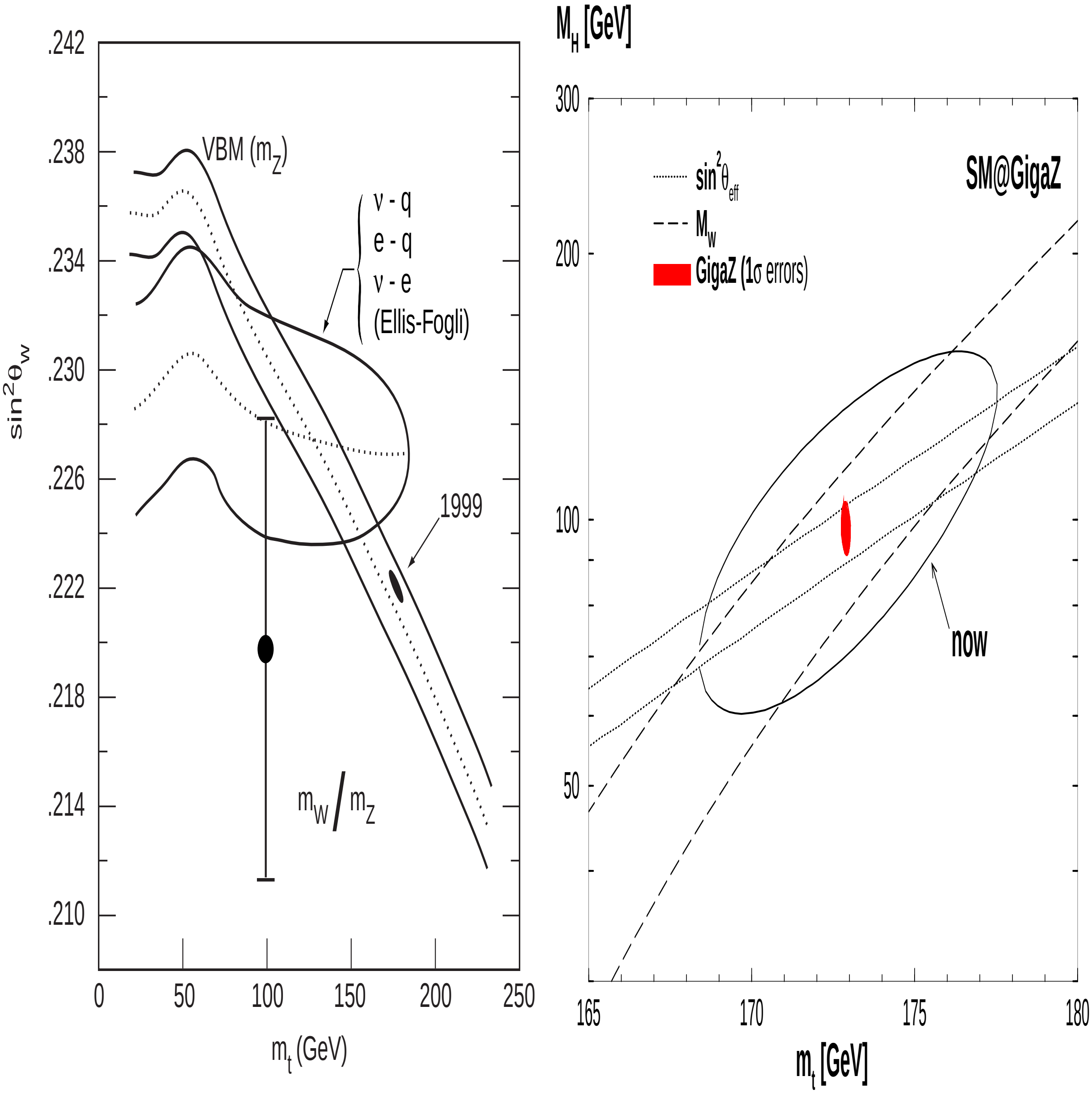


Figure 22: *A global fit on the mixing angle just before the start of LEP and how it compares with the situation in 1999!. If a linear collider is built and is run as a Z factory, the second plot shows how much improvement one expects.*

APPENDIX

A More formulae for the dimensionally regularised integrals

After Feynman parameterisation all integrals (before Wick rotation) will be of the form

$$I^{(N)}_{\mu\nu\cdots\rho} = \int \frac{d^n r}{(2\pi)^n} \frac{r_\mu r_\nu \cdots r_\rho}{(r^2 - 2r.P - M^2)^N} \tag{A.1}$$

They can all be deduced from our original

$$I^{(N)} = \int \frac{d^n r}{(2\pi)^n} \frac{1}{(r^2 - 2r.P - M^2)^N} \tag{A.2}$$

by taking partial derivatives with respect to $P_\mu P_\nu \cdots P_\rho$

$$I^{(N')}_{\mu\nu\cdots\rho} \propto \frac{\partial}{\partial P_\mu}\frac{\partial}{\partial P_\nu}\frac{\partial}{\partial P_\rho} I^{(N)} \tag{A.3}$$

with $\Delta = M^2 + P^2$

$$\begin{aligned}
I^{(N)} &= \frac{(-1)^N i\pi^{n/2}}{(2\pi)^n \Gamma(N)} \Gamma(N - n/2)\Delta^{-(N-n/2)} = \tilde{I}^{(N)}\Gamma(N - n/2) \\
I^{(N)}_\mu &= I^{(N)} P_\mu \\
I^{(N)}_{\mu\nu} &= \tilde{I}^{(N)} \left(\Gamma(N - n/2)P_\mu P_\nu - \frac{1}{2} g_{\mu\nu}\Delta\Gamma(N - 1 - n/2)\right) \\
I^{(N)}_{\mu\nu\rho} &= \tilde{I}^{(N)} \left(\Gamma(N - n/2)P_\mu P_\nu P_\rho - \frac{\Delta}{2}(g_{\mu\nu}P_\rho + g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu)\Gamma(N - 1 - n/2)\right) \\
I^{(N)}_{\mu\nu\rho\sigma} &= \tilde{I}^{(N)} \left(\Gamma(N - n/2)P_\mu P_\nu P_\rho P_\sigma + \frac{\Delta^2}{4}(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho})\Gamma(N - 2 - n/2)\right. \\
&- \left.\frac{\Delta}{2}(g_{\mu\nu}P_\rho P_\sigma + g_{\mu\rho}P_\nu P_\sigma + g_{\mu\sigma}P_\nu P_\rho + g_{\nu\rho}P_\mu P_\sigma + g_{\nu\sigma}P_\mu P_\rho + g_{\rho\sigma}P_\mu P_\nu)\Gamma(N - 1 - n/2)\right)
\end{aligned} \tag{A.4}$$

And use

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \qquad \Gamma\left(\frac{\epsilon - 1}{2}\right) = -\frac{2}{\epsilon} - (1 - \gamma_E) + \mathcal{O}(\epsilon) \tag{A.5}$$

A.1 The $i\epsilon$ prescription

The propagators are defined from the Green's functions which are solutions of the homogeneous dynamical equations of a field theory describing a particle of mass m. The choice of boundary conditions for these solutions is equivalent to the ϵ prescription. With the general expression for the propagator

$$\frac{i}{p^2 - m^2 + i\epsilon} \tag{A.6}$$

I always define the imaginary part of the distribution as

$$\frac{1}{p_0 - \omega_p \pm i\epsilon} = PP\frac{1}{p_0 - \omega_p} \pm i\pi\delta(p_0 - \omega_p) \tag{A.7}$$

this choice is also consistent with our definition of the logarithm

$$\ln(x) = \ln(|x|) + i\pi\theta(-x) \tag{A.8}$$

I have included this section for those of you who will want to go further in the renormalisation of the electroweak theory.

Feynman Rules for the Generalised Non-Linear Gauge Fixing Condition

We begin by presenting our conventions and notations for the bosonic sector of the $SU(2) \times U(1)$ model.

The $SU(2)$ gauge fields are $\boldsymbol{W}_\mu = W^i_\mu \tau^i$, while the hypercharge field is denoted by $\boldsymbol{B}_\mu = \tau_3 B_\mu$. The normalisation for the Pauli matrices is $\mathrm{Tr}(\tau^i \tau^j) = 2\delta^{ij}$. The radiation Lagrangian is expressed *via* the field strength, $\boldsymbol{W}_{\mu\nu}$

$$\begin{aligned} \boldsymbol{W}_{\mu\nu} &= \frac{1}{2}\left(\partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu + \frac{i}{2} g [\boldsymbol{W}_\mu, \boldsymbol{W}_\nu]\right) \\ &= \frac{\tau^i}{2}\left(\partial_\mu W^i_\nu - \partial_\nu W^i_\mu - g\epsilon^{ijk} W^j_\mu W^k_\nu\right) \end{aligned} \tag{A.9}$$

and

$$\boldsymbol{B}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)\tau_3 \tag{A.10}$$

such that the pure gauge kinetic term writes

$$\mathcal{L}_{\text{Gauge}} = -\frac{1}{2}\left[\mathrm{Tr}(\boldsymbol{W}_{\mu\nu}\boldsymbol{W}^{\mu\nu}) + \mathrm{Tr}(\boldsymbol{B}_{\mu\nu}\boldsymbol{B}^{\mu\nu})\right] \tag{A.11}$$

The Higgs doublet, Φ, with hypercharge $Y = 1$, is written as

$$\begin{pmatrix} \varphi^+ \\ \frac{1}{\sqrt{2}}(v + H + i\varphi_3) \end{pmatrix}$$

and the covariant derivative acting on this doublet is such that

$$\mathcal{D}_\mu \Phi = \left(\partial_\mu + \frac{i}{2}(g\boldsymbol{W}_\mu + g' Y B_\mu)\right)\Phi \tag{A.12}$$

The Higgs potential is introduced

$$\mathcal{V}_{SSB} = \lambda\left[\Phi^\dagger\Phi - \frac{\mu^2}{2\lambda}\right]^2 \tag{A.13}$$

with $\mu^2, \lambda > 0$ such that spontaneous symmetry breaking ensues. The $W^\pm$ and Higgs masses are

$$M_W = \frac{gv}{2} \qquad M_H^2 = 2\mu^2 = 2\lambda v^2 \quad v = 246 GeV \tag{A.14}$$

Our convention for the fields and couplings is

$$Z_\mu = c_W W^{(3)}_\mu - s_W B_\mu \tag{A.15}$$

$$A_\mu = s_W W^{(3)}_\mu + c_W B_\mu \tag{A.16}$$

$$g = e/s_W \tag{A.17}$$

$$g' = e/c_W \tag{A.18}$$

$$g_Z = e/s_W c_W \tag{A.19}$$

We propose, when one is dealing with multiparticle final states that involve photons and weak bosons, to use the generalised non-linear gauge fixing condition, both for the W

$$\mathcal{L}_{\xi_W} = -\frac{1}{\xi_W}|(\partial_\mu + ie\tilde{\alpha}A_\mu + ig\cos_W \tilde{\beta}Z_\mu)W^{\mu+} + i\xi_W \frac{g}{2}(v + \tilde{\delta}H - i\tilde{\kappa}\varphi_3)\varphi^+|^2 \tag{A.20}$$

and Z

$$\mathcal{L}_{\xi_Z} = -\frac{1}{2\xi_Z}(\partial.Z + \xi_Z \frac{g}{2\cos_W}(v + \tilde{\varepsilon}H)\varphi_3)^2 \tag{A.21}$$

The most practical choice for the ξ_i is $\xi_W = \xi_Z = \xi = 1$. We do not touch the gauge fixing for the photon:

$$\mathcal{L}_\xi = -\frac{1}{2\xi}(\partial.A)^2 \tag{A.22}$$

As we pointed earlier these gauge fixing conditions have to be paralleled with the gauge fixing constraints one imposes in the background-field method. In the latter, upon splitting the fields ψ into into their classical, ψ_{cl}, and quantum, ψ_Q, parts: $\psi = \psi_{cl} + \psi_Q$ and specialising to the case where the gauge parameters ξ are all equal, one has for the $SU(2) \times U(1)$ theory (see for instance[**?**])

$$\begin{aligned}
\mathcal{L}^{bckgrd} &= -\frac{1}{\xi}|(\partial.W_Q^+ + ig(W_{cl}^{(3)}.W_Q^+ - W_Q^{(3)}.W_{cl}^+) + \frac{i}{2}\xi S^+|^2 \\
&- \frac{1}{2\xi}|(\partial.A_Q + ie(W_{cl}^+.W_Q^- - W_Q^+.W_{cl}^-) + ie\xi(\varphi_Q^+\varphi_{cl}^- - \varphi_{cl}^+\varphi_Q^-)|^2 \\
&- \frac{1}{2\xi}|(\partial.Z_Q + igc_W(W_{cl}^+.W_Q^- - W_Q^+.W_{cl}^-) \\
&+ i\xi\frac{1}{2s_W c_W}\Big((c_W^2 - s_W^2)(\varphi_Q^+\varphi_{cl}^- - \varphi_{cl}^+\varphi_Q^-) + i(H_Q\varphi_{3cl} - (v + H_{cl})\varphi_{3Q})\Big)|^2 \\
S^+ &= \varphi_Q^+(v + H_{cl} - i\varphi_{3cl}) - \varphi_{cl}^+(H_Q - i\varphi_{3Q})
\end{aligned} \tag{A.23}$$

The identification with the non-linear gauge-fixing constraint is the following. For the $\gamma\gamma$ processes we have been studying, one does not need a gauge-fixing for the photon (and the Z). These are then considered purely classical as is the corresponding neutral Goldstone and the Higgs. On the other hand, since there is no separation, in the non-linear gauge, between classical and quantum fields one interprets the $W^\pm$ and their Goldstones as "quantum". Then making $W_{cl}^\pm, \varphi_{cl}^\pm \to 0$ (but $W_Q^{(3)}, H_Q \to 0$) leads to the charged part of the non-linear gauge constraint with

$$\tilde{\alpha} = \tilde{\beta} = \tilde{\delta} = \tilde{\kappa} = 1 \tag{A.24}$$

These are the values that bring the most simplifications in practical calculations. Note, however that if we *also* fix the gauge in the neutral sector then the identification is less transparent, as mixtures necessarily occur. In A.21, for instance, and with $\tilde{\varepsilon} = 1$, H is to be interpreted classical whereas φ_3 is necessarily quantum like the Z.

This is the gauge we have taken in this paper (although we did not need to specify $\tilde{\varepsilon}$).

At this point it is worth comparing with specific examples of non-linear gauges that have been used for loop calculations. The condition used in[**?**] can be recovered by setting

$$\tilde{\alpha} = \tilde{\beta} = 1 \qquad \tilde{\delta} = \tilde{\kappa} = \tilde{\epsilon} = 0 \tag{A.25}$$

This condition gets rid of $W^{\pm}\varphi^{\mp}\gamma$ and has the advantage of keeping the same Lorentz structure for the tri-linear $WW\gamma$ and WWZ vertices. However, the vertices $W^{\pm}\varphi^{\mp}Z$ and $W^{\pm}\varphi^{\mp}H(A,Z)$ are present. The condition taken in[**?**] corresponds to

$$\tilde{\alpha} = \tilde{\delta} = \tilde{\epsilon} = 1 \quad \tilde{\beta} = \tilde{\kappa} = 0 \tag{A.26}$$

Here both $W^{\pm}\varphi^{\mp}\gamma$ and $W^{\pm}\varphi^{\mp}HA$ vanish.

In[**?**] both $W^{\pm}\varphi^{\mp}\gamma$ and $W^{\pm}\varphi^{\mp}Z$ are made to vanish. One can see that this is arrived at by taking $\tilde{\alpha} = 1$, $\tilde{\delta} = \tilde{\epsilon} = \tilde{\kappa} = 0$, while

$$\tilde{\beta} = -\frac{s_W^2}{c_W^2} \tag{A.27}$$

This corresponds to a $U(1)_Y$ covariant derivative. However, contrary to what is claimed in [**?**], with this choice, $W^{\pm}\varphi^{\mp}H(Z,\gamma)$ still remain (but luckily these vertices have no incidence on the calculation $\gamma\gamma \to ZZ$ in[**?**]).

B Constructing the Ghost Lagrangian

To construct the ghost Lagrangian, we will require that the full effective Lagrangian, or rather the full action, be invariant under the BRS transformation (the measure being invariant). This implies that the full quantum Lagrangian

$$\mathcal{L}_Q = \mathcal{L}_{YM} + \mathcal{L}_{matter} + \mathcal{L}_{GF} + \mathcal{L}_{Gh} \tag{B.28}$$

be such that

$$\delta_{BRS}\,\mathcal{L} = \delta_{BRS}\,\mathcal{L}_{YM} + \delta_{BRS}\,\mathcal{L}_{matter} + \delta_{BRS}\,\mathcal{L}_{GF} + \delta_{BRS}\,\mathcal{L}_{Gh} \tag{B.29}$$

Since $\mathcal{L}_{YM}$ and $\mathcal{L}_{matter}$ are trivially invariant under the BRS: by construction they are invariant under a gauge transformation, the requirement of BRS invariance on the quantum Lagrangian implies that

$$\delta_{BRS}\,\mathcal{L}_{GF} = -\delta_{BRS}\,\mathcal{L}_{Gh} \tag{B.30}$$

The transformations on the gauge-fixing Lagrangian will trigger the compensating ghost terms.

In our case, denoting the gauge-fixing functions as $G^i \;\; i = \pm, \gamma, Z$, one has

$$\mathcal{L}_{GF} = G^- G^+ + \frac{1}{2}|G^Z|^2 + \frac{1}{2}|G_\gamma|^2 \tag{B.31}$$

In the original BRS implementation it was required that the anti-ghost be defined from

$$\delta_{BRS}\,\bar{\theta}^i = G^i \tag{B.32}$$

Hence using the nilpotency of the BRS variation $(\delta_{BRS}\,)^2\mathcal{F} = 0$ where $\mathcal{F}$ is any function of fields, one has that

$$\delta_{BRS}\,\mathcal{L}_{GF} = \delta_{BRS}\,\left(\bar{\theta}^+\delta_{BRS}\,G^+ + \bar{\theta}^-\delta_{BRS}\,G^- + \bar{\theta}^Z\delta_{BRS}\,G^Z + \bar{\theta}^\gamma\delta_{BRS}\,G^\gamma\right) \tag{B.33}$$

By identification we see that $\mathcal{L}_{Gh}$ must be of the form

$$\mathcal{L}_{Gh} = -\left(\bar{\theta}^+\delta_{BRS}\,G^+ + \bar{\theta}^-\delta_{BRS}\,G^- + \bar{\theta}^Z\delta_{BRS}\,G^Z + \bar{\theta}^\gamma\delta_{BRS}\,G^\gamma\right) + \delta_{BRS}\,\tilde{\mathcal{L}}_{Gh} \tag{B.34}$$

that is, one recovers the Fadeev-Popov prescription, but only up to an overall function, $\delta_{BRS}\,\tilde{\mathcal{L}}_{Gh}$, which is BRS invariant:

$$\mathcal{L}_{FP} = -\bar{\theta}^i\delta_{BRS}\,G^i \equiv -\bar{\theta}^i\,\frac{\partial_\omega G^i}{\partial\omega^j}\,\theta^j \tag{B.35}$$

(G^i does not depend on the θ's and $\partial\omega \equiv \delta_{BRS}$) This shows that the BRS requirement gives a more general Lagrangian than the FP construction, which in fact does not work for NL gauge-fixings.

B.1 Auxilliary Fields

$\mathcal{L}_{GF}$ may be written in a more efficient way by introducing the auxilliary fields B^i

$$\mathcal{L}_{GF} = \xi_W B^+B^- + \frac{\xi_Z}{2}|B^Z|^2 + \frac{\xi_\gamma}{2}|B^\gamma|^2 + B^-G^+ + B^+G^- + B^\gamma G^\gamma \tag{B.36}$$

Note that a priori one can define the parameter ξ in the GF: $G^+ = \partial^\mu W^+_\mu + i\xi M_W\phi^+ + ...$ that may be different from ξ_W. From the equations of motion for the B-fields one recovers the usual $\mathcal{L}_{GF}$. With $\xi_W = \xi_Z = \xi_\gamma = \xi$, one gets

$$B^i = -\frac{G^i}{\xi} \qquad \mathcal{L}_{GF} = -\frac{1}{2\xi}|G^i|^2 \tag{B.37}$$

Of course, $B^i = \delta_{BRS}\,\bar{\theta}^i$.

The use of the auxilliary B fields allows to discuss the theory (transformations) independently of the gauge parameters.

C Specific form of the BRS transformations

The action of the BRS transformations is derived as a generalisation of the usual gauge transformations to which we now turn.

$$U = exp\left(-ig\frac{\vec{\tau}.\vec{\omega}}{2} - ig'\beta\frac{Y}{2}\right) \tag{C.38}$$

whose infinitesimal form leads to

$$\begin{aligned} \delta W_\mu^i &= \partial_\mu \omega^i + g\,\epsilon_{ijk}\,\omega^j W_\mu^k \\ \delta W_{\mu\nu}^i &= g\,\epsilon_{ijk}\,\omega^j W_{\mu\nu}^k \\ \delta B_\mu &= \partial_\mu \beta \end{aligned} \tag{C.39}$$

Defining

$$\begin{aligned} \omega^\pm &= \frac{1}{\sqrt{2}}(\omega^1 \mp i\omega^2) \\ \omega^\gamma &= s_W\omega^3 + c_W\beta \\ \omega^Z &= c_W\omega^3 - s_W\beta \end{aligned} \tag{C.40}$$

one obtains

$$\begin{aligned} \delta W_\mu^\pm &= \partial_\mu\omega^\pm \pm\; ie\left[\left((A_\mu + \frac{c_W}{s_W}Z_\mu\right)\omega^\pm - \left(\omega_\gamma + \frac{c_W}{s_W}\omega_Z\right)W^\pm\right] \\ \delta Z_\mu^\pm &= \partial_\mu\omega^Z + igc_W\left(W_\mu^+\omega^- - W_\mu^-\omega^+\right) \\ \delta A_\mu^\pm &= \partial_\mu\omega^\gamma + ie\left(W_\mu^+\omega^- - W_\mu^-\omega^+\right) \end{aligned} \tag{C.41}$$

Likewise by considering the gauge transformation on the Higgs doublet one gets

$$\begin{aligned} \delta H &= -\frac{ig}{2}\left(\omega^-\phi^+ - \omega^+\phi^-\right) - \frac{e}{2s_Wc_W}\omega^Z\phi^3 \\ \delta\phi^3 &= -\frac{g}{2}\left(\omega^+\phi^- + \omega^-\phi^+\right) + \frac{e}{2s_Wc_W}\omega^Z(v+H) \\ \delta\phi^\pm &= \mp\left\{\frac{g}{2}\left(v+H\pm i\phi^3\right)\omega^\pm + e\phi^\pm\left(\omega^\gamma + \frac{c_W^2 - s_W^2}{2s_Wc_W}\omega^Z\right)\right\} \end{aligned} \tag{C.42}$$

It is then straightforward to get the BRS transformations, by identifying the parameters ω to the θ's and δ to δ_{BRS} .

To find the transformation for the ghost fields, notice that the BRS transformation is nilpotent. For instance from $(\delta_{BRS}\ )^2 W_\mu^i = 0$ one gets $\delta_{BRS}\ \theta$. Indeed more generally one has, for any group,

$$\delta_{BRS}\ A_\mu^i = D_\mu\theta^i = \partial_\mu\theta^i + [A_\mu,\theta]^i \rightarrow \delta_{BRS}\ \theta^i = -\frac{1}{2}[\theta,\theta]^i \tag{C.43}$$

§

§care should be taken that δ_{BRS} being a fermion operator the graded Leibniz rule applies: $\delta_{BRS}\ (XY) = (\delta_{BRS}\ X)Y \pm X(\delta_{BRS}\ Y)$ where the minus sign applies if X has an odd number of ghosts or antighots, note also that $(\theta^i)^2 = 0$

In our case this implies

$$\delta_{BRS}\, \theta_B = 0 \qquad \delta_{BRS}\, \theta^i = +\frac{1}{2} g\, \epsilon_{ijk}\, \theta^j \theta^k \tag{C.44}$$

and thus

$$\begin{aligned} \delta_{BRS}\, \theta^{\pm} &= \pm i\, g\, \theta^{\pm}(s_W \theta^{\gamma} + c_W \theta^Z) \\ \delta_{BRS}\, \theta^{\gamma} &= -i\, e\, \theta^+ \theta^- \\ \delta_{BRS}\, \theta^Z &= -i\, g\, c_W\, \theta^+ \theta^- \end{aligned} \tag{C.45}$$

C.1 Anti-Ghosts

One could have also introduced anti-ghosts through an anti-BRS transformation $\bar{\delta}_{BRS}$, with the identification $\bar{\delta}_{BRS} \to \bar{\theta}$, *i.e*, $\bar{\delta}_{BRS}\,(A_\mu) = \delta_{BRS}\,(A_\mu(\theta \to \bar{\theta})$, but then what is $\delta_{BRS}\,(\bar{\theta})$ or for that matter $\delta_{\bar{BRS}}\,(\theta)$? We therefore postulates an auxilliary set of fields, B^i, such that

$$\delta_{BRS}\,(\bar{\theta}^i) = B^i \tag{C.46}$$

In our case one has the identification $\delta_{BRS}\,(\bar{\theta}^{\pm,\gamma,Z}) = B^{\mp,\gamma,Z}$. $Q_{em} = \theta^- = Q_{em}\bar{\theta}^+ = -1$.

In the usual description one does not introduce the anti-BRS transformation, only the BRS with the anti-ghosts. Had we not introduced the auxilliary Bfields the BRS transformations of the anti-ghosts would have been complicated functions of the fields.

D The Generalised Ward Identities

The Ward identities are particularly easy to derive if one works with the B-fields if one considers BRS transformations on some specific Green's functions (vacuum expectation values of time ordered products). For two point function of any two fields A and B, we use the short-hand notation:

$$< A\, B >=< 0|\,(TA(x)B(y))\,|0 > \tag{D.47}$$

For example take the generic Green's function $< \bar{\theta}^i\, B^j >$ which in fact is zero (it has a non vanishing ghost number). Subjecting it to a BRS transformation one gets:

$$\begin{aligned} \delta_{BRS}\, < \bar{\theta}^i\, B^j > &= < (\delta_{BRS}\, \bar{\theta}^i)\, B^j > - < \bar{\theta}^i\, (\delta_{BRS}\, B^j) >=< B^i\, B^j >= 0 \\ or &\quad < G^i\, G^j >= 0 \end{aligned} \tag{D.48}$$

Where in the last part we have used the equation of motion for the B^i's. The above relation leads directly to a constraint on the (bare) two point functions of the gauge vector boson, the gauge-goldtone mixing and the goldstone two point functions in the linear gauge. In the non-linear case one sees that this constraint involves also three-point functions, whose contribution vanishes for on-shell (they do not have the pole structure contained in the two-point functions)

.

One can also derive the constraint on the the ghost two-point function. In this case we start with a Green's function of a gauge-field and an anti-ghost, and then apply the same trick:

$$\begin{aligned}\delta_{BRS} \ < \bar{\theta}^i \ A^j_\mu > \ &= \ < (\delta_{BRS} \ \bar{\theta}^i) \ A^j > - < \bar{\theta}^i \ (\delta_{BRS} \ A^j_\mu) > \\ &= \ < B^i \ A^j_\mu > - < \bar{\theta}^i \ (\partial_\mu \theta^i) > -g \ \epsilon_{jkl} \ < \bar{\theta}^i(x) \ \left(\theta^k(y) A^l_\mu(y)\right) >= 0\end{aligned} \tag{D.49}$$

likewise one can use the equation of motion for the auxilliary field B. We now give the Feynman rules for the generalised non-linear gauge fixing condition:

Propagators

$$\Pi^W_{\mu\nu} = \frac{-i}{k^2 - M_W^2}\left[g_{\mu\nu} + \frac{(\xi_W - 1)k_\mu k_\nu}{k^2 - \xi_W M_W^2}\right] \tag{D.50}$$

$$\Pi^Z_{\mu\nu} = \frac{-i}{k^2 - M_Z^2}\left[g_{\mu\nu} + \frac{(\xi_Z - 1)k_\mu k_\nu}{k^2 - \xi_Z M_Z^2}\right] \tag{D.51}$$

$$\Pi^\gamma_{\mu\nu} = \frac{-i}{k^2}\left[g_{\mu\nu} + \frac{(\xi_\gamma - 1)k_\mu k_\nu}{k^2}\right] \tag{D.52}$$

$$\Pi^H = \frac{i}{k^2 - m_H^2} \tag{D.53}$$

$$\Pi^{\varphi_3} = \frac{i}{k^2 - \xi_Z M_Z^2} \tag{D.54}$$

$$\Pi^{\varphi^\pm} = \frac{i}{k^2 - \xi_W M_W^2} \tag{D.55}$$

$$\Pi^{\vartheta^\gamma} = \frac{i}{k^2} \tag{D.56}$$

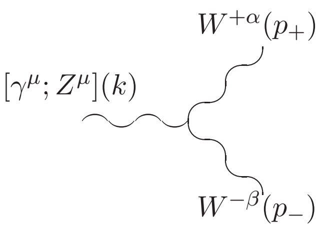
$$\Pi^{\vartheta^Z} = \frac{i}{k^2 - \xi_Z M_Z^2} \tag{D.57}$$

$$\Pi^{\vartheta^\pm} = \frac{i}{k^2 - \xi_W M_W^2} \tag{D.58}$$

As is obvious, for all calculations $\xi_i = 1$ is to be prefered.

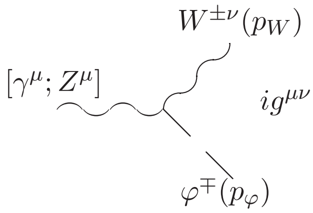
Trilinear vertices

In the following all momenta are taken to be incoming.



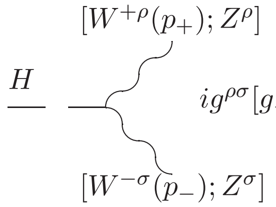
$$-ie\left[1; \frac{c_W}{s_W}\right]\left[g_{\alpha\beta}(p_- - p_+)_\mu \ + \ \left(1 + \frac{\tilde{\alpha}}{\xi_W}\right)(k_\alpha g_{\mu\beta} - k_\beta g_{\mu\alpha}) + \left(1 - \frac{\tilde{\alpha}}{\xi_W}\right)(g_{\mu\alpha}p_{+\beta} - g_{\mu\beta}p_{-\alpha}); \tilde{\alpha} \to \tilde{\beta}\right]$$

The form of this vertex calls for some comments. First, when $\tilde{\alpha}$ and $\tilde{\beta}$ are equal the vertices have, apart from an overall constant, the same Lorentz structure. The first term, that does not depend on any of the gauge-fixing parameters, corresponds to the convection current. This is the same current that one obtains for scalars and indeed, apart from the $g_{\alpha\beta}$ term that counts the vector degrees of freedom, this is exactly as in scalar electrodynamics. When we further take the most "practical values" $\tilde{\alpha} = \tilde{\beta} = 1$ (that correspond to taking a covariant derivative along the T_3 direction) and with $\xi_W = 1$ the third term vanishes and the second is nothing else but the spin current with the correct value for the magnetic moment of a spin-1 gauge particle.

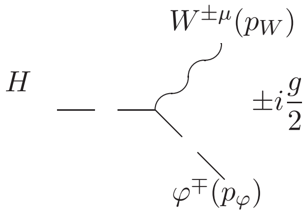


$$ig^{\mu\nu}[eM_W(1-\tilde{\alpha}); -gM_Z(1-c_W^2(1-\tilde{\beta}))]$$

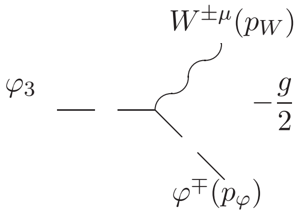
One can make both the $W^\pm\varphi^\mp Z$ and $W^\pm\varphi^\mp\gamma$ vanish. While the vanishing of the photon part is for the value $\alpha = 1$ that makes the $WW\gamma$ (and as we will see the the $WW\gamma\gamma$) simple, the vanishing of the $W^\pm\varphi^\mp Z$ requires $\tilde{\beta} = -s_W^2/c_W^2$ that does not make the other vertices simpler. Note that these vertices do not depend on ξ_W. The remaining tri-linear vertices that we list below can not be made to vanish.



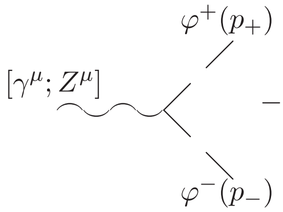
$$ig^{\rho\sigma}[gM_W; g_Z M_Z]$$



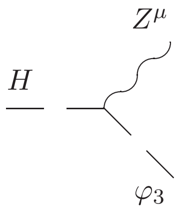
$$\pm i\frac{g}{2}((1-\tilde{\delta})p_\varphi - (1+\tilde{\delta})p_H)^\mu$$



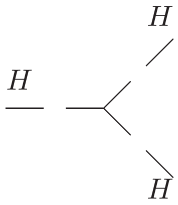
$$-\frac{g}{2}((1-\tilde{\kappa})p_\varphi - (1+\tilde{\kappa})p_{\varphi_3})^\mu$$



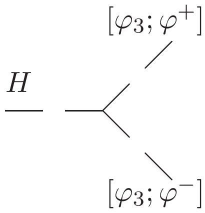
$$-i\left[e; g_Z\frac{c_W^2 - s_W^2}{2}\right](p_+ - p_-)^\mu$$



$$\frac{g_Z}{2}((1+\tilde{\varepsilon})p_H - (1-\tilde{\varepsilon})p_{\varphi_3})^\mu$$

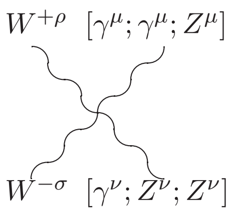


$$-\frac{3igM_H^2}{2M_W}$$



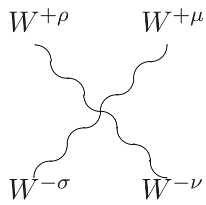
$$\left[-\frac{ig}{2M_W}(M_H^2 + 2\xi_W\tilde{\delta}M_W^2); (g \to g_Z, M_W \to M_Z, \xi_W \to \xi_Z, \tilde{\delta} \to \tilde{\varepsilon}\right]$$

Quartic vertices

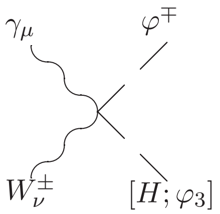


$$-ie^2[1; c_W/s_W; c_W^2/s_W^2](2g^{\mu\nu}g^{\rho\sigma}$$
$$-(g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma})[(1-\tilde{\alpha}^2/\xi_W); (1-\tilde{\alpha}\tilde{\beta}/\xi_W); (1-\tilde{\beta}^2/\xi_W)])$$

Again for the values that correspond to the T_3 covariant derivative and $\xi_W = 1$ there only remains the same part that one finds for the scalars (apart from the factor counting the vector degrees of freedom).

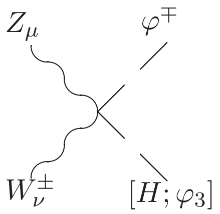


$$ig^2(2g^{\mu\rho}g^{\nu\sigma} - (g^{\mu\sigma}g^{\nu\rho} + g^{\mu\nu}g^{\rho\sigma}))$$



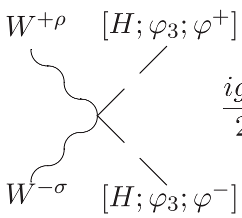
$$[i(1-\tilde{\alpha}\tilde{\delta}); \mp(1-\tilde{\alpha}\tilde{\kappa})]\frac{ge}{2}g_{\mu\nu}$$

Note that it is not sufficient to take $\tilde{\alpha} = 1$ to get rid of this vertex.

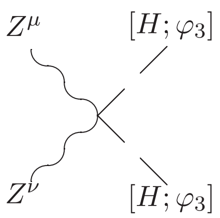


$$[-i\left(1-c_W^2(1-\tilde{\beta}\tilde{\delta})\right); \pm\left(1-c_W^2(1-\tilde{\beta}\tilde{\kappa})\right)]\frac{gg_Z}{2}g_{\mu\nu}$$

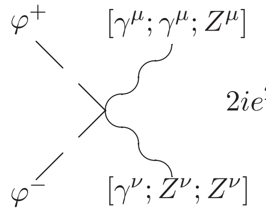
Note that if $\tilde{\delta} = \tilde{\kappa} = 1$ then the same condition that makes the $W^\pm\varphi^\mp Z$ vanish, eliminates this vertex too.



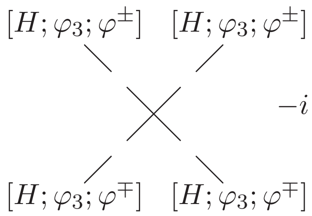
$$\frac{ig^2}{2}g^{\rho\sigma}$$



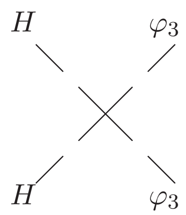
$$\frac{ig_Z^2}{2}g^{\mu\nu}$$



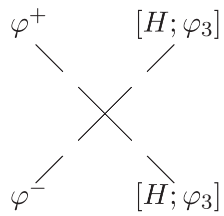
$$2ie^2 \left[1; \frac{c_W^2 - s_W^2}{2s_W c_W}; \left(\frac{c_W^2 - s_W^2}{2s_W c_W} \right)^2 \right] g^{\mu\nu}$$



$$-i\frac{g^2 M_H^2}{2M_W^2} \left[\frac{3}{2}; \frac{3}{2}; 1 \right]$$

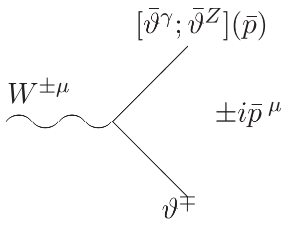


$$-\frac{ig^2}{4M_W^2}(M_H^2 + 2M_Z^2 \tilde{\varepsilon}^2 \xi_Z)$$

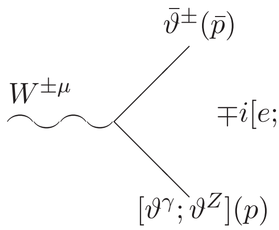


$$-\frac{ig^2}{4M_W^2}[M_H^2 + 2M_W^2 \tilde{\delta}^2 \xi_W; M_H^2 + 2M_W^2 \tilde{\kappa}^2 \xi_W]$$

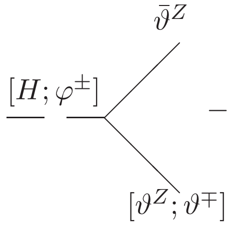
Ghosts vertices



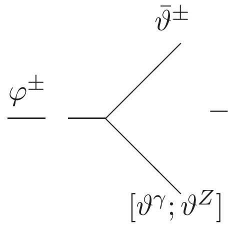
$$\pm i\bar{p}^{\,\mu}[e; gc_W]$$



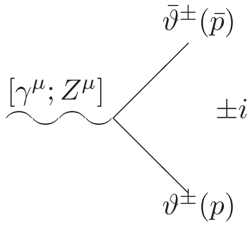
$$\mp i[e; gc_W](\bar{p} + [\tilde{\alpha}; \tilde{\beta}]p)^\mu$$



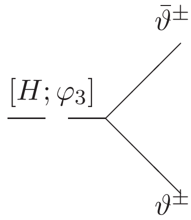
$$-\frac{im_Z \xi_Z}{2}[g_Z(1 + \tilde{\varepsilon}); -g]$$



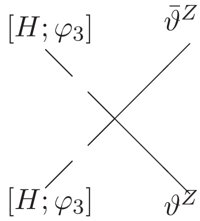
$$-iM_W \xi_W[e; \frac{g_Z}{2}(\tilde{\kappa} + c_W^2 - s_W^2)]$$



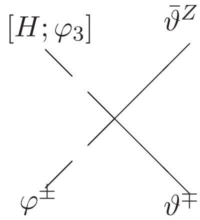
$$\pm i[e; gc_W](\bar{p} - [\tilde{\alpha}; \tilde{\beta}]p)^{\mu}$$



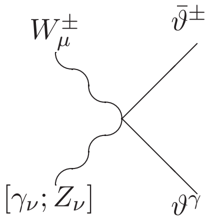
$$[-i(1+\tilde{\delta}); \pm(1-\tilde{\kappa})]\frac{gM_W\xi_W}{2}$$



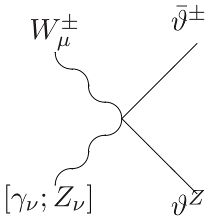
$$[-; +]i\xi_Z\tilde{\varepsilon}\frac{g_Z^2}{2}$$



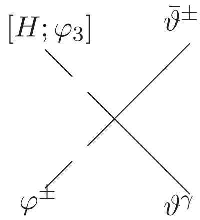
$$[i; \mp 1]\tilde{\varepsilon}\xi_Z\frac{gg_Z}{4}$$



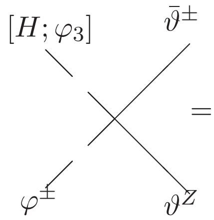
$$-ieg_{\mu\nu}[e\tilde{\alpha}; gc_W\tilde{\beta}]$$



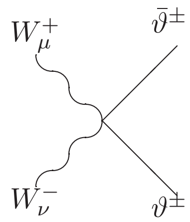
$$-\frac{ie^2c_W}{s_W}g_{\mu\nu}[\tilde{\alpha}; \tilde{\beta}\frac{c_W}{s_W}]$$



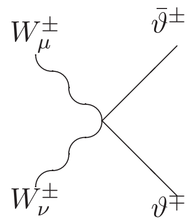
$$-\frac{e^2\xi_W}{2s_W}[i\tilde{\delta}; \pm\tilde{\kappa}]$$



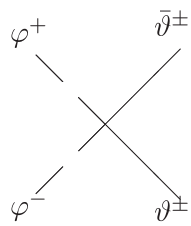
$$= -\frac{gg_Z\xi_W}{4}[i(\tilde{\kappa} + \tilde{\delta}(c_W^2 - s_W^2)); \pm(\tilde{\delta} + \tilde{\kappa}(c_W^2 - s_W^2))]$$



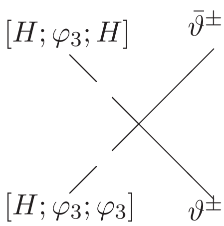
$$-i(e^2\tilde{\alpha} + g^2c_W^2\tilde{\beta})g_{\mu\nu}$$



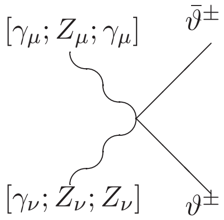
$$2i(e^2\tilde{\alpha} + g^2c_W^2\tilde{\beta})g_{\mu\nu}$$



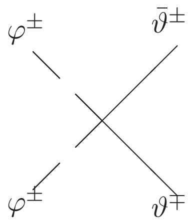
$$i\frac{g^2\xi_W}{4}(\tilde{\delta} + \tilde{\kappa})$$



$$-\frac{g^2\xi_W}{4}[2i\tilde{\delta}; 2i\tilde{\kappa}; \pm(\tilde{\kappa} - \tilde{\delta})]$$



$$i[2e^2\tilde{\alpha}; 2g^2c_W^2\tilde{\beta}; g^2c_Ws_W(\tilde{\alpha}+\tilde{\beta})]g_{\mu\nu}$$



$$i\frac{g^2\xi_W}{2}(\tilde{\kappa}-\tilde{\delta})$$

E Constructing the Ghost Lagrangian

To construct the ghost Lagrangian, we will require that the full effective Lagrangian, or rather the full action, be invariant under the BRS transformation (the measure being invariant). This implies that the full quantum Lagrangian

$$\mathcal{L}_Q = \mathcal{L}_{YM} + \mathcal{L}_{matter} + \mathcal{L}_{GF} + \mathcal{L}_{Gh} \tag{E.59}$$

be such that

$$\delta_{BRS}\,\mathcal{L} = \delta_{BRS}\,\mathcal{L}_{YM} + \delta_{BRS}\,\mathcal{L}_{matter} + \delta_{BRS}\,\mathcal{L}_{GF} + \delta_{BRS}\,\mathcal{L}_{Gh} \tag{E.60}$$

Since $\mathcal{L}_{YM}$ and $\mathcal{L}_{matter}$ are trivially invariant under the BRS: by construction they are invariant under a gauge transformation, the requirement of BRS invariance on the quantum Lagrangian implies that

$$\delta_{BRS}\,\mathcal{L}_{GF} = -\delta_{BRS}\,\mathcal{L}_{Gh} \tag{E.61}$$

The transformations on the gauge-fixing Lagrangian will trigger the compensating ghost terms.

In our case, denoting the gauge-fixing functions as $G^i \;\; i = \pm, \gamma, Z$, one has

$$\mathcal{L}_{GF} = G^-G^+ + \frac{1}{2}|G^Z|^2 + \frac{1}{2}|G_\gamma|^2 \tag{E.62}$$

In the original BRS implementation it was required that the anti-ghost be defined from

$$\delta_{BRS}\,\bar{\theta}^i = G^i \tag{E.63}$$

Hence using the nilpotency of the BRS variation $(\delta_{BRS}\,)^2\mathcal{F} = 0$ where $\mathcal{F}$ is any function of fields, one has that

$$\delta_{BRS}\,\mathcal{L}_{GF} = \delta_{BRS}\,\left(\bar{\theta}^+\delta_{BRS}\,G^+ + \bar{\theta}^-\delta_{BRS}\,G^- + \bar{\theta}^Z\delta_{BRS}\,G^Z + \bar{\theta}^\gamma\delta_{BRS}\,G^\gamma\right) \tag{E.64}$$

By identification we see that $\mathcal{L}_{Gh}$ must be of the form

$$\mathcal{L}_{Gh} = -\left(\bar{\theta}^{+}\delta_{BRS}\, G^{+} + \bar{\theta}^{-}\delta_{BRS}\, G^{-} + \bar{\theta}^{Z}\delta_{BRS}\, G^{Z} + \bar{\theta}^{\gamma}\delta_{BRS}\, G^{\gamma}\right) + \delta_{BRS}\, \tilde{\mathcal{L}}_{Gh} \tag{E.65}$$

that is, one recovers the Fadeev-Popov prescription, but only up to an overall function, $\delta_{BRS}\, \tilde{\mathcal{L}}_{Gh}$, which is BRS invariant:

$$\mathcal{L}_{FP} = -\bar{\theta}^{i}\delta_{BRS}\, G^{i} \equiv -\bar{\theta}^{i}\, \frac{\partial_{\omega} G^{i}}{\partial \omega^{j}}\, \theta^{j} \tag{E.66}$$

(G^i does not depend on the θ's and $\partial\omega \equiv \delta_{BRS}$) This shows that the BRS requirement gives a more general Lagrangian than the FP construction, which in fact does not work for NL gauge-fixings.

E.1 Auxilliary Fields

$\mathcal{L}_{GF}$ may be written in a more efficient way by introducing the auxilliary fields B^i

$$\mathcal{L}_{GF} = \xi_W B^{+} B^{-} + \frac{\xi_Z}{2}|B^Z|^2 + \frac{\xi_\gamma}{2}|B^\gamma|^2 + B^{-}G^{+} + B^{+}G^{-} + B^{\gamma}G^{\gamma} \tag{E.67}$$

Note that a priori one can define the parameter ξ in the GF: $G^{+} = \partial^\mu W^{+}_\mu + i\xi M_W \phi^{+} + ...$ that may be different from ξ_W. From the equations of motion for the B-fields one recovers the usual $\mathcal{L}_{GF}$. With $\xi_W = \xi_Z = \xi_\gamma = \xi$, one gets

$$B^i = -\frac{G^i}{\xi} \qquad \mathcal{L}_{GF} = -\frac{1}{2\xi}|G^i|^2 \tag{E.68}$$

Of course, $B^i = \delta_{BRS}\, \bar{\theta}^i$.

The use of the auxilliary B fields allows to discuss the theory (transformations) independently of the gauge parameters.

F Specific form of the BRS transformations

The action of the BRS transformations is derived as a generalisation of the usual gauge transformations to which we now turn.

$$U = exp\left(-ig\frac{\vec{\tau}.\vec{\omega}}{2} - ig'\beta\frac{Y}{2}\right) \tag{F.69}$$

whose infinitesimal form leads to

$$\begin{aligned} \delta W^i_\mu &= \partial_\mu \omega^i \,+\, g\, \epsilon_{ijk}\, \omega^j W^k_\mu \\ \delta W^i_{\mu\nu} &= g\, \epsilon_{ijk}\, \omega^j W^k_{\mu\nu} \\ \delta B_\mu &= \partial_\mu \beta \end{aligned} \tag{F.70}$$

Defining

$$\begin{aligned} \omega^{\pm} &= \frac{1}{\sqrt{2}}(\omega^1 \mp i\omega^2) \\ \omega^{\gamma} &= s_W \omega^3 + c_W \beta \\ \omega^{Z} &= c_W \omega^3 - s_W \beta \end{aligned} \tag{F.71}$$

one obtains

$$\begin{aligned} \delta W_\mu^{\pm} &= \partial_\mu \omega^{\pm} \pm\ ie \left[\left((A_\mu + \frac{c_W}{s_W} Z_\mu \right) \omega^{\pm} - \left(\omega_\gamma + \frac{c_W}{s_W}\omega_Z \right) W^{\pm} \right] \\ \delta Z_\mu^{\pm} &= \partial_\mu \omega^Z + igc_W \left(W_\mu^+ \omega^- - W_\mu^- \omega^+ \right) \\ \delta A_\mu^{\pm} &= \partial_\mu \omega^\gamma + ie \left(W_\mu^+ \omega^- - W_\mu^- \omega^+ \right) \end{aligned} \tag{F.72}$$

Likewise by considering the gauge transformation on the Higgs doublet one gets

$$\begin{aligned} \delta H &= -\frac{ig}{2} \left(\omega^- \phi^+ - \omega^+ \phi^- \right) - \frac{e}{2 s_W c_W} \omega^Z \phi^3 \\ \delta \phi^3 &= -\frac{g}{2} \left(\omega^+ \phi^- + \omega^- \phi^+ \right) + \frac{e}{2 s_W c_W} \omega^Z (v + H) \\ \delta \phi^{\pm} &= \mp \left\{ \frac{g}{2} \left(v + H \pm i\phi^3 \right) \omega^{\pm} + e\phi^{\pm} \left(\omega^\gamma + \frac{c_W^2 - s_W^2}{2 s_W c_W} \omega^Z \right) \right\} \end{aligned} \tag{F.73}$$

It is then straightforward to get the BRS transformations, by identifying the parameters ω to the θ's and δ to δ_{BRS} .

To find the transformation for the ghost fields, notice that the BRS transformation is nilpotent. For instance from $(\delta_{BRS}\ )^2 W_\mu^i = 0$ one gets $\delta_{BRS}\ \theta$. Indeed more generally one has, for any group,

$$\delta_{BRS}\ A_\mu^i = D_\mu \theta^i = \partial_\mu \theta^i + [A_\mu, \theta]^i \to \delta_{BRS}\ \theta^i = -\frac{1}{2}[\theta, \theta]^i \tag{F.74}$$

¶

In our case this implies

$$\delta_{BRS}\ \theta_B = 0 \qquad \delta_{BRS}\ \theta^i = +\frac{1}{2} g\ \epsilon_{ijk}\ \theta^j \theta^k \tag{F.75}$$

and thus

$$\begin{aligned} \delta_{BRS}\ \theta^{\pm} &= \pm i\ g\ \theta^{\pm}(s_W \theta^\gamma + c_W \theta^Z) \\ \delta_{BRS}\ \theta^{\gamma} &= -i\ e\ \theta^+ \theta^- \\ \delta_{BRS}\ \theta^{Z} &= -i\ g\ c_W\ \theta^+ \theta^- \end{aligned} \tag{F.76}$$

¶care should be taken that δ_{BRS} being a fermion operator the graded Leibniz rule applies: $\delta_{BRS}\ (XY) = (\delta_{BRS}\ X)Y \pm X(\delta_{BRS}\ Y)$ where the minus sign applies if X has an odd number of ghosts or antighots, note also that $(\theta^i)^2 = 0$

F.1 Anti-Ghosts

One could have also introduced anti-ghosts through an anti-BRS transformation $\bar{\delta}_{BRS}$, with the identification $\bar{\delta}_{BRS} \rightarrow \bar{\theta}$, *i.e*, $\bar{\delta}_{BRS}\ (A_\mu) = \delta_{BRS}\ (A_\mu(\theta \rightarrow \bar{\theta})$, but then what is $\delta_{BRS}\ (\bar{\theta})$ or for that matter $\bar{\delta}_{BRS}\ (\theta)$? We therefore postulates an auxilliary set of fields, B^i, such that

$$\delta_{BRS}\ (\bar{\theta}^i) = B^i \tag{F.77}$$

In our case one has the identification $\delta_{BRS}\ (\bar{\theta}^{\pm,\gamma,Z}) = B^{\mp,\gamma,Z}$. $Q_{em} = \theta^- = Q_{em}\bar{\theta}^+ = -1$.

In the usual description one does not introduce the anti-BRS transformation, only the BRS with the anti-ghosts. Had we not introduced the auxilliary Bfields the BRS transformations of the anti-ghosts would have been complicated functions of the fields.

G The Generalised Ward Identities

The Ward identities are particularly easy to derive if one works with the B-fields if one considers BRS transformations on some specific Green's functions (vacuum expectation values of time ordered products). For two point function of any two fields A and B, we use the short-hand notation:

$$< A\ B > = < 0|\left(TA(x)B(y)\right)|0 > \tag{G.78}$$

For example take the generic Green's function $< \bar{\theta}^i\ B^j >$ which in fact is zero (it has a non vanishing ghost number). Subjecting it to a BRS transformation one gets:

$$\begin{aligned} \delta_{BRS}\ < \bar{\theta}^i\ B^j > \ &= \ < (\delta_{BRS}\ \bar{\theta}^i)\ B^j > - < \bar{\theta}^i\ (\delta_{BRS}\ B^j) > = < B^i\ B^j > = 0 \\ or \ &\ < G^i\ G^j > = 0 \end{aligned} \tag{G.79}$$

Where in the last part we have used the equation of motion for the B^i's. The above relation leads directly to a constraint on the (bare) two point functions of the gauge vector boson, the gauge-goldtone mixing and the goldstone two point functions in the linear gauge. In the non-linear case one sees that this constraint involves also three-point functions, whose contribution vanishes for on-shell (they do not have the pole structure contained in the two-point functions) .

One can also derive the constraint on the the ghost two-point function. In this case we start with a Green's function of a gauge-field and an anti-ghost, and then apply the same trick:

$$\begin{aligned} \delta_{BRS}\ < \bar{\theta}^i\ A^j_\mu > \ &= \ < (\delta_{BRS}\ \bar{\theta}^i)\ A^j > - < \bar{\theta}^i\ (\delta_{BRS}\ A^j_\mu) > \\ &= \ < B^i\ A^j_\mu > - < \bar{\theta}^i\ (\partial_\mu \theta^i) > -g\ \epsilon_{jkl}\ < \bar{\theta}^i(x)\ \left(\theta^k(y) A^l_\mu(y)\right) > = 0 \end{aligned} \tag{G.80}$$

likewise one can use the equation of motion for the auxilliary field B.

H Two-point functions

Formulas for one-point/two-point functions

$$\begin{aligned} n &= 4 - 2\epsilon \\ C_{UV} &= \frac{1}{\epsilon} - \gamma + \log 4\pi \end{aligned} \tag{H.81}$$

$$\begin{aligned} J(M^2) &\equiv \int \frac{d^n k}{i(2\pi)^n} \frac{1}{(k^2 - M^2)} \\ I_0(M_0^2, M_1^2) &\equiv \int \frac{d^n k}{i(2\pi)^n} \frac{1}{(k^2 - M_0^2)((k-q)^2 - M_1^2)} \\ I_1^{\mu\nu}(M_0^2, M_1^2) &\equiv \int \frac{d^n k}{i(2\pi)^n} \frac{k^\mu q^\nu}{(k^2 - M_0^2)((k-q)^2 - M_1^2)} = q^\mu q^\nu \tilde{I}_1(M_0^2, M_1^2) \\ I_2^{\mu\nu}(M_0^2, M_1^2) &\equiv \int \frac{d^n k}{i(2\pi)^n} \frac{k^\mu k^\nu}{(k^2 - M_0^2)((k-q)^2 - M_1^2)} \\ &\equiv g^{\mu\nu} \tilde{I}_2^T(M_0^2, M_1^2) + q^\mu q^\nu \tilde{I}_2^L(M_0^2, M_1^2) \end{aligned} \tag{H.82}$$

$$\begin{aligned} J(M^2) &= \frac{1}{(4\pi)^2} \left\{ M^2 (C_{UV} + 1 - \log M^2) \right\} \\ I_0(M_0^2, M_1^2) &= \frac{1}{(4\pi)^2} \left\{ C_{UV} - F_0(M_0, M_1) \right\} \\ \tilde{I}_1(M_0^2, M_1^2) &= \frac{1}{(4\pi)^2} \left\{ \frac{C_{UV}}{2} - F_1(M_0, M_1) \right\} \\ \tilde{I}_2^L(M_0^2, M_1^2) &= \frac{1}{(4\pi)^2} \left\{ \frac{C_{UV}}{3} - F_2(M_0, M_1) \right\} \\ \tilde{I}_2^T(M_0^2, M_1^2) &= \frac{1}{(4\pi)^2} \left\{ \frac{(C_{UV}+1)}{2} \left(\frac{M_0^2 + M_1^2}{2} - \frac{q^2}{6} \right) \right. \\ &\left. -\frac{1}{2} \left[M_0^2 F_0(M_0, M_1) + (M_1^2 - M_0^2) F_1(M_0, M_1) - q^2 F(M_0, M_1) \right] \right\} \end{aligned} \tag{H.83}$$

$$F_n(M_0, M_1) = \int_0^1 dx x^n \log\left[M_0^2 (1-x) + M_1^2 x - q^2 x (1-x) \right] \tag{H.84}$$

$$F \equiv F_1 - F_2 \tag{H.85}$$

Some useful formulae:

$$\begin{aligned} F_2(M, M) &= \frac{F_0(M, M)}{3} - \frac{M^2}{3q^2} (F_0(M, M) - \log M^2) - \frac{1}{18} \\ F_1(M, M) &= \frac{F_0(M, M)}{2} \end{aligned} \tag{H.86}$$

$$F_2(M_0,M_1) = \frac{1}{3}\Bigg\{2F_1 - \frac{1}{6} + \frac{2(M_0^2-M_1^2)}{q^2}F_1(M_0,M_1) - \frac{M_0^2}{q^2}F_0(M_0,M_1) + \frac{M_1^2}{q^2}\log M_1^2 + \frac{M_0^2-M_1^2}{2q^2}\Bigg\}$$

$$F_1(M_0,M_1) = \frac{F_0}{2} + \frac{M_0^2-M_1^2}{2q^2}F_0(M_0,M_1) + \frac{M_0^2}{2q^2}(1-\log M_0^2) - \frac{M_1^2}{2q^2}(1-\log M_1^2) \quad \text{(H.87)}$$

$$\begin{aligned} F_1(M_0,M_1) &= -F_1(M_1,M_0) + F_0(M_1,M_0) \\ F_2(M_0,M_1) &= F_2(M_1,M_0) - 2F_1(M_1,M_0) + F_0(M_1,M_0) \\ F(M_0,M_1) &= F(M_1,M_0) \end{aligned} \quad \text{(H.88)}$$

Self energies for neutral bosons

In the non-linear gauge only the boson contribution to the self energies is modified(at least at 1-loop), the fermion contribution will not be given here, it can be found in Aoki et al. In the limit $\tilde{\alpha}, \tilde{\beta}, \tilde{\epsilon} \to 0$, the formulas for the transverse part of the self-energy agree with Aoki et.al. For short we define

$$F(V_1,V_2) \equiv F(M_{V_1}, M_{V_2})$$

.

$$\Pi_{\mu\nu} = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Pi_T + q_\mu q_\nu \Pi_L \quad \text{(H.89)}$$

- $\gamma\gamma$

$$\Pi_T^{\gamma\gamma} = \frac{\alpha}{4\pi}q^2\left[C_{UV}(3+4\tilde{\alpha}) - F_0(W,W)(1+4\tilde{\alpha}) - 12F(W,W) + \frac{2}{3}\right] \quad \text{(H.90)}$$

or

$$\Pi_T^{\gamma\gamma} = \frac{\alpha}{4\pi}q^2\left[7C_{UV} - 5F_0(W,W) - 12F(W,W) + \frac{2}{3} - 4(1-\tilde{\alpha})(C_{UV} - F_0(W,W))\right] \quad \text{(H.91)}$$

$$\Pi_L^{\gamma\gamma} = 0 \quad \text{(H.92)}$$

- $\gamma - Z$

$$\begin{aligned} \Pi_T^{\gamma Z} = \frac{\alpha}{4\pi}\frac{c_w}{s_w} \quad & \left[C_{UV}\left(q^2(3+2\tilde{\alpha}+2\tilde{\beta}+\frac{1}{6c_w^2}) + 2(1-\tilde{\alpha})M_Z^2\right)\right. \\ + \quad & F_0(W,W)\left(q^2(-1-2\tilde{\alpha}-2\tilde{\beta}-\frac{1}{2c_w^2}) - 2(1-\tilde{\alpha})M_Z^2\right) \\ - \quad & \left. 4q^2(3-\frac{1}{2c_w^2})F(W,W) + \frac{2}{3}q^2\right] \end{aligned} \quad \text{(H.93)}$$

or

$$\Pi_T^{\gamma Z} = \frac{\alpha}{4\pi}\frac{c_w}{s_w} \quad \left[C_{UV}\left(7+\frac{1}{6c_w^2}\right)q^2 - 4q^2\left(3-\frac{1}{2c_w^2}\right)F(W,W) + \frac{2}{3}q^2 - q^2\left(1+\frac{1}{2c_w^2}\right)F_0(W,W)\right.$$
$$-2(1-\tilde{\alpha})(q^2-M_Z^2)(C_{UV}-F_0(W,W))$$
$$\left.-2(1-\tilde{\beta})q^2(C_{UV}-F_0(W,W))\right] \qquad \text{(H.94)}$$

$$\Pi_L^{\gamma Z} = \frac{\alpha}{4\pi}\frac{c_w}{s_w}2(1-\tilde{\alpha})\frac{M_W^2}{c_w^2}\left[C_{UV}-F_0(W,W)\right] \qquad \text{(H.95)}$$

As it should, $\Pi_L^{\gamma Z} = 0$ for $\alpha = 1$.

• $Z - Z$

$$\Pi_T^{ZZ} = \frac{\alpha}{4\pi s_w^2 c_w^2}\left(T_{ZZ} + (1-\tilde{\beta})\Delta T_{ZZ}\right) \qquad \text{(H.96)}$$

$$T^{ZZ} = \left\{C_{UV}\left[q^2\left(7c_w^4 - \frac{1-2c_w^2}{6}\right) - 2M_W^2 - M_Z^2\right] + \frac{2}{3}q^2c_w^4 - \frac{q^2}{12}\right.$$
$$-8q^2c_w^4F_0(W,W) + q^2(F_0(W,W) - 4F(W,W))\left(3c_w^4 + \frac{1-4c_w^2}{4}\right) + 2M_W^2F_0(W,W)$$
$$+ \frac{q^2}{2}F(H,Z) - \frac{M_H^2}{2}F_0(H,Z) - \frac{M_Z^2-M_H^2}{2}F_1(H,Z)$$
$$\left.+ M_Z^2F_0(H,Z) + \frac{1}{4}(M_H^2\log M_H^2 + M_Z^2\log M_Z^2)\right\} \qquad \text{(H.97)}$$

$$\Delta T^{ZZ} = -4c_w^4\left(q^2-M_Z^2\right)\left(C_{UV}-F_0(W,W)\right) \qquad \text{(H.98)}$$

We have,

$$\Pi^T_{NonLinear}(M_Z^2) = \Pi^T_{Linear}(M_Z^2)$$

$$\Pi_L^{ZZ} = \frac{\alpha}{16\pi s_w^2 c_w^2}\left[C_{UV}(-4M_Z^2) - \frac{q^2}{3} + 2q^2F(H,Z) + 4M_Z^2F_0(H,Z)\right.$$
$$- 2M_H^2F_0(H,Z) - 2(M_Z^2-M_H^2)F_1(H,Z) + (M_H^2\log M_H^2 + M_Z^2\log M_Z^2)$$
$$+ q^2\left[C_{UV}\tilde{\epsilon}^2 - (1-\tilde{\epsilon})^2F_0(H,Z) - 4F_2(H,Z) + 4(1-\tilde{\epsilon})F_1(H,Z)\right]$$
$$\left.- 8M_W^2\left(C_{UV}-F_0(W,W)\right)\left(1-2c_w^2(1-\tilde{\beta})\right)\right] \qquad \text{(H.99)}$$

H.1 Self energy- W boson

Note: Here the transverse part contains the term in $g^{\mu\nu}$ only and the longitudinal part the term in $q^\mu q^\nu$.

$$\Pi^{WW} = g^{\mu\nu}\Pi_T^{WW} + q^\mu q^\nu \Pi_L^{WW}$$

$$\begin{aligned}
\Pi_T^{WW} = \frac{\alpha}{4\pi s_W} \quad & \Bigg\{ C_{UV}\left(\frac{19}{6}q^2 + 2M_W^2 - M_Z^2\right) - \frac{q^2}{6} \\
& +4s_W\left[q^2(F(AW) - F_0(A,W)) - M_W^2 F_1(AW)\right] \\
& +c_W\Bigg[4q^2(F(ZW) - F_0(ZW)) - 4(M_W^2 - M_Z^2)F_1(ZW) \\
& +\left(-7M_Z^2 + \frac{M_Z^2}{c_W}\right)F_0(ZW)\Bigg] + \frac{q^2}{2}\left(F(HW) + F(ZW)\right) + M_W^2 F_0(H,W) \\
& -\frac{1}{2}\left[M_H^2 F_0(HW) + M_Z^2 F_0(ZW) + (M_W^2 - M_H^2)F_1(HW) + (M_W^2 - M_Z^2)F_1(ZW)\right] \\
& +\frac{5M_W^2}{2}\log M_W^2 + \frac{M_Z^2}{4}\log M_Z^2 + \frac{M_H^2}{4}\log M_H^2 + 2M_W^2\log M_Z^2 \\
& -(q^2 - M_W^2)\left(2s_W\tilde{\alpha}(C_{UV} - F_0(AW)) + 2c_W\tilde{\beta}(C_{UV} - F_0(ZW))\right)\Bigg\}
\end{aligned} \tag{H.100}$$

As for the Z,

$$\Pi^T_{NonLinear}(M_W^2) = \Pi^T_{Linear}(M_W^2)$$

$$\begin{aligned}
\Pi_L^{WW} = \frac{\alpha}{4\pi} \quad & \Bigg\{ \frac{1}{s_W}\left(\frac{13}{3}C_{UV} - \frac{8}{3}\right) - 2F_1(AW) - 8F_2(AW) \\
& \frac{c_W}{s_W}(-2F_1(ZW) - 8F_2(Z,W)) + \frac{1}{s_W}(-F_2(HW) - F_2(ZW)) \\
& +(1-\tilde{\alpha})(-12C_{UV} + 8F_0(AW) + 8F_1(AW) + 4) \\
& +(1-\tilde{\alpha})^2(5C_{UV} - 6F_0(AW) + 2F_1(AW) - 2) \\
& +(1-\tilde{\beta})\frac{c_W}{s_W}(-12C_{UV} + 8F_0(ZW) + 8F_1(ZW) + 4) \\
& +(1-\tilde{\beta})^2\frac{c_W}{s_W}(5C_{UV} - 6F_0(ZW) + 2F_1(ZW) - 2) \\
& \frac{1}{4s_W}\Bigg[(1-\tilde{\delta})(-2C_{UV} + 4F_1(HW)) + (1-\tilde{\kappa})(-2C_{UV} + 4F_1(ZW)) \\
& +(1-\tilde{\delta})^2(C_{UV} - F_0(HW)) + (1-\tilde{\kappa})^2(C_{UV} - F_0(ZW))\Bigg]\Bigg\}
\end{aligned} \tag{H.101}$$

H.2 Self energy Higgs boson

The tadpole is ignored as one can always add a counterterm to cancel it.

$$\Pi_T^H = \frac{\alpha}{4\pi s_W} \quad \Bigg\{ C_{UV} \left[\left(-q^2 + \frac{M_H^2}{4} \right) \left(1 + \frac{1}{2c_W} \right) + \frac{9M_W^2}{2} + \frac{9M_Z^2}{4c_W} + \frac{15M_H^4}{8M_W^2} \right]$$

$$-F_0(W,W) \left(-q^2 + 3M_W^2 + \frac{M_H^4}{4M_W^2} \right) - \frac{F_0(Z,Z)}{2c_W} \left(-q^2 + 3M_Z^2 + \frac{M_H^4}{4M_Z^2} \right)$$

$$-\frac{9M_H^4}{8M_W^2} F_0(H,H) - 3\frac{M_W^2}{2}(1 + \log M_W^2) - 3\frac{M_Z^2}{4c_W}(1 + \log M_Z^2) \tag{H.102}$$

$$+\frac{M_H^2}{4} \left[1 - \log M_W^2 + \frac{1}{2c_W}(1 - \log M_Z^2) + \frac{3M_H^2}{2M_W^2}(1 - \log M_H^2) \right]$$

$$+(q^2 - M_H^2) \left[(-C_{UV} + F_0(W,W))\, \tilde{\delta} + (-C_{UV} + F_0(Z,Z))\, \frac{\tilde{\epsilon}}{2c_W} \right] \Bigg\}$$

$$\Pi_T^H(fermions) = \frac{\alpha}{4\pi s_W} \sum_f \frac{-2m_f^2}{M_W} N_c^f \Bigg\{ C_{UV}(6m_f^2 - q^2)$$

$$+2m_f^2 - 2m_f^2 \log m_f^2 - F_0(f,f)(4m_f^2 - q^2) \Bigg\} \tag{H.103}$$

Again at $q^2 = M^2$ the transverse self-energy is independent of the gauge parameter

$$\Pi^T_{NonLinear}(M_H^2) = \Pi^T_{Linear}(M_H^2)$$

More details on the W self energy: The result is given explicitly for the diagrams including photon,Z and only scalars

$$\Pi^{WW} = \Pi^A + \Pi^Z + \Pi^S$$

Note that the diagram with only W's is included in the scalar contribution.

Contribution from individual diagrams (photons)

$$\begin{aligned}
\Pi_{WA} \;=\; & g^{\alpha\beta}\Bigg\{ C_{UV}\left(\frac{19}{6}q^2 + \frac{9}{2}M_W^2\right) + q^2\left[-4F_0(W,A) - \frac{1}{6} + 5F(W,A)\right] \\
& + M_W^2(1 - \log M_W^2 - F_0(W,A) + \frac{1}{2} - 5M_W^2 F_1(W,A)) \\
& - \tilde{\alpha}\Bigg[C_{UV}\left(\frac{11}{6}q^2 - \frac{3M_W^2}{2}\right) - \frac{q^2}{6} + q^2(F(W,A) - 2F_0(W,A)) + \\
& M_W^2(\frac{1}{2} - F_1(W,A) - 2(1 - \log M_W^2))\Bigg]\Bigg\} \\
& + q^\alpha q^\beta\Bigg[8\frac{C_{UV} - 1}{3} - 8F_2(W,A) + (1 - \tilde{\alpha})(-12C_{UV} + 8F_0(AW) + 8F_1(AW) + 4) \\
& + 2(1 - \tilde{\alpha})\left(\frac{C_{UV}}{3} - F_2(W,A)\right) + (1 - \tilde{\alpha})^2(5C_{UV} - 6F_0(W,A) + 2F_1(W,A) + 2)\Bigg]
\end{aligned} \tag{H.104}$$

$$\begin{aligned}
\Pi_{CC} \;=\; & q^2 g^{\alpha\beta}(1 - \tilde{\alpha})\left(\frac{C_{UV} + 1}{6} - F(AW)\right) \\
& + g^{\alpha\beta}(1 - \tilde{\alpha})^2\left(-\frac{C_{UV} + 1}{2} + F_1(AW)\right) \\
& + q^\alpha q^\beta\left[C_{UV} - 2F_1(A,W) - 2(1 - \tilde{\alpha})\left(\frac{C_{UV}}{3} - F_2(A,W)\right)\right]
\end{aligned} \tag{H.105}$$

$$\Pi_{WG} = g^{\alpha\beta}M_W^2(1 - \tilde{\alpha})^2\left(-C_{UV} + F_0(AW)\right) \tag{H.106}$$

$$\Pi_C = g^{\alpha\beta}(-2M_W^2\tilde{\alpha})\left(C_{UV} + 1 - \log M_W^2\right) \tag{H.107}$$

Total Contribution from diagrams with photons:

$$\begin{aligned}
\Pi_L^A = \frac{\alpha}{4\pi} \; & \Bigg\{\frac{11}{3}C_{UV} - \frac{8}{3} - 2F_1(A,W) - 8F_2(A,W) \\
& + (1 - \tilde{\alpha})(-16C_{UV} + 8F_0(A,W) + 16F_1(A,W) + 4) \\
& + (1 - \tilde{\alpha})^2(5C_{UV} - 6F_0(A,W) + 2F_1(A,W) - 2)\Bigg\}
\end{aligned} \tag{H.108}$$

$$\Pi_T^A = \frac{\alpha}{4\pi} \left\{ C_{UV}\left(\frac{10}{3}q^2 + 3M_W^2\right) + 4q^2(F(A,W) - F_0(A,W)) \right.$$
$$+M_W^2(-4F_1(A,W) + 1 - \log M_W^2)$$
$$\left. +2\tilde{\alpha}\left(-C_{UV} + F_0(A,W)\right)(q^2 - M_W^2) \right\} \quad \text{(H.109)}$$

or pulling out the term in $1 - \tilde{\alpha}$

$$\Pi_T^A = \frac{\alpha}{4\pi} \left\{ C_{UV}\left(\frac{4}{3}q^2 + 5M_W^2\right) + q^2(4F(A,W) - 2F_0(A,W)) \right.$$
$$+M_W^2(-4F_1(A,W) - 2F_0(A,W) + 1 - \log M_W^2)$$
$$\left. +2(1-\tilde{\alpha})(C_{UV} - F_0(A,W))(q^2 - M_W^2) \right\} \quad \text{(H.110)}$$

Contribution from diagrams with Z:

$$\Pi_L^Z = \frac{\alpha c_W}{4\pi s_W} \left\{ \frac{11}{3}C_{UV} - \frac{8}{3} - 2F_1(Z,W) - 8F_2(Z,W) \right.$$
$$+(1-\tilde{\beta})(-16C_{UV} + 8F_0(Z,W) + 16F_1(Z,W) + 4)$$
$$\left. +(1-\tilde{\beta})^2(5C_{UV} - 6F_0(Z,W) + 2F_1(Z,W) - 2) \right\} \quad \text{(H.111)}$$

$$\Pi_T^Z = \frac{\alpha c_W}{4\pi s_W} \left\{ C_{UV}\left(\frac{4}{3}q^2 + 5M_W^2 + 3M_Z^2 - \frac{M_Z^2}{c_W}\right) \right.$$
$$-F_0(Z,W)\left(2q^2 + 2M_W^2 + 3M_Z^2 - \frac{M_Z^2}{c_W}\right)$$
$$+4q^2F(Z,W) - 4(M_W^2 - M_Z^2)F_1(Z,W)$$
$$-4M_Z^2F_0(Z,W) + M_W^2(1 - \log M_W^2) + 2M_Z^2\log M_Z^2$$
$$\left. +2(1-\tilde{\beta})(C_{UV} - F_0(Z,W))(q^2 - M_W^2) \right\} \quad \text{(H.112)}$$

$$\Pi_T^Z = \frac{\alpha c_W}{4\pi s_W} \left\{ C_{UV}\left(\frac{4}{3}q^2 + 5M_W^2 + 3M_Z^2 - \frac{M_Z^2}{c_W}\right) \right.$$
$$+q^2(4F(Z,W) - 2F_0(Z,W)) - F_0(Z,W)\left(2M_W^2 + 7M_Z^2 - \frac{M_Z^2}{c_W}\right)$$
$$-4(M_W^2 - M_Z^2)F_1(Z,W) + M_W^2(1 - \log M_W^2) + 2M_Z^2\log M_Z^2$$
$$\left. +2(1-\tilde{\beta})(C_{UV} - F_0(Z,W))(q^2 - M_W^2) \right\} \quad \text{(H.113)}$$

Diagrams with scalars

$$\pi_T^S = \frac{\alpha}{4\pi s_W} \left\{ C_{UV}(-\frac{q^2}{6} - 4M_W^2) - \frac{q^2}{6} - M_W^2 + M_W^2 F_0(W,H) \right.$$
$$-\frac{1}{2}\left(M_H^2 F_0(H,W) + (M_W^2 - M_H^2)F_1(HW) - q^2 F(HW)\right)$$
$$-\frac{1}{2}\left(M_Z^2 F_0(Z,W) + (M_W^2 - M_Z^2)F_1(ZW) - q^2 F(ZW)\right)$$
$$\left. +\frac{M_Z^2}{4}\log M_Z^2 + \frac{M_H^2}{4}\log M_H^2 + \frac{7M_W^2}{2}\log M_W^2 \right\} \tag{H.114}$$

$$\pi_L^S = \frac{\alpha}{4\pi s_W} \left\{ C_{UV}(\tilde{\delta}^2 + \frac{4}{3} - 2\tilde{\delta}) - \tilde{\delta}^2 F_0(HW) - 4F_2(HW) + 4\tilde{\delta}F_1(HW) \right.$$
$$\left. +C_{UV}(\tilde{\kappa}^2 + \frac{4}{3} - 2\tilde{\kappa}) - \tilde{\kappa}^2 F_0(ZW) - 4F_2(ZW) + 4\tilde{\kappa}F_1(ZW) \right\} \tag{H.115}$$

$$\Pi_T^{WW} = \frac{\alpha}{4\pi s_W} \left\{ C_{UV}\left(\frac{7}{6}q^2 + 7M_W^2 + 3M_Z^2 c_W - M_Z^2\right) - \frac{q^2}{6} \right.$$
$$+s_W\left[q^2(4F(AW) - 2F_0(A,W)) - M_W^2(4F_1(AW) + 2F_0(AW))\right]$$
$$+c_W\left[q^2(4F(ZW) - 2F_0(ZW)) - (M_W^2 - M_Z^2)(4F_1(ZW) + 2F_0(ZW))\right.$$
$$\left.+\left(-9M_Z^2 + \frac{M_Z^2}{c_W}\right)F_0(ZW)\right] + \frac{q^2}{2}\left(F(HW) + F(ZW)\right)$$
$$-\frac{1}{2}\left[M_H^2 F_0(HW) + M_Z^2 F_0(ZW) + (M_W^2 - M_H^2)F_1(HW) + (M_W^2 - M_Z^2)F_1(ZW)\right]$$
$$+\frac{5M_W^2}{2}\log M_W^2 + \frac{M_Z^2}{4}\log M_Z^2 + \frac{M_H^2}{4}\log M_H^2 + 2M_W^2 \log M_Z^2$$
$$+2s_W(1-\tilde{\alpha})(q^2 - M_W^2)(C_{UV} - F_0(AW))$$
$$\left. +2c_W(1-\tilde{\beta})(q^2 - M_W^2)(C_{UV} - F_0(ZW)) \right\} \tag{H.116}$$

In the case $\tilde{\alpha} = 0$, we get

$$\Pi_T^{\gamma\gamma} = \frac{\alpha}{4\pi}q^2\left[3C_{UV} - F_0(W,W) - 12F(W,W) + \frac{2}{3}\right] \tag{H.117}$$

which agrees with 5.21.

$$\Pi_T^{\gamma Z} = \frac{\alpha}{4\pi}\frac{c_w}{s_w} \Bigg[C_{UV}\left(q^2(3+\frac{1}{6c_w^2})+2M_Z^2\right) - 4q^2(3-\frac{1}{2c_w^2})F(W,W)+\frac{2}{3}q^2$$
$$- \quad F_0(W,W)\left(q^2(1+\frac{1}{2c_w^2})+2M_Z^2\right)$$
$$+2\tilde{\alpha}(q^2-M_Z^2)(C_{UV}-F_0(W,W))+2\tilde{\beta}q^2(C_{UV}-F_0(W,W))\Bigg] \quad \text{(H.118)}$$

Two-point functions Vector- χ

$$\begin{aligned}
\Pi_{W\chi_+} &= iq^\alpha\frac{\alpha M_W}{16\pi s_W^2}\Bigg\{C_{UV}\left(2-\frac{3}{c_W^2}\right)\\
&+ C_{UV}\left(\frac{M_H^2}{M_W^2}\tilde{\delta}+\tilde{\kappa}+\tilde{\delta}+2\tilde{\delta}^2+s_W^2(18\tilde{\alpha}^2-12\tilde{\alpha})+4\tilde{\beta}(4-3c_W^2)+18c_W^2\tilde{\beta}^2\right)\\
&+ 4s_W^2\left(4F_1(A,W)-F_0(A,W)\right)\\
&+ 4s_W^2\tilde{\alpha}\ \left(-6F_1(A,W)+6F_0(A,W)-5\tilde{\alpha}F_0(A,W)+\tilde{\alpha}F_1(A,W)\right)\\
&+ F_1(Z,W)\left(s_W^2(-16-\frac{2}{c_W^2})-2\tilde{\kappa}-24c_W^2\tilde{\beta}+4c_W^2\tilde{\beta}^2\right)\\
&+ F_0(Z,W)\left(2-4c_W^2+4\frac{s_W^2}{c_W^2}-16\tilde{\beta}+24c_W^2\tilde{\beta}-20c_W^2\tilde{\beta}^2\right)\\
&+ \frac{M_H^2}{M_W^2}\left(2F_1(W,H)-F_0(W,H)(1+\tilde{\delta})\right)+2\left(2F_0(W,H)-F_1(W,H)\right)\\
&+ 2\tilde{\delta}\left(F_1(W,H)-F_0(W,H)(1+\tilde{\delta})\right)\\
&+ 8\tilde{\alpha}s_W^2(1-\tilde{\alpha})-8\tilde{\beta}\left(1-c_W^2(1-\tilde{\beta})\right)\Bigg\} \qquad \text{(H.119)}
\end{aligned}$$

$$\begin{aligned}
\Pi_{Z\chi_3} &= iq^\alpha\frac{\alpha M_Z}{8\pi s_W^2c_W^2}\Bigg\{c_W^2(-3+4c_W^2(1-\tilde{\beta})-\tilde{\kappa})\left(C_{UV}-F_0(W,W)\right)\\
&+ \frac{M_H^2}{2M_Z^2}\left[F_0(H,Z)-2F_1(H,Z)\right]-\left[\frac{3C_{UV}}{2}-F_1(H,Z)-F_0(H,Z)\right]\\
&+ \tilde{\epsilon}\left[\frac{C_{UV}}{2}-F_1(H,Z)+\frac{M_H^2}{2M_Z^2}(C_{UV}-F_0(H,Z)\right]\\
&+ \tilde{\epsilon}^2\left(C_{UV}-F_0(H,Z)\right)\Bigg\} \qquad \text{(H.120)}
\end{aligned}$$

$$\Pi_{A\chi_3} = -iq^\alpha\frac{\alpha M_W}{2\pi s_W}(1-\tilde{\alpha})\left(C_{UV}-F_0(W,W)\right) \qquad \text{(H.121)}$$

Two-point functions $\chi - \chi$

$$
\begin{aligned}
\Pi_{\chi_3\chi_3} &= \frac{\alpha}{16\pi s_W^2}\Bigg\{C_{UV}\left[\frac{3M_H^4}{2M_W^2} + M_H^2(1+\frac{1}{2c_W^2}) + 6M_W^2 + \frac{3M_Z^2}{c_W^2} - 2q^2(2+\frac{1}{c_W^2})\right] \\
&+ 4q^2F_0(W,W) + F_0(H,Z)\left(\frac{2q^2}{c_W^2} + \frac{2M_H^2}{c_W^2} - \frac{M_Z^2}{c_W^2} - \frac{M_H^4}{M_W^2}\right) \\
&+ (6M_W^2 + M_H^2)(1-\log M_W^2) + (\frac{2M_Z^2}{c_W^2} + \frac{3M_H^2}{2c_W^2})(1-\log M_Z^2) \\
&+ M_H^2(\frac{M_H^2}{2M_W^2} + \frac{1}{c_W^2})(1-\log M_H^2) - 4M_W^2 - 2\frac{M_Z^2}{c_W^2} - 4\tilde{\kappa}q^2(C_{UV} - F_0(W,W)) \\
&+ \frac{\tilde{\epsilon}}{c_W^2}(C_{UV} - F_0(H,Z))\left[2(q^2+M_H^2)\ +\ 3M_Z^2\tilde{\epsilon}\right]\Bigg\} \qquad \text{(H.122)}
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{\chi_+\chi_+} &= \frac{\alpha}{16\pi s_W^2}\Bigg\{C_{UV}\left[\frac{3M_H^4}{2M_W^2} + M_H^2(1+\frac{1}{2c_W^2}) + 6M_W^2 + \frac{3M_Z^2}{c_W^2} - 2q^2(2+\frac{1}{c_W^2})\right] \\
&+ 2\tilde{\delta}(q^2+M_H^2) + 2\tilde{\kappa}q^2 + (3\tilde{\delta}^2 - \tilde{\kappa}^2)M_W^2 \\
&+ 32M_W^2 s_W^2(\tilde{\beta} - \tilde{\alpha}) + 16M_W^2(s_W^2\tilde{\alpha}^2 + c_W^2\tilde{\beta}^2)\Big] \\
&+ 8s_W^2F_0(A,W)(q^2 + M_W^2 - 2M_W^2(1-\tilde{\alpha})^2) \\
&+ F_0(H,W)\left[2(q^2+M_H^2)(1-\tilde{\delta}) - M_W^2(1+3\tilde{\delta}^2) - \frac{M_H^4}{M_W^2}\right] \\
&+ F_0(Z,W)\Bigg[q^2\left(\frac{2}{c_W^2} - 8s_W^2 + 2(1-\tilde{\kappa})\right) + M_Z^2\left(-8 - 8c_W^2 + 2 - \frac{1}{c_W^2}\right) \\
&+ M_W^2(-1 + \tilde{\kappa}^2 - 8s_W^2 + 32(1-\tilde{\beta}) - 16c_W^2(1-\tilde{\beta})^2)\Bigg] \\
&- M_W^2\log M_W^2(4 + \frac{1}{c_W^2}) + 2M_H^2(1-\log M_W^2) + \frac{M_W^2}{c_W^2} - 8M_W^2 s_W^2(1-\tilde{\alpha})^2 \\
&+ M_H^2(1-\log M_H^2)\left(1 + \frac{M_H^2}{2M_W^2}\right) + \frac{M_H^2}{2c_W^2} \\
&- \log M_Z^2\left[\frac{1}{c_W^2}(2M_Z^2 + \frac{M_H^2}{2}) + M_Z^2(1-8s_W^2)\right] \\
&+ M_Z^2 - 8M_Z^2\left(1 - c_W^2(1-\tilde{\beta})\right)^2\Bigg\} \qquad \text{(H.123)}
\end{aligned}
$$

In the linear gauge fixing,

$$
\begin{aligned}
\Pi_{\chi+\chi+} &= \frac{\alpha}{16\pi s_W^2}\Bigg\{C_{UV}\left[\frac{3M_H^4}{2M_W^2}+M_H^2(1+\frac{1}{2c_W^2})+6M_W^2+\frac{3M_Z^2}{c_W^2}-2q^2(2+\frac{1}{c_W^2})\right]\\
&+ 8s_W^2F_0(A,W)(q^2-M_W^2)+F_0(H,W)\left[2q^2+2M_H^2-M_W^2-\frac{M_H^4}{M_W^2}\right]\\
&+ F_0(Z,W)\Bigg[q^2\left(\frac{2}{c_W^2}-8s_W^2+2\right)-M_Z^2(+6+8c_W^2+\frac{1}{c_W^2})\\
&+ M_W^2(-1-8s_W^2+32+16c_W^2)\Bigg]\\
&- M_W^2\log M_W^2(4+\frac{1}{c_W^2})+2M_H^2(1-\log M_W^2)+M_H^2(1-\log M_H^2)\left(1+\frac{M_H^2}{2M_W^2}\right)\\
&- \log M_Z^2\left[\frac{1}{c_W^2}(2M_Z^2+\frac{M_H^2}{2})+M_Z^2(1-8s_W^2)\right]+\frac{M_H^2}{2c_W^2}+8M_W^2-6M_Z^2\Bigg\} \qquad \text{(H.124)}
\end{aligned}
$$

H.3 Tadpole

The tadpole does not depend on the non-linear gauge parameters.

$$
\begin{aligned}
T &= \frac{g}{16\pi^2M_W}\Bigg\{(\frac{M_H^2}{2}+3M_W^2)M_W^2(C_{UV}+1-\log M_W^2)\\
&+ (\frac{M_H^2}{2}+3M_Z^2)\frac{M_Z^2}{2}(C_{UV}+1-\log M_Z^2)\\
&+ \frac{3M_H^4}{4}(C_{UV}+1-\log M_H^2)-2M_W^4-M_Z^4\Bigg\} \qquad \text{(H.125)}
\end{aligned}
$$

this can be recast into :

$$
\begin{aligned}
T &= \frac{2M_W}{g}\frac{\alpha}{16\pi s_W^2}\Bigg\{C_{UV}\left[\frac{3M_H^4}{2M_W^2}+M_H^2(1+\frac{1}{2c_W^2})+6M_W^2+\frac{3M_Z^2}{c_W^2}\right]\\
&+ (M_H^2+6M_W^2)\left(1-\log(M_W^2)\right)+\frac{1}{2c_W^2}(M_H^2+6M_Z^2)\left(1-\log(M_Z^2)\right)\\
&+ \frac{3M_H^4}{2M_W^2}\left(1-\log(M_H^2)\right)-4M_W^2-2\frac{M_Z^2}{c_W^2}\Bigg\} \qquad \text{(H.126)}
\end{aligned}
$$

However various two-point functions do receive a contribution from the tadpole which depends on the NLG parameters.

$$
\Pi^{tadpole}_{\chi_3\chi_3} = \frac{-g}{2M_Zc_W}(1+2\tilde{\epsilon}\frac{M_Z^2}{M_H^2})T \qquad \text{(H.127)}
$$

$$
\Pi^{tadpole}_{\chi+\chi+} = \frac{-g}{2M_W}(1+2\tilde{\delta}\frac{M_W^2}{M_H^2})T \qquad \text{(H.128)}
$$