t-channel <u>total</u> cross sections at high energy

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1 Spin-1 exchange: $e^+e^- \to \nu_e \bar{\nu}_e$

It is sufficient to only consider the GI t-channel contribution, the s-channel is GI on its own. (it is the same as the one for $\nu_{\mu,\tau}$.). Here the only mass is that of the W boson. With θ the scattering angle, the differential cross section writes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16 s_W^4} \frac{1}{s} \frac{\cos^4(\theta/2)}{\left(\sin^2(\theta/2) + M_W^2/s\right)^2}, \quad d\Omega = 2\pi d(\cos\theta) \tag{1}$$

θ is the scattering angle. s_W is the sine of the weak angle. α is $\alpha!$, $\alpha^2/s_W^4 \to (g_W^2/4\pi)^2$. s is the Mandelstam s. The integrated cross section can be obtained easily even in the case of a cut, c, on the scattering angle (which can be trivially written in terms of a p_T cut)

$$\begin{aligned}
\sigma(\sqrt{s}, M_W, c) &= \hat{\sigma}_{e\nu_e} \int_{-1+c}^{1-c} \frac{(1+x)^2}{(1-x+2\mu^2)^2}, \quad \hat{\sigma}_{e\nu_e} = \frac{\pi\alpha^2}{8 s_W^4} \frac{1}{s}, \quad \mu^2 = M_W^2/s, \\
&= \hat{\sigma}_{e\nu_e} \left(4 - 4c + \frac{c^2}{-2+c-2\mu^2} + \frac{(c-2)^2}{c+2\mu^2} + 4(1+\mu^2)\log\left(\frac{2\mu^2+c}{2\mu^2+2-c}\right)\right) \\
&= 4\hat{\sigma}_{e\nu_e} \left(\frac{1}{c+2\mu^2} + \log\left(\mu^2 + c/2\right) + \quad \mu^2 \log\left(\mu^2 + c/2\right) \right. \\
&\quad \left. + 1 - c + \frac{c^2}{4(-2+c-2\mu^2)} + \frac{c(c-4)}{4(c+2\mu^2)} - (1+\mu^2)\log\left(1+\mu^2-c/2\right)\right)
\end{aligned} \tag{2}$$

I have written the above as the sum of the leading (universal, spin-1) singularity in red, subleading log singularity (spin-1/2) in blue and the remaining process dependent constant term in which we can take the limit $\mu \to 0$ and $c \to 0$ at will. In that limit only the $+1$ (second line) will remain. It is crucial to note that the regulators, μ, c enter in the leading and subleading singularity in the combination, $2\mu^2 + c$. This shows the very important observation that the leading behaviour of another heavier mass μ^2_{heavy} corresponds to a cut, applied to the original calculation performed with μ^2. For this to make sense of course (i.e with a cut),

$$0 < c < 1 \qquad c \equiv 2(\mu^2_{\text{heavy}} - \mu^2) \tag{3}$$

The argument applies equally for a spin-1/2 exchange.

We recover the asymptotic, high energy, constant cross section with no cut as ($c = 0$, $s \gg M_W^2$) as

$$\sigma^{\text{asymp}}_{e\nu_e} = \frac{\pi\alpha^2}{4 s_W^4} \frac{1}{M_W^2} = 2\hat{\sigma}_{e\nu_e}\left(\frac{s}{M_W^2}\right) \tag{4}$$

We may note that $\cos^4(\theta/2)$ may be considered as process dependent, the leading singularity may be easily extracted by setting $\theta \to 0$ in the numerator and leaving only the pure forward process (spin-1) independent, singularity. In that case we would have to only integrate

$$
\begin{aligned}
\tilde{\sigma}(\sqrt{s}, M_W, c) &= \sigma_{e\nu_e} \int_{-1+c}^{1-c} \frac{4}{(1-x+2\mu^2)^2} \\
&= 4\sigma_{e\nu_e} \left(\frac{1}{c+2\mu^2} - \frac{1}{(2+2\mu^2-c)} \right)
\end{aligned} \tag{5}
$$

WARNING

- the identification is $c = 2M_V^2/s$, implying $M_V^2 < s/2$
- if $s = 4T^2$ and $M^2 = \kappa^2 T^2$, this implies $\kappa < \sqrt{2}$
- As Equations 2 and 5 show, do not expect perfect agreement. The process dependent pieces (non coloured red in particular) may play some role. but the $1/T^2$ will be present with a good approximation, see later for a full comparison.
- The proposal was made for high T. I don t understand why we are talking about low T (before EWPHT??) where the T^2 term is negligible for the Z and W. Gluons and photons do not (especially through $t-$channel) produce DM or any non SM coupled to SM particle. It is almost the same for W and Z of the SM unless through mixing that vanishes beyond $2M_W$.
- if the identification $s = 4T^2$ is not made then c can be large for s too small
- I have not made the the convolution over s and over T since the problem is the cross section, but I would really separate the different T regimes which would even solve the issue of resonance, which does NOT occur at high T

1.1 p_T connection

Let us look again at Equation 2. From kinematics

$$
\sin^2\theta = \frac{4p_T^2}{s} \tag{6}
$$

From small enough c (the cut in Equation 2) we have that

$$
c = \frac{2p_T^2}{s} \tag{7}
$$

If we only keep the leading terms (divergent) in $1/\mu^2, 1/c(1/p_T^2)$,

$$
\begin{aligned}
\sigma(\sqrt{s}, M_W, p_T) &\simeq 4\,\sigma_{e\nu_e} \left(\frac{1}{2}\frac{1}{M_W^2+p_T^2} + 1 + \log\frac{M_W^2+p_T^2}{s} \right. \\
&\simeq = \frac{\pi\alpha^2}{2s_W^4} \left(\frac{1}{2M_W^2(T)} + \frac{1}{s}\left(1+\log\frac{M_W^2(T)}{s}\right)\right),
\end{aligned} \tag{8}
$$

We see that if the code (MO/Calchep) insists on running with a ($T = 0$) W mass and a p_T cut, then the p_T cut that corresponds to the correct T mass (sourced either from $v(T)$ or Debye screening is

$$p_T^2 = M_W^2(T) - M_W^2(T = 0) \tag{9}$$

so either we provide the (known) expressions for $M_W^2(T)$ (this needs that we provide a function/Table for $v(T)$) and the known expression (that depends on the particles in the bath), as we see later this identification is the same for any type of interaction having a spin-1 exchange that may be large once angular integration has been performed.

The cross section agrees extremely well with the result of a Monte-Carlo CalcHEP (allowing for a fine grid/enough points/steps in the simulation).

Surely a cut on the scattering gets rid of the forward region but is of course cut-dependent. Equation 2 can implement any mass (and cut), even a temperature/energy dependent mass. Since the temperature dependence of the cross section can be genuinely introduced after the convolution, to see the effect of the temperature I will map the temperature to energy $T \leftrightarrow \sqrt{s}$ and look at the energy dependence of the cross section. To be very general, the mass of the W gets a contribution in the symmetric high energy region from the Debye screening, general formulae exist, it is bath dependent (if in the model there are more degrees of freedom than those of the SM, the formulae can be adapted and they exist, I wrote the formulae in one of my message before/after the summer). In the SSB phase there is another contribution sourced by $v(T)$. I implement the running mass (MWd) in quadrature as

$$M_{Wd}^2 \equiv M_{Wd}^2(\sqrt{s}, g, v) = v \times M_W^2 \times \theta(2M_W - \sqrt{s}) + g \times g_W^2 \times s \tag{10}$$

the SSB term has a threshold about that of the EW phase transition, which I take as a step function for simplicity, but this can be implemented more correctly especially in the case of the SM, but at very high energy this will not be the leading term and if we are in the symmetric phase it will be zero. The factor v, is $v = 1$, but in my code it can be substitued by a more realistic $v(T)$ smooth function. if I make a $\sqrt{s} \leftrightarrow T$, then $g = 11/12$ in the SM and 9/4 in the MSSM for example. Not that the Debye screening here for the transverse mode. There are other thermal corrections but they are sub-leading. In this example the exchange is entirely due to the transverse mode. In any case the longitudinal contributions are smaller in any case.

The cross section taking into account this s/T^2 dependence I obtain by simply plugging this mass in the total integrated cross section, I call these cross sections siggMd($\sqrt{s}, g, v$), the ones obtained with an angular cut but a non-T corrected mass is called sigg($\sqrt{s}, c$), the total cross section is then sigg($\sqrt{s}, 0$).

A more correct convolution to get the temperature dependence is needed to confirm these finding. But in any case a correct/justified argument between introducing an effective cut and the temperature dependence of the mass must be made.

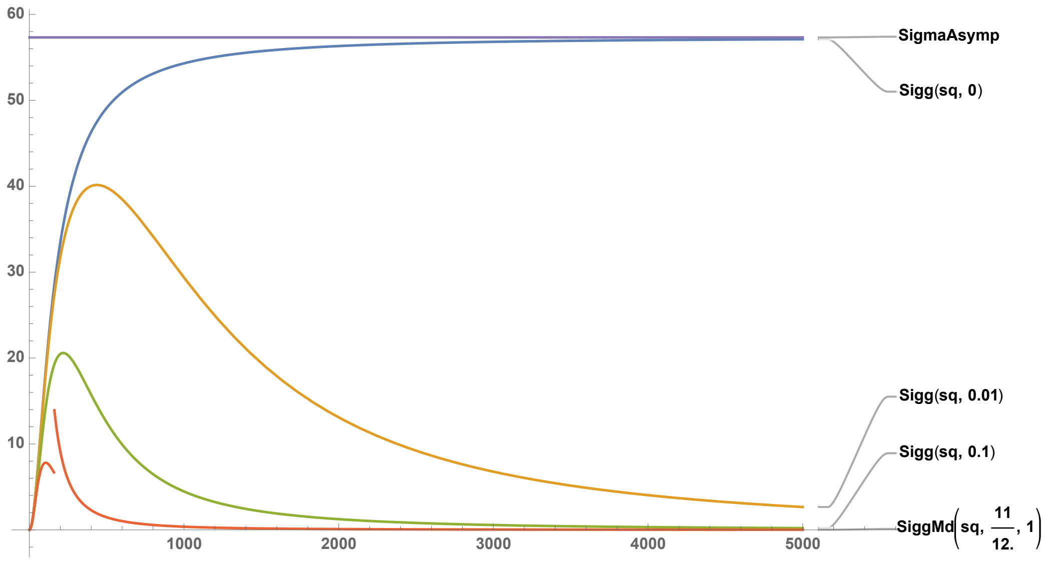

Figure 1: *Total cross sections as function of* $\sqrt{s}$ *labelled according to what is stated in the text. The unit of* σ *is pb. The total, no cut, with the* $T = 0$ *mass, Sigg(sq,0), catches the asymptotic value very well. The effect of the cut is quite dramatic with a sharp fall of the cross section with increasing cut, c. The effect of the thermal mass has an even more dramatic effect,here only SM is assumed in the bath* $g = 11/12$*. Note that the discontinuity is due to the implementation of the EWPT.*

1.2 A smoothy mass

Here I consider a smooth $v(T)$, it is still some approximation but the numerical values are not so remote from an exact $v(T)$, a simple arc (circle) function has been used with the correct value at $T = 0$, the W mass dependence is show below

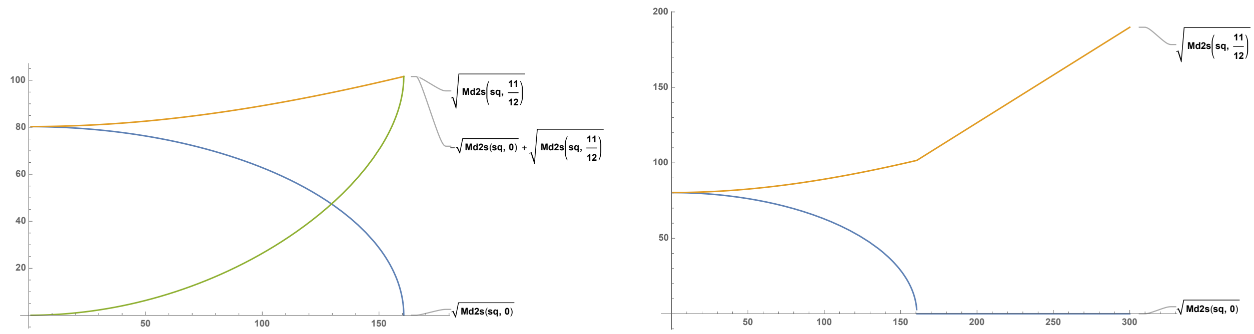

Figure 4: *The* W *mass as a function of energy/temperature. On the left, the difference (green) represents the contribution of the thermal mass. In blue is the mass from the vev.*

The ensuing cross sections get smoothed

1.3 Link with p_T and implementation in MO

Let us look again at Equation 2. From kinematics

$$\sin^2\theta = \frac{4p_T^2}{s} \tag{11}$$

From small enough c (the cut in Equation 2) we have that

$$c = \frac{2p_T^2}{s} \tag{12}$$

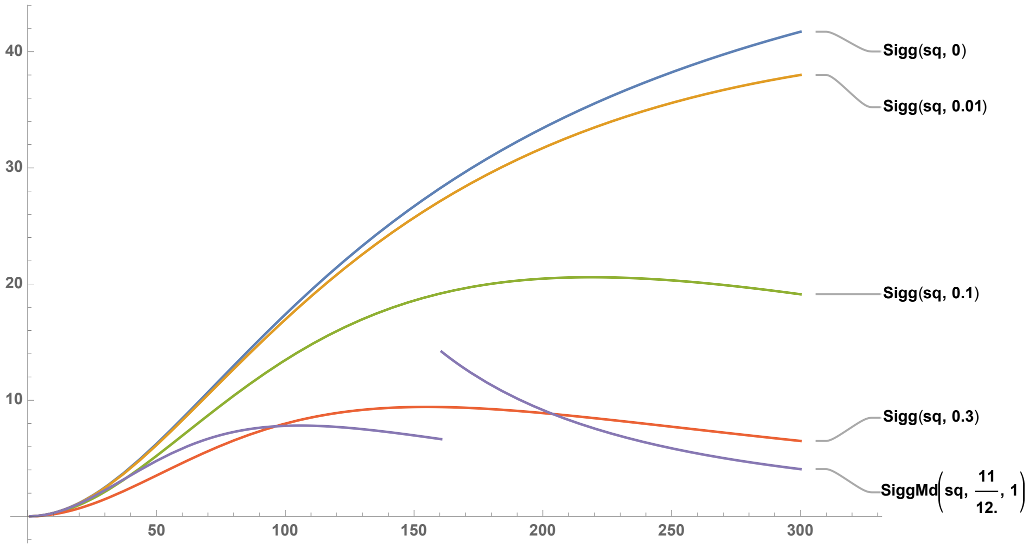

Figure 2: *As in the previous figure but with a close up with $\sqrt{s}$ up to "only" GeV. A rough order of magnitude of the SM T-dep is rendered by a cut which is relatively large,* $c = 0.3$

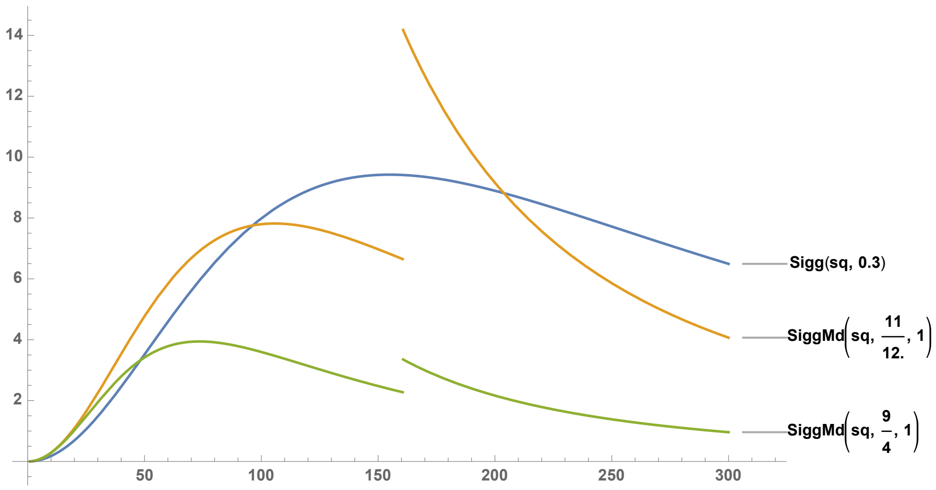

Figure 3: *As in the previous graph but with an MSSM thermal mass as am example. There is a large difference with the SM.*

If we only keep the leading terms (divergent) in $1/\mu^2, 1/c(1/p_T^2,$

$$\begin{aligned}
\sigma(\sqrt{s}, M_W, p_T) &\simeq 4\,\sigma_{e\nu_e} \left(\frac{1}{2}\frac{1}{M_W^2 + p_T^2} + \right. \\
&\simeq \;= \frac{\pi\alpha^2}{2s_W^4}\left(\frac{1}{2M_W^2(T)} + \frac{1}{s}\left(1 + \log\frac{M_W^2(T)}{s}\right)\right),
\end{aligned} \tag{13}$$

We see that if the code (MO/Calchep) insists on running with a $(T = 0\ )$ W mass and a p_T cut, then the p_T cut that corresponds to the correct T mass (sourced either from $v(T)$ or Debye screening is

$$p_T^2 = M_W^2(T) - M_W^2(T = 0) \tag{14}$$

so either we provide the (known) expressions for $M_W^2(T)$ (this needs that we provide a function/Table for $v(T)$) and the known expression (that depends on the particles in the bath), as we

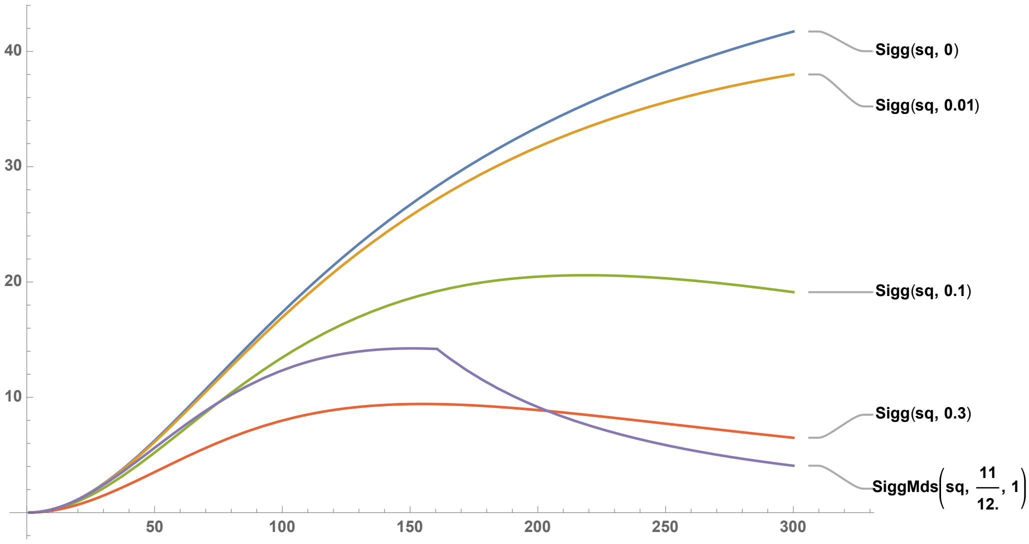

Figure 5: *As in Figure 2*

see later this identification is the same for any type of interaction having a spin-1 exchange that may be large once angular integration has been performed. While I do not expect any mechanism for FIMP creation from SM particles mediated by t-channel photons or gluons, the above cut trick and identification works also with a heavy photon/gluon

The thermal masses of the gauge masses (which do not capture all of the T dependence, and if we shut an eye on the longitudinal modes and difference between the frequency of the plasma,...) write as

$$M_V^2(T)\|_D = \frac{1}{6}g^2T^2(N + N_S + N_F/2) \tag{15}$$

N is the SU(N), N_S is the number of relativistic active, still in the bath at temperature T, scalar species, N_F same for fermions. Need to check this formula again. Anyway, obviously, this is with the approximation of massless species (I don t want to say anything here about H). So to do better some decoupling with temperature needs to be implemented. We can do better than DS, and certainly not assume that only SM particles are in the bath. The $v \to v(T)$ we can tabulate.

2 $\gamma\gamma \to W^+W^-$ and $\epsilon^-\gamma \to \nu_e W^-$

The $e^+e^- \to \nu_e\bar{\nu}_e$ example is simple enough and avoids many difficulties about gauge invariance. $\gamma\gamma \to W^+W^-$ issues of gauge invariance and the concomitant unitarity violation are crucial. For example while at very high energy the total cross section is contant, due to the $t-$ channel W exchange, one can not simply select only the $t-$channel contribution, unitarity will be broken at any value of the scattering angle. Keeping only the t-channel will give, at high energy, an amplitude which has s^2 growth due to an incomplete cancellation. But the constant cross section should be obtained more simply. First gauge invariance shows that the production of any longitudinal mode is a factor $M_W/\sqrt{s}$ smaller (at amplitude level) than that of a transverse mode. For $s \gg M_W^2$, with λ_1, λ_2 describing the helicity of the photon(s) and $\lambda_\pm$ that of the charged Ws, the leading amplitude is the all transverse, same helicity

$$\mathcal{M}_{\lambda_1,\lambda_2;\lambda^+\lambda^-} = 4e^2\delta_{\lambda_1\lambda_2}\delta_{\lambda^+\lambda^-}\frac{1}{(1-\beta^2\cos^2\theta)}, \quad \beta = \sqrt{1-4M_W^2/s} \tag{16}$$

Note that here as expected from Bose symmetry, the peak is both at $\theta = 0$ and $\theta = \pi$, the leading cross section is from adding incoherently the forward and backward peak

$$\frac{1}{(1-\beta^2\cos^2\theta)} = \frac{1}{2}\left(\frac{1}{(1-\beta x)} + \frac{1}{(1+\beta x)}\right) \quad x = \cos\theta \tag{17}$$

So we are back to the $e^+e^- \to \nu_e \nu_e$ (upon squaring the amplitude and integrating), the total cross section is from the integration

$$\begin{aligned}\sigma(\sqrt{s}, M_W, c) &= \int_{-1+c}^{1-c} \frac{1}{(1-\beta^2 x^2)^2} \\ &= \frac{1-c}{1-\beta^2(1-c)^2} + \frac{\text{ArcTanh}(\beta(1-c))}{\beta}, \quad \beta = \sqrt{1-4M_W^2/s} \end{aligned}\tag{18}$$

The leading and the log term are exactly what we have derived in the case of fermion fermion scattering (with $M^2 \to M_W^2 + p_T^2$ both at leading (constant asymptotic) and subleading (log. ! term). The leading constant cross section in the case of $\gamma\gamma \to W^+W^-$ is (as expected)

$$\sigma_{\gamma\gamma\to WW} = \frac{8\pi\alpha^2}{M_W^2} \tag{19}$$

To a good approximation we can apply the same treatment as was done before.
For $e\gamma \to W\nu$ we find

$$\sigma_{e\gamma\to W\nu} = \frac{\pi\alpha^2}{s_W^2}\frac{1}{M_W^2} \tag{20}$$

Therefore spin-1 exchange expressions are universal, we only need to correct for symmetry factor, the number of helicities taking place and the strength of the coupling. Coupling $e\nu W$ is left handed, allowing only one-helicity for the electron, the coupling is $\frac{g_W}{\sqrt{2}}$, hence $(\frac{g_W}{\sqrt{2}})^4$. For $e\gamma$, count 2 helicities for the photon and a factor $(e\frac{g_W}{\sqrt{2}})^2$. For $\gamma\gamma$ count 4 helicities, 1 contribution at $x = 1$ and one at $x = -1$ and a factor e^4.

3 t-channel spin-1/2

The classic example is $\gamma\gamma \to f\bar{f}$. With obvious notations (from my days with photon colliders)

$$\frac{\mathrm{d}\sigma_{QED}}{\mathrm{d}x} = \frac{2\pi\alpha^2 e_f^4 N_c \beta}{s(1-\beta^2x^2)^2}\left((1+\lambda_1\lambda_2)(1-\beta^4) + (1-\lambda_1\lambda_2)\beta^2(1-x^2)(2-\beta^2(1-x^2))\right). \tag{21}$$

We see that the $J_Z = 0$ has a strong chiral factor which totally counterbalances the square term. The $J_Z = 2$ are exactly the log terms we saw for the spin-1 exchange. Therefore the same combination of p_T^2 and m_f^2 appears as with p_T^2 and M_V^2 and therefore calls for the same prescription as for the spin-1.

Added January 26, 2022
Since Sasha has brought up this example, including now a new parameter κ...Let me deal with it , **exactly and analytically**, even if the main arguments have been give above.
To start with, QED being a vector theory we can implement the fermion mass by hand. Call them

hard terms. No problem with GI. Therefore we can compare the result of the cut with the result of implementing the mass of the fermions. For gauge invariance, the mass of the internal, t-channel, particle and the mass of the external fermion are the same. This example can not deal with giving the photon a mass, at least in `CalcHEP`, if one were so picky as to consider everything in a bath. For $\gamma\gamma \to f\bar{f}$ the cross section is regulated, at high energy, by introducing a cut on the cosine angle in the forward direction, or/and considering a non zero fermion mass. I will consider both at the same time, the QED example is easy enough. One of the integrals we need is the spin-1 integral (but here the accompanying chiral factor will turn it into a subdominant piece). We have already encountered this integral in Equation 18. We adapt it by simply turning $M_W \to m_f$

$$\begin{aligned} D_2(\sqrt{s}, m_f, c) &= \int_{-1+c}^{1-c} \frac{\mathrm{d}x}{(1-\beta^2 x^2)^2} \\ &= \frac{1-c}{1-\beta^2(1-c)^2} + \frac{\mathrm{ArcTanh}(\beta(1-c))}{\beta}, \quad \beta = \sqrt{1-4m_f^2/s} \end{aligned} \tag{22}$$

We also need, the spin-1/2 integral

$$\begin{aligned} D_1(\sqrt{s}, m_f, c) &= \int_{-1+c}^{1-c} \frac{\mathrm{d}x}{(1-\beta^2 x^2)} \\ &= 2\frac{\mathrm{ArcTanh}(\beta(1-c))}{\beta}, \quad \beta = \sqrt{1-4m_f^2/s} \end{aligned} \tag{23}$$

and the constant (numerator) term

$$D_0(c) = \int_{-1+c}^{1-c} \mathrm{d}x = 2(1-c) \tag{24}$$

To turn x^n in the numerators of Equation 21, that come in, the angular momentum conservation, combination $\beta^2(1-x^2)$,

$$\beta^2(1-x^2) = (\beta^2-1) + (1-\beta^2 x^2) \tag{25}$$

which helps reduce to the fundamental $D_{0,1,2}$ functions. Specialising to the unpolarised cross section, I get

$$\sigma_{QED}(\sqrt{s}, m_f, c) = \frac{2\pi\alpha^2 e_f^4 N_c \beta}{s} \left(2\big(1+(1-\beta^2)\big) D_1(\sqrt{s}, m_f, c) - 2(1-\beta^2)^2 D_2(\sqrt{s}, m_f, c) - D_0(\sqrt{s}, m_f, c) \right) \tag{26}$$

I have a good numerical agreement with `CalcHEP`.

We can now play the game of comparing the effect to a cut on the scattering angle (that mimics a thermal mass but only in the t-channel and avoid issues of new threshold etc..) with that of providing a heavy mass (temperature dependent) by hand which is conceptually and technically easy in this QED example. Note that whatever implementation, the cross sections will both be $1/s$ and will both be $1/T^2$. Since the exercise is useful for high T when the masses have only a thermal dependence, let us recall the thermal masses for the leptons and and the quarks. The former only have a QED component, the latter have both a QCD and a QED. I will neglect the latter for obvious reasons

$$\begin{aligned} m_l^2 &= \frac{1}{8}e^2 T^2 = \frac{\pi}{2}\alpha T^2 = \kappa_l\ \alpha\ T^2 = (\kappa_l^{Sasha})^2\ T^2 \sim (0.105T)^2 \sim 0.011\ T^2 \\ m_q^2 &= \frac{1}{6}g_s^2 T^2 = \frac{2\pi}{3}\alpha_s T^2 = \kappa_q\ \alpha_s\ T^2 = (\kappa_q^{Sasha})^2\ T^2 \sim (0.501T)^2 \sim 0.251\ T^2 \\ \Big(M_W^2|_{\rm SM} &= \frac{11}{12}g_W^2 = \frac{11}{12}\frac{4\pi}{s_W^2}\alpha T^2 = (\kappa_W^{Sasha})^2\ T^2 \sim (0.605T)^2 \sim 0.366\ T^2 \Big) \end{aligned} \tag{27}$$

Moreover, recall that our universal c cut thermal mass inspired is such that $c = m^2_{\rm therm}/2s$. I have implemented $\sqrt{s} = 2T$ on the basis that the photons in the bath have energy/ momentum T, such that the total energy of the incident photons is $2T$. This means that c is temp. independent. Note also that apart from the overall $1/s$ of the cross section, the s (or T^2) dependence is in the ratio m^2/s (through β). Upon turning these quantities into temperature, it means that all temperature dependence cancels. Therefore whatever implementation $\sigma = {\rm cste}/T^2$. Are the corresponding, cste, temperature independent constants so different by implementing the full thermal mass or by working with $m_f = 0$ and imposing a *a thermal induced cut* which would be more universal?

The cross section including the full thermal mass can be calculated with $c = 0$. Since now the fundamental parameters are κ_f and α, α_s. I define

$$\sigma_{QED}^{\text{Full therm mass}}(\kappa, \alpha, T) = \sigma_{QED}(2T, m_f(T), 0) \tag{28}$$

Whereas the cut inspired is

$$\sigma_{QED}^{\text{Cut therm}}(\kappa, \alpha, T) = \sigma_{QED}(2T, 0, 2m_f^2(T)/4T^2) \tag{29}$$

As expected I find, the temperature independent results

$$\begin{aligned} \frac{\sigma_{QED}^{\text{Full therm mass}}(\kappa, \alpha, T)}{\sigma_{QED}^{\text{Cut therm}}(\kappa, \alpha, T)} &= 1.0108 \quad \text{for leptons} \\ &= 1.16108 \quad \text{for quarks} \end{aligned} \tag{30}$$

For leptons the agreement is excellent! A bit less so for quarks as the thermal mass is larger but it is still quite good. The cut cuts more than the full threshold mass. But still, one may wonder considering the approximations whether it is worth to bother. Remember with the spin-1 we were orders of magnitude and here the $1/T^2$ is recovered in both cases.

3.1 s *vs* T^2 *vs* $4T^2$

If one insists on using for this QED example the full mass (not the $t-$cut approach) then we hit the issue of threshold in any case, even more dramatically with the identification $s = T^2$. Indeed note that

$$\frac{4m_q^2}{T^2} > 1 \tag{31}$$

meaning the particles are not produced.

4 Issue of masses of incoming/outgoing particles

Up to now I have concentrated on the case where the incoming and outgoing particles were massless. No need to try to understand the most general case if we do not understand the basics and if we do not justify the implementation. This is not experimental science and does not rest on guessing. To not cover cases we will not encounter, I will first specialise to the case of $P_1 P_1 \to P_1' P_1'$ with the exchange of spin-1, of mass M_V, from the singularity/kinematic point of view this is the same as $P_1 \bar{P}_1 \to P_1' \bar{P}_1'$, the simplifying assumption is that the incoming particles have the same mass, m_i, as do the outgoing particle m_f. We will keep in mind, later, that $P'(m_f)$ are DM and not in the bath. P_1 may or may not be in the bath. We may wonder in which gauge we are going to do the calculation. The easiest if the Feynman gauge. One may argue that this should be supplemented by the Godstone (in the SSB phase), but this is beside the point. Goldstones are spin-0, they will not catch the spin-1 singularity.

Doing it in the Unitary gauge, added $k_\mu \kappa_\nu / M_W^2$, the latter will only bring suppressed chiral factor, and in all cases we are after the leading singularity of a spin-1 t-channel exchange.

With s the usual total energy (squared), the s Mandelstam variable, let us define the usual β's, such that

$$\beta_{i,f} = \sqrt{1 - 4m_{i,f}^2/s} \tag{32}$$

we have the kinematics in the cms frame as

$$p_{1,2} = \sqrt{\frac{s}{2}}\Big(1, 0, 0, \pm\beta_i\Big) \qquad p'_{1,2} = \sqrt{\frac{s}{2}}\Big(1, 0, \pm\beta_f s_\theta, \pm\beta_f c_\theta\Big) \tag{33}$$

By further defining

$$\beta_{if}^2 = \frac{2\beta_i\beta_f}{\Gamma_{if}^2}, \quad \Gamma_{if}^2 = \beta_i^2 + \beta_f^2 \tag{34}$$

one can write

$$t = (p_1 - p'_1)^2 = -\frac{s}{4}\Gamma_{if}^2(1 - \beta_{if}^2 \cos\theta), \quad t - M_V^2 = -\frac{s\Gamma_{if}^2}{4}\Big((1 - \beta_{if}^2\cos\theta) + \mu_{vif}^2\Big), \quad \mu_{vif}^2 = \frac{4M_V^2}{s\Gamma_{if}^2} \tag{35}$$

Note in passing that when $i = f$, with $\beta_i = \beta_f$, $\beta_{if}^2 = 1$ (*exactly!* independently of how heavy or light the external masses are, this is already a clue that the external masses are subdominant).

We are now in familiar territory. With $m_i = m_f = 0$ we find the results of $e^+e^- \to \nu_e\bar{\nu}_e$. The integrated cross section

$$\begin{aligned} I_\sigma &= \int_{-1+c}^{1-c} \frac{\mathrm{d}x}{\Big((1 - \beta_{if}^2 x) + \mu_{vif}^2\Big)^2} \\ &= \frac{2(1-c)}{(1 + \mu_{vif}^2 - \beta_{if}(1-c))(1 + \mu_{vif}^2 + \beta_{if}(1-c))} \end{aligned} \tag{36}$$

Specialising to the no cut result in order to see if the in/out masses provide a “cut-off", we realise that

$$\beta_{if}^2 = 2\frac{\beta_i\beta_f}{\beta_i^2 + \beta_f^2} = 1 + \mathcal{O}(m_i^4/s^2, m_f^4/s^2, (m_i^2 m_f^2)/s^2) \tag{37}$$

So this is really a sub-leading effect, the masses will then gives effect as s/T^4 !!

This shows that the cut-off is indeed provided by the vector boson mass with a small correction due to the in/out masses, indeed the cut-off is

$$2\frac{M_V^2\Big(1 + 2(m_i^2 + m_f^2)/s\Big)}{s} \tag{38}$$

This is to be expected. It is the vector boson which responsible for the long range (in the limit $M_V \to 0$) force, it is M_V that controls the range, independently of how massive the external particles are .

4.1 `CalcHEP` Numerical examples of why masses of external particles are subdominant

Note that for $\sqrt{s} \gg M_W$ (we don t have to be in the extremely asymptotic region as the asymptote is reached quickly). With the SM values of the masses

$$\sigma(e^+e- \to \nu_e\bar{\nu_e}) = \sigma(e^-\mu^+ \to \nu_e\bar{\nu_\mu} = \sigma(e^-t \to \nu_e b) = \frac{\pi\alpha^2}{4s_W^2}\frac{1}{M_W^2} \simeq 57.2\text{pb} \tag{39}$$

Note that the mass of the top does not regulate. To with a plot of the total cross section $p = 10^3 - 10^4$GeV, the constant cross section becomes 57.5 pb when m_t is changed from $m_t = 172.5$ to $m_t = 1000$! Compare this by changing $M_W = 80.38$ to $10M_W$, the result is reduced by 2 orders of magnitude (0.571) and $10M_W$ is less than $m_t = 1000$GeV! In the opposite if we take $M_W/10$, we get 5.7210^3pb. Here the vector boson mass is $8GeV$ compared to $m_t = 172.5$ GeV! In practice the thermal mass of the gauge boson will be larger in any case. So complicating the issue with external masses does not regulate or change the result much. The important mass is the gauge boson mass/ the mass of the exchanged spin-1 in the t-channel.

5 The $s-$ channel component and the interference with t-channel

We now consider the full $e^+e^- \to \nu_e \bar{\nu}_e$ taking into the s-channel. I will work with $\Gamma_Z = 0$. If some are so keen about GI, then $\Gamma_Z = 0$. My equations should then be simpler.

The interference term writes as

$$\frac{\mathrm{d}\sigma_{WZ}}{\mathrm{d}\cos\theta} = \sigma_{e\nu_e} \frac{1}{1 - M_Z^2/s} \frac{c_{2W}}{c_W^2} \frac{\cos^4(\theta/2)}{\left(\sin^2(\theta/2) + M_W^2/s\right)}, \quad \sigma_{e\nu_e} = \frac{\pi\alpha^2}{8 s_W^4} \frac{1}{s} \tag{40}$$

Upon integration we obtain

$$\sigma_{WZ} = -\sigma_{e\nu_e} \frac{1}{1 - M_Z^2/s} \frac{c_{2W}}{c_W^2} \left(2(1+\mu_W^2)^2 \log \frac{2\mu_W^2 + c}{2 + 2\mu_W^2 - c} + (1-c)(3 + 2\mu_W^2) \right) \tag{41}$$

The pure s-channel which also corresponds to the GI invariant part since it is nothing but exactly the same as $e^+e^- \to \nu_{\mu,\tau}\bar{\nu}_{\mu,\tau}$

$$\frac{\mathrm{d}\sigma_{ZZ}}{\mathrm{d}\cos\theta} = \sigma_{e\nu_e} \frac{1}{(1 - M_Z^2/s)^2} \frac{1}{c_W^4} \left(s_W^4 \sin^4(\theta/2) + \frac{c_{2W}^2}{4} \cos^4(\theta/2) \right) \tag{42}$$

For further reference, to weigh the importance of the cut in a more model independent way, note that the contribution is split between $e_R e_R$ production and $e_L e_L$ (same as the one in the $t-channel$) , and hence different θ dependence. The final state is of course LL. Note also that if one is so worried about hight T behaviour, in that phase of no symmetry breaking we do not have any $e_R e_L$, gauge symmetry does not allow this. But this is beside the point for the moment. Integration over the scattering angle is trivial especially for the pure s-channel.

$$\begin{aligned} \sigma_{ZZ}(c, M_Z^2) &= \frac{\sigma_{e\nu_e}}{c_W^4} \frac{1}{(1 - M_Z^2/s)^2} \frac{(1-c)(1 - c/2 + c^2/4)}{6} (4 s_W^4 + (1 - 2 s_W^2)^2) \\ &= \frac{\sigma_{e\nu_e}}{c_W^4} \frac{1}{(1 - M_Z^2/s)^2} \frac{(1-c)(1 - c/2 + c^2/4)}{6} \frac{1 + (1 - 4 s_W^2)^2}{2} \end{aligned} \tag{43}$$

Here the contribution is split between the universal leading log (spin-1/2) and the process independent non singular contribution.

The previous result is written in terms g_L/g_R coupling and $g_V^2(el.) g_A^2(el)$. We recognise the factors of $\Gamma(Z \to e^+e^-)\Gamma(Z \to \nu\bar{\nu}$, as it should be. (Sorry a correction from what I sent this morning) This shows that the cut (anyway c<1) affects the $e_L e_L$ and $e_R e_R$ production the same way. For instance with $c = 0.1$ the cross section is cut by 14% while for $c = 0.2$ it is already cut by 27%. A θ independent cross section will be cut by, respectively, 10%, 20%.

As you see the cuts here are totally misleading (I mean for s-channel only). There is no identification of a cut with a mass as was shown for t-channel. If there is any temperature dependence (forgetting about couplings and masses of in/out), the temperature is hidden in M_Z^2, which in all logic should be replaced by $M_Z^2(T)$. Now if you worry about high T, then $M_Z^2(T) = M_W^2(T) = g^2 T^2$, ($g^2(< 1)$ is very important. The propagator will ten be $(T^2(1 - g^2))^2$, at these high temperatures (crucial for t-channel argument) there is NO resonance. The resonance occurs at much lowers energies when $M_Z^2(T = 0)$ dominates and a more $T = 0$ is at play. For high T the resonance is lost. We are mixing issues. For the t-channel argument high temperature is important and the trick of $p_T^{\rm cut}$ is OK. There are 2 regimes. To me at least, I will think more.

In the case of a Z_p exchange probably emerging from an extra $U(1)$, it is quite unlikely that is involved in a t-channel where the producing SM particle is turned into DM through t-channel (in a way there is no $ZpSMDM$ independently of $Z_2, ..$), so interference (for GI sake) is a non issue. For such a case, specifically $U(1)$ coupled to conserved current giving a mass by hand is totally allowed! Here I would not even recommend a tCUt by just change the mass by hand. $U(1)$ bosons, even if in the bath, will get a much smaller mass than Z, W, the gauge/spin-1 contribution is nil and they would have to couple to a lot a lot of fermion in order to make this thermal meaningful. What will be important is whether they have mass at $T = 0$, that's what will be important at the end.