The major goal of the modern particle physics is to verify the existing models of elementary particles, to establish their limitations, and to discover possible signs of new physics. In order to achieve this goal several ingredients are indispensable. On the one hand, the theoretical predictions for various physical observables calculated in the framework of the Standard Model are required. On the other hand, these observables are to be measured in the collider experiments. There is no obvious discrepancy between the experiment and the theory. However, the discrepancy could appear if the comparison is made at higher precision level. The current precision level of the experimental data demands very precise predictions from the theory.

For the energy scale of the collider experiments, the Standard Model interactions are in the weak-coupling regime. This is a very fortunate situation since there is a small parameter and the observables could in principle be calculated order by order in powers this small parameter. In order to achieve a high enough precision the leading order and the first subleading correction are not sufficient. Thus, higher-order corrections are to be calculated. The latter is an extremely complicated task which involves numerous conceptual and technical obstacles. Each jump by one order in perturbative calculations became possible in the past owing to a breakthrough in the theoretical understanding, development of innovative calculation techniques, substantial increase of the available computational resources, and development of novel computational tools.

The crucial ingredients in the theoretical predictions of the collider experiments are scattering amplitudes. They are indispensable in calculation of the cross-sections and event-shapes which are measured in the collider experiments. At each order of the perturbative expansion, the scattering amplitudes are given by sums of Feynman diagrams. According to the laws of quantum mechanics, each Feynman diagram represents a possible history of the scattering process. Each Feynman diagram is a rational function (so called, integrand) which is to be integrated over a number of loop momenta. The number of loop integrations equals to the order of the perturbative expansion.

The three major obstacles in dealing with Feynman diagrams are as follows. Firstly, the number of contributing Feynman diagrams rapidly grows with the order of perturbative expansion (so called, the loop order) and with the number of scattered particles (so called, legs of the amplitude). In the state-of-the-art calculations their number could reach hundreds of thousands or even more. At this step, one deals with the integrand of the scattering amplitude, which is a rational function of the loop and external momenta, and all Feynman diagrams have to be assembled in a more digestible expression by means of algebraic manipulations. Thus, a huge collection of rational functions has to be handled. This requires usage of the powerful computational tools and symbolic algebra systems. Several more efficient approaches to deal with Feynman diagrams and integrands of scattering amplitudes have been elaborated over the years. They largely rely on the ideas of the generalised unitarity. The generalised unitarity provides guidance how to find a concise representation of the amplitude integrand. The integrand is localized by taking several consecutive resides and more concise expressions for the residues are found. In total, the multi-loop scattering amplitudes are assembled from smaller pieces which are the tree-level scattering amplitudes. Depending on the problem at hand one or another method to process the amplitude integrand could turn out to be the most cost-efficient. All these approaches rely on the extensive usage of the computer algebra systems or specially elaborated computer codes.

The second obstacle is related to multidimensional integrations. Namely, the individual Feynman diagrams or the amplitude integrands are to be integrated over all loop momenta. Not only the multidimensional integration is a complicated task both numerically and analytically, but also the number of loop integrals to be evaluated can be extremely high. The individual loop integrals are called Feynman integrals. Fortunately, they are not all independent. There are numerous linear relations among the Feynman integrals contributing to the scattering amplitude. Thus, it may seem reasonable to identify a minimal set of independent Feynman integrals and to express the scattering amplitude in their terms. Reduction of the amplitude to a minimal basis can in principle always be done algorithmically. However, in order to achieve this goal, huge systems of linear equations with rational function coefficients have to be solved. This is a complicated computational problem. Its complexity grows with the loop order and with the number of kinematic scales of the family of Feynman integrals. The improvements can be achieved along two directions. Firstly, one can try to decrease the size of the linear systems. The linear equations of the system are obtained by seeding with integers the so-called integration by parts relations which are linear recurrence relations among Feynman integrals with shifted powers of their propagators. A reasonable choice of these recurrence relations could lead to a smaller system after seeding it with the integers. The methods of computational algrebraic geometry were found to be useful in this regard. Secondly, one can try to elaborate more efficient computational methods for solving the linear systems. In particular, recently elaborated approaches based on finite field arithmetic demonstrated their superiority over traditional ones. They employ the seeding of the rational coefficients of the system with the integer numbers taken from the finite prime fields and solving the resulting linear systems over the finite fields. Several evaluations over prime fields are required to restore the rational function. The computation over prime fields are much more cost-efficient than over rational numbers or over rational functions. These ideas are implemented in several computer codes. More recently, the ideas from intersection theory have been shown relevant for solving the linear relations among Feynman integrals and identifying a minimal set of independent ones.

Finally, after a minimal set of the Feynman integrals is identified, they have to be evaluated by doing loop integrations. There are several ways to rewrite the loop integrations as well-defined multidimensional integrals. Evaluation of these integrals is a complicated task both numerically and analytically. The main advantage of the numerical approaches is their universality. Whereas the analytic approach requires tedious calculations for each new family of Feynman integrals. Opting for the numerical approach to scattering amplitudes, we need to be able to evaluate the scattering amplitudes efficiently and with controllable accuracy in tens of thousands of kinematic points. This is required for computation of the observables with the Monte Carlo generators. The Feynman integrals are functions of the kinematic invariants, which specify the phase-space point of the scattering process, as well as of a regularization parameter which is necessary to handle the ultra-violet, and soft/collinear divergences. The singularities of the Feynman integrals in the regularization parameter have to be revealed prior to numerical integrations. This is achieved by the sector decomposition method. It provides a series expansion of the Feynman integrals in the regularisation parameter and expresses each coefficient of the expansion as a sum of well-defined multidimensional integrals which only depend only on the phase-space point. The latter can be numerically integrated. This approach is implemented in several computer codes. Obviously, it is difficult to achieve high precision in multidimensional integrations. Also, numerical stability through the phase space is not under full control.

In general, the Feynman integrals are multivariable functions with complicated branch cut structure. The analytic calculations of the Feynman integrals are of interest for several reasons both practical and theoretical.

Since the final goal is to evaluate efficiently and with high precision the scattering amplitudes at a large number of the phase-space points, the obtained analytic expression should serve this purpose. A good analytic understanding could provide a more refined set-up for efficient numerical evaluations. Analytic expressions could be easier to evaluate and to control precision of evaluations. They also provide insight into the spurious singularities in the phase space, which could be present in individual Feynman integrals (but not in their sums – the scattering amplitudes) and spoil a naive numerical loop integration with large cancellations among several Feynman integrals.

A prerequisite for analytic calculation of the Feynman integrals is to identify the relevant space of special functions. The complexity of the special functions increases with the loop order and the number of variables (i.e. the number of scales in the scattering amplitudes). At the moment there is no understanding what are the spacial functions which are required to express an arbitrary Feynman integral. We note in passing that the answer to this problem is not known even in a simpler situation of the scaleless Feynman integrals which evaluate to transcendental numbers. In the latter case, the complexity of the transcendental numbers grows with the loop order. The multiple zeta values are sufficient at low loop orders, but much more complicated numbers pop up at higher loop orders. Despite of the fact the complete picture is obscured so far, numerous multi-loop and multi-scale examples have been worked out explicitly. This provides some insight into the relevant spaces of special functions. The question about the special functions is extremely interesting from the mathematical point of view. It has attracted attention of both mathematicians and theoretical physicists. Very advanced and abstract mathematics happened to be relevant to address it properly.

The most efficient methods for doing loop integrations of the multi-scale Feynman integrals analytically are based on the differential equations. Similar to Feynman integrals, their derivatives in the kinematic invariants are also not linear independent. This leads to linear systems of the first order differential equations (in several variables). Of course, the system of differential equations depends on the choice of the basis of Feynman integrals, which are unknowns in the system. A change of this basis transforms the system of differential equations. It has been noticed that a clever choice of the basis can enormously simplify the system such that it is solved straightforwardly. In other words, a difficult task of solving a generic system of the differential equations is traded for the task of finding an appropriate basis of the Feynman integrals. In many cases there is a guiding principle how to choose the basis. It is based on the analysis of the leading singularities of the Feynman integrals. They are multidimensional residues of the integrand which localize all loop integrations. The transformed system of differential equations is solved in terms of the iterated integrals. If the transformation is achieved by an algebraic transformation, which is suggested by the leading singularity analysis, then the iterated integrals are of polylogarithmic type. Their mathematical structure is well-understood. Very efficient algebraic methods to handle them are known. The relevant space of special functions is a subspace of the generalised polylogarithms. Namely, each scattering kinematics (perhaps up to a certain loop order) is specified by the so-called alphabet, which is a finite collection of algebraic functions of the kinematic invariants. Roughly speaking, the alphabet is the allowed arguments of the generalised polylogarithms which contribute to the Feynman integral. Usually the Feynman integrals with not too many legs and with zero masses in internal lines (or with a small number of massive lines) belong to this class. This is what is needed for dealing with massless QCD processes (which is a reasonable approximation in the high-energy limit where the weak-coupling expansions are valid) of not very high multiplicity.

However, the polylogarithmic case does not exhaust all possible Feynman integrals. In some cases the system of differential equations cannot be simplified by an algebraic transformation. In this situation the polylogarithmic iterated integrals are not sufficient, and one has to consider iterated integrals over modular forms and the elliptic extension of the polylogarithms. Even more complicated iterated integrals are known to appear, which are related to higher dimensional elliptic curves. Their mathematical properties are not well understood as those of the polylogarithms and computer codes for their numerical evaluation are not so well elaborated. The elliptic type special functions is not an artifact of Feynman integrals, they also do contribute to the scattering amplitudes. For example, they are relevant for scattering amplitudes with top-quark loop corrections.

Differential equations is a useful tool not only for analytic calculations by for the numerical approaches as well. Provided that the previous calculation steps are done analytically, such that the differential equations for a minimal set of Feynman integrals are available, the differential equations can be solved numerically. In this way, the complicated issue of the relevant special functions is completely avoided. A much higher numerical precision and efficiency can be achieved in solving the differential equations as compared to multidimensional numerical integrations. Numerical solution of the differential equations could be a reasonable compromise between numerics and analytics. These ideas and their extensions have been implemented in several computer codes. Although, numerical stability is not guaranteed and calculation speed is still an issue for complicated Feynman integrals.

Provided the relevant space of special functions is identified, one could look for a minimal algebraically independent set of special functions which are required to express the amplitude. This reduction to a minimal set of special functions is usually the most economic since the unphysical artifacts of the intermediate calculation steps are eliminated. In particular, it takes into account automatically numerous cancellations among contributing Feynman integrals, which are obscure in the numerical approaches. Huge intermediate expressions is a bottleneck for any calculation in practice. This observation suggests that the technological advancement could be achieved if the unphysical expressions of the intermediate calculation steps can be harnessed to avoid blow up in their size. A practical implementation of this idea is to seed the kinematics with integers, to perform all algebraic calculations over finite prime fields, and to reconstruct the scattering amplitude from a number of evaluations. Summarising, an expression of the amplitude in terms of a basis of special functions is the physical one and is the most concise with all spurious singularities canceled out. This seems to be the best representation for the scattering amplitudes, and it is the starting point for the high precision phenomenology. The drawback of this approach is that it requires tedious analytic analysis on a case-by-case basis which cannot be completely automatized for large classes of scattering amplitudes.

The Standard Model scattering amplitudes is not the only scope of applications of the calculation techniques outlined above. They are equally applicable to high-order perturbative calculations in effective field theories in various dimensions, and in general for any quantities (correlation functions, anomalous dimensions, Wilson coefficients, ...) expressible in terms of Feynman-type integrals. In particular, the knowledge about quantum amplitudes in gauge theories found very successful applications in calculations of nonliner corrections in classical gravity.

The last two decades were marked by the technological revolution in perturbative calculations. This became possible due to new theoretical ideas, rapid development of the computer algebra systems, and substantial increase of the computational resources. High orders of the perturbative expansion can be extremely bulky expressions which are very difficult to handle. Thus, they require usage of adequate computational tools. Without them many novel and old theoretical ideas could not found their practical implementation. An important source of inspiration for multiloop calculations in the realistic gauge field theories are the supersymmetric gauge field theories with extended supersymmetry. They are the toy models in a sense that their dynamics is restricted by various symmetries. In particular the scattering amplitudes and other observables posses a number of appealing mathematical properties. They are a perfect theoretical laboratory for testing novel calculation strategies and polishing the computational tools. Theoretical ideas and advanced computational tools go hand in hand allowing for calculating more and more complicated scattering amplitudes.