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Glossary

Abbreviations

2PRR	Two-particle reducible
ACU	Analyticity, Crossing symmetry, and Unitarity. Basic bootstrap axioms.
AM	Atkinson–Mandelstam.
EFT	Effective field theory.
FG	Froissart–Gribov (projection)
LC	Landau Curves.
ML	Machine Learning.
MP	Multi-particle.
NN	Neural Network.
QFT	Quantum field theory.
SDP	Semidefinite program.
SDPB	Semidefinite program solver [1, 2]

Special terms

ϱ -bootstrap. In the text, I refer to the technology based on semidefinite optimization introduced in [3–5] as the ϱ -bootstrap.

Atkinson-Mandelstam (AM) bootstrap Constructive bootstrap strategy developped in [PT18, PT22, PT26, PT29] based on works of Mandelstam [6] and Atkinson [7–10] that generates amplitudes with nontrivial particle production and non-perturbative control over the amplitude at high energies and spins.

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Chapter 1

Introduction

*Le idee spesso sono come un boomerang:
partono in una direzione
ma poi vanno a finire altrove.*

— G. Parisi, *In un volo di storni.*

1.1 The bootstrap

In 1968, Gabriele Veneziano wrote down a formula to describe hadronic scattering: the Veneziano amplitude [11], a crossing-symmetrized version of

$$A(s, t) = -\frac{\Gamma(-s + m^2)\Gamma(-t + m^2)}{\Gamma(-s - t + 2m^2)}, \quad (1.1.1)$$

This remarkable formula resums an infinite tower of resonances while realizing crossing symmetry in a genuinely nontrivial way. In doing so it embodies the 1960s notion of duality [12]: resonances exchange in one channel reproduce singularities in the other channel.

The formula emerged at a time when the status of the strong interactions was unclear. The rapidly growing “hadron zoo” seen in collider experiments contrasted sharply with the relative simplicity of quantum electrodynamics (QED) for which quantum field theory (QFT) had been worked out, and it helped fuel the bootstrap hypothesis and the idea of “nuclear democracy”: hadrons were not organized around a small set of elementary degrees of freedom, but were instead viewed as mutually composite, each built out, in some sense, of all the others. The tone of the period is well captured by the strong stance of Chew, who wrote [13] “*So that there can be no misunderstanding of the position I am espousing, let me say at once that I believe the conventional association of fields with strongly interacting particles to be empty*”, and continued: “*Which of the strongly interacting particles are elementary? ... I am convinced that there can be only one sensible answer, and that is that none of them is elementary.*”

In a well documented history¹, shortly after the Veneziano model was proposed it was realized independently by Susskind, Nambu, and Nielsen that the Veneziano amplitude is the tree-level four-point amplitude of a relativistic string. From there, the subject expanded into what we now call string theory.

It is sometimes said that string theory was discovered “by chance”. There was obviously an element of chance in this discovery: the amplitude was written down to fit hadron data, while the string interpretation came only later, as a serendipitous re-reading. But there is an opposing lesson as well. Veneziano was guided by an heuristics attempting

¹See e.g. [14] and references therein.

to combine consistency requirements of quantum mechanics and relativity within the S-matrix framework, and those requirements turned out to be sufficiently restrictive to force a nontrivial structure. In that sense, the episode illustrates how far one can get by pushing these general principles. From the standpoint of nuclear democracy—and the broader aim of doing without elementary constituents—the string interpretation gave this program an unexpected twist: the bootstrap agenda led to a candidate “ultimate” constituent for matter.²

Today the bootstrap is again a very active field, propelled in particular by the modern conformal bootstrap [16] and follow-ups. In several ways the modern bootstrap (S-matrix and conformal) revives an ambition similar to the original: to extract dynamics from general consistency conditions. If this line of thought were to one day show that any UV-complete theory of quantum gravity must exhibit string-like dynamics, Parisi’s boomerang would have truly come back. If not, the throw would still have been worth it: non-returning boomerangs are efficient, too.

Rigidity. With the development of QFT, we have learned that elementary interactions are largely fixed by the requirements of locality, unitarity, and Lorentz invariance. Gauge theories, and gravity, are rigid structures in this sense: they are not arbitrary dynamical choices, but essentially forced once one insists on local, consistent couplings of massless particles [17]. This rigidity has no analogue in ordinary quantum mechanics: many potentials can define a consistent theory with consistent scattering. In relativistic quantum field theory, forces can only come from particle-exchanges. For strong interactions, crossing symmetry identifies the “potential scattering” in one channel to particle exchange in another, and the same degrees of freedom that scatter must also source the interactions that mediate the scattering. As we alluded to already, the Veneziano amplitude is an early and clean illustration of this broader point: not only the kinematics of QFT is rigid, so is the dynamics.

Another interesting illustration of this rigidity is the Froissart bound. Consider the following standard semi-classical reasoning [18, 19]. Take two particles interacting by the exchange of a massive particle m , propagating over a typical distance $R \sim 1/m$ and corresponding to a polynomial interaction growing with energy. The interaction radius is determined by

$$(gE)^n \exp(-mR) \sim 1 \quad (1.1.2)$$

which yields $R \sim \ln(E)$, and hence a cross section $\sigma(s)$

$$\sigma(s) \sim \ln(s)^2 \quad (1.1.3)$$

As is well known, this is precisely what the actual careful derivation yields, see Froissart [20] and Martin [21]. One remarkable fact about the Froissart-Martin bound is that we actually do not know a concrete mechanism that can yield a saturation of the bound. These physical principles: quantum mechanical evolution, Lorentz invariance, and causality, embodied (in reverse order) in the axioms of “Analyticity, Crossing-Symmetry, Unitarity” – ACU –, are so constraining at the level of the S-matrix that finding a solution to them even for the simplest of *two-to-two* scattering was basically never achieved, see Kupsch [22]. As he reviews, when combined at the most accurate level of their formulation, the principles of ACU have only been proven so far to allow for a behavior

$$\sigma(s) \sim \log(s)^{-3} \quad (1.1.4)$$

²As Weinberg puts it [15], *This whole subject [the S-matrix bootstrap] is filled with delicious ironies.*

as a result of a series of paper from Atkinson, [7–10]. Absence of proof does not mean impossibility, but the message from above here too comes through: that taken together, the principles of ACU are particularly constraining.

This motivates the question: how much rigidity can be leveraged on the dynamics of QFTs from these axioms?

Computing power One basic reason why these questions and the S-matrix bootstrap is hard is that the unitarity equations are intrinsically nonlinear

$$\hat{S}\hat{S}^\dagger = \mathbb{1} \implies 2i\text{Im}\hat{T} = \hat{T}\hat{T}^\dagger, \quad \hat{S} = \mathbb{1} + i\hat{T} \quad (1.1.5)$$

Historically, the most direct implementation of this condition—based on Mandelstam’s analyticity framework [6] — on which parts of the works presented in this thesis are based — was simply out of reach computationally, and one had to rely on uncontrolled approximations such as the strip approximation [23, 24].

Since 2016, inspired by the numerical and analytical CFT bootstraps [25–27],³ Refs. [3–5] introduced a powerful, model-independent numerical technology to compute nonperturbative bounds on low-energy observables, launching a return of the S-matrix bootstrap.

In the text, we refer to the technology introduced in [3–5] as the **ϱ -bootstrap** (see Glossary). In many problems, this framework has produced sharp, reproducible numerical results and has been a major driver of progress in studying S-matrix bootstrap bounds, and scattering amplitudes from QCD to string theory. These techniques are based on a minimalist way to turn the nonlinear problem above into a semi-definite problem, that could be treated with numerical algorithms (most notably, the software SDPB developed for the conformal bootstrap [1, 2]). Expressed in the partial-wave basis, unitarity implies the simple statement that $|S_J(s)| \leq 1$. This can be easily rewritten as a semi-definite condition on a matrix made of $\text{Re}(S_J(s))$ and $\text{Im}(S_J(s))$, explained around Eq. (4.1.2). Besides bounds on low-energy physical observables, this technology also produces the amplitudes that realize the bounds, for which physical consistency conditions are satisfied in a range of energies and spins (partial-waves). These developments generated a large activity with many results, both formal: charting the space of consistent QFTs [29–31], and phenomenological, related to pion and glueball scattering [32–40] and to the Froissart growth [41]. The same methods could even be applied to massless scattering, with for instance graviton scattering in $d = 9, 10, 11$ supergravity theories [42, 43], showing from first principles that string theory occupies a distinguished position in the space of quantum-gravity low-energy UV completions. For an exhaustive review, see [44].

1.2 Particle production

The semi-definite simplification previously mentioned came at a cost: in semi-definite optimization, constraints tend to be saturated, leading to apparent purely elastic partial waves $|S_J(s)| \leq 1 \rightsquigarrow |S_J(s)| \equiv 1$. As is discussed in several places in the present text, if this were true for all spins J and energy squared s , this would be in contradiction with Aks’ theorem [45], that proves from axiomatic QFT that in $d > 2$, *some* amount of particle-production (implying $|S_J(s)| < 1$ for some J and s) is required.

Thus, it is important to develop alternative approaches to the bootstrap to shed light on this question. An important step in this direction was achieved with a rigorous, dual

³On the subject of the history of the CFT bootstrap, the reader can refer to the recent account [28].

bootstrap introduced in [46]. It at least confirmed that dual bounds do not exclude primal bounds.⁴

Beyond computing power, how to treat multi-particle production is another reason why the bootstrap stalled in the 1960s, and is still unsolved today. The analytic structure of multi-particle phase-space is an extremely difficult problem in $d > 2$ spacetime dimensions. At the level of the 2-to-2 amplitude, Eq. (1.1.5) projected on 2-particle scalar identical states gives unitarity for the scattering amplitude $T(s, t)$

$$2i \operatorname{Im} T(s, t) = \int T(s, t') T(s, t'')^* dt' dt'' + \sum_{n \geq 3} T_{2 \rightarrow n} T_{n \rightarrow 2}^* \quad (1.2.1)$$

where the operator product $\hat{T} \hat{T}^\dagger$ was expanded by inserting a complete basis of n -particle states $\mathbb{1} \sim \sum_{n \geq 3} |n\rangle \langle n|$ with on-shell phase-space integrals. From this equation we see directly that channels are coupled: in order to close the $2 \rightarrow 2$ system, one needs $2 \rightarrow 3$. But to close the $2 \rightarrow 3$ one could write an equation similar to the one above that would then require $3 \rightarrow 3$ and so on.

This state of affairs is also clearly reflected in hadronic physics experiments [47], where the scattering is mostly inelastic at high energies, see Fig. 1.1. In quantum gravity as well,

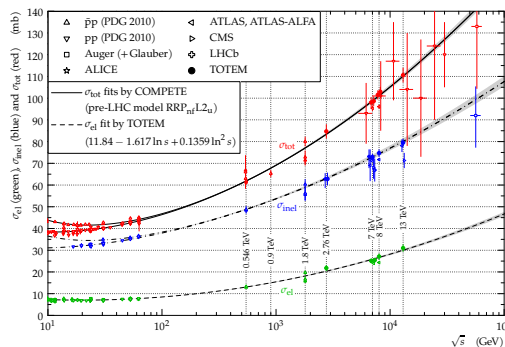


Figure 1.1: Elastic, and total, inelastic pp and $p\bar{p}$ cross sections versus $\sqrt{s}$. Reproduced from Fig. 4 of Ref. [48].

scattering is expected to become strongly inelastic at high energies, with large-multiplicity final states dominated by gravitational radiation [49–51]. At sufficiently small impact parameter, trans-Planckian collisions are expected to form a trapped surface and hence a black hole [52, 53], leading to strong absorption and suppression of the elastic channel [54]. This question is directly relevant for modern bootstrap studies where inelasticity has been hypothesized to play a role in nonperturbative bounds [43].

Some of the works in this thesis aim to provide an independent method to construct nonperturbative amplitudes with a mass gap, and develop a new constructive approach to the bootstrap that includes multi-particle production.

1.3 Summary of results

This thesis is about the practical implications of *analyticity*, *crossing*, and *unitarity* for UV completion. I describe works based on two complementary approaches.

⁴“Primal” constructions such as the ρ bootstrap or the constructive approaches explored in this thesis build amplitudes explicitly. “Dual” constructions typically rule out the possibility of certain observable to exists and do not necessarily give rise to a concrete amplitude.

- A **nonperturbative constructive ACU completion** viewpoint: work in dispersive variables where crossing is built in, and solve unitarity as a *nonlinear equation* (the Mandelstam equation). This makes particle production and spectral explicit and controlled objects that build up the amplitude.
- A **Regge/dual-model** viewpoint: work in the language of complex spin and meromorphic “stringy” amplitudes to understand what weakly coupled (tree-level) UV completions can look like, and what they *cannot* look like.

A recurrent theme in both parts is that extremal-looking answers often hide important physics in sectors that are not directly visible in a some $2 \rightarrow 2$ diagnostic. The manuscript organizes this into two tensions and a set of constructive numerical studies.

Part I: constructive amplitudes and controlled particle production. Chapter 2 fixes conventions and reviews quickly the ACU axioms. Chapter 3 then turns these axioms into a concrete program: solve unitarity through the Mandelstam equation and obtain explicit amplitudes. The main results are:

- results on numerical solvers (iterative [PT18, PT22] and gradient-descent/machine-learning [PT26, PT29]) for the Mandelstam equation in 3.2
- explicit four-dimensional solutions with particle production and controlled ACU completion and fine details (double-discontinuity support, fixed-angle decay, and an emergent nonperturbative Regge behavior).
- A proposal for a numerically solvable sub-sector with microscopic interpretation (2-particle-recursively reducible – 2PRR), with nonzero particle production.
- a sharpened understanding of the tension between the ϱ -bootstrap and Aks theorem.

In the text, I refer to this approach as the **Atkinson-Mandelstam (AM) bootstrap** (see section Glossary), in relation to the papers of Atkinson who proved [7–10] that the Mandelstam set-up admitted formally solutions and that inspired our iterative approach.

Controlled particle production. The works [PT18, PT22, PT26, PT29] produce scattering amplitudes that respect ACU in a form made precise below [6]. They manage to generate *numerically UV-complete* amplitudes with particle-production because they are based on a formulation, initially due to Mandelstam, which cleanly separates two-body physics from multi-particle production. In this formulation, multi-particle production is split in two parts:

- an iterated 2-body part, constructed directly (2PRR), and
- a genuinely multi-particle (MP) part, that is constrained by 2-particle unitarity indirectly.

In his original works, Mandelstam proposed to treat this genuine MP part would be fixed and the amplitude solved from it. This is the motto “scattering from production” put forward in [PT18, PT22].

In our works, we go beyond the works of Mandelstam in several ways, technical and conceptual (treatment of S -wave sector). In particular, in work [PT29], we depart from the motto above by using the genuine MP part as a variational degree of freedom of the amplitude, which can be adjusted so as to explore the space of ACU-consistent amplitudes

more efficiently. This is an important conceptual tool, which allows to scan over the space of all multi-particle data, while staying in a purely two-to-two framework.⁵

Thanks to this clean Mandelstam formulation and our numerical implementations, we are able i) produce ACU complete amplitudes and ii) explore the space of multi-particle production.

Details on numerical implementations are mostly avoided in this text, though we present a few discussion elements about the use of Machine-Learning methods in Sec. 3.2.2.1.

Note that, in practice, in [PT29], implementing this “variational multi-particle” was made easy by the use of machine-learning methods, but in principle it is not unthinkable that it could be adapted to other methods as well (iterative, semi-definite) though it would require specific development.

The Aks tension: extremality versus production. In the ϱ -bootstrap, extremal points often look nearly elastic in the resolved partial waves ($|S_J| \simeq 1$ over large energy windows). In $d > 2$, Aks’s theorem tells you this cannot be exact for all J and s , unless the amplitude is trivial. The conceptual resolution that we advocate for in the works reported on in this thesis is not mysterious: Aks’s theorem is qualitative, and production can be pushed into corners (higher spins, higher energies, or sectors not parameterized explicitly).

In the Atkinson-Mandelstam language, the hiding place is explicit: crossing turns elastic physics in one channel into an inelastic contribution in the crossed channel, and the only way to suppress its imprint on fixed spins is to tune the genuinely multi-particle data. This is what we call *Aks screening*. The constructive evidence are given in the comparison chapter, Chap. 7.1.

Elastic bounds across dimension and a locality diagnostic [PT28] Chapter 4 studies ϱ -bootstrap bounds as a function of spacetime dimension for high dimensions $d = 3, \dots, 12$ including half-integers. The main structural point that we had to solve is related to the fact that varying d changes what elastic unitarity allows at threshold, and extremal solutions actually use these options heavily. This shows up concretely as:

- two branches of extremal solutions and sharp kinks in $d = 5$ and $d = 7$ that correlate with changes in locality of threshold behavior;
- the breakdown (for $d \geq 5$) of naive Wilson-coefficient positivity unless one performs the appropriate IR subtractions;
- An efficient SDPB mathematica wrapper package was developed by S. Metayer for the project.

These results open several interesting perspective, in particular that of a large- d bootstrap for the branches after the kinks.

Part II: Regge theory, dual models, no-go’s. Part II (Chap. 5) uses dual models as explicit solutions of the S-matrix bootstrap where crossing and positivity are realized at tree-level (weak-coupling) by an given spectrum of infinite higher-spin exchanges together with their residues. The main takeaways are:

⁵Of course, this does not bypass the need to apply further $n \rightarrow m$ constraints to ultimately constrain the MP part, which is otherwise only constrained indirectly by the requirement that it does not violate 2-to-2 unitarity.

- explicit toy models of trajectory bending (Coon amplitude), together with their previously unappreciated non-meromorphy;
- a constructive “Regge bootstrap” where crossing is enforced pointwise and the Veneziano amplitude emerges as a distinguished dip in a tolerance landscape;
- two no-go theorems: under standard boundedness assumptions, a unitary meromorphic amplitude cannot have only finitely many Regge trajectories.

The no-go theorem directly relates to a tension observed in the recent literature on approaches to corner meromorphic S-matrices [55–59].

Other contributions (Part III). Part III discusses briefly works on (i) eikonal gravitational scattering and its reformulations, and (ii) pre-2020-works on ambitwistor string theory and double-copy formulation, belonging to “on-shell” scattering amplitudes methods.

Part I

ACU completion, particle production

Chapter 2

The S-matrix

道生一，一生二，二生三，三生物。

*“The Tao gives birth to One;
One gives birth to Two;
Two gives birth to Three;
Three gives birth to the ten thousand things.”*

— *Tao Te Ching*, ch. 42

Ancient texts are an inexhaustible source of wisdom, and it is not uncommon in the field of the modern S-matrix to re-discover long-forgotten facts.⁶ Going further back, a tongue-in-cheek reading of the epigraph above is a foreshadowing of the bootstrap: the Two, quantum mechanics and relativity, give rise to the Three: Analyticity, Crossing and Unitarity, which generate to the space of 10,000 consistent quantum relativistic interactions.⁷

There exist many detailed accounts on the S-matrix theory, both ancient and modern, and this chapter will not be a detailed account. It will solely collect useful definitions to later present results in a somewhat standalone manner. For more details, the reader can for instance refer to textbooks at [60], and the classic [61]. Another very useful reference are the 1969 lectures by Gribov collected in a book [62]. This book contains an interesting foreword by Dokshitzer and Frankfurt on Gribov’s works. Modern lectures also exist, such as [63, 64] and the *S-matrix toolkit* [65], whose conventions we mostly follow.

In Sec. 2.1 we introduce basic definitions (partial waves, double-discontinuity, Froissart-Gribov projection). In Sec. 2.2 we define briefly the axioms of analyticity, crossing and unitarity. Then in Sec. 2.3 we review Aks’s theorem.

In the works on which the thesis is based, there are a few new elements that could be considered as genuine S-matrix theory material itself, that are to our knowledge new are: non-positivity of Mandelstam kernel (technical point); a refinement of Aks’s theorem with inelasticity screening; Proof that dual-model amplitudes need to have infinitely many Regge trajectories (discussed in Chap. 5).

2.1 Definitions

In most of the thesis we discuss the scattering of 2 to 2 identical gapped scalars of mass squared m^2 . We write the S-matrix operator as in Eq. (1.1.5) $\hat{S} \equiv \mathbb{1} + i\hat{T}$, and define the

⁶See the Note added in [4].

⁷Five chapters later one also finds a useful warning when reading 1960s literature: *The further you go, the less you know.*

invariant $2 \rightarrow 2$ amplitude $T(s, t)$ from the connected matrix element as

$$\langle p_3, p_4 | \hat{T} | p_1, p_2 \rangle \equiv (2\pi)^d \delta^{(d)}(p_1 + p_2 - p_3 - p_4) T(s, t), \quad (2.1.1)$$

where p_i are the momenta of scattered particles $p_i^2 = m^2$, and $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_3)^2$, and $u = 4m^2 - s - t$. We further assume some $\mathbb{Z}_2$ symmetry so that there is no cubic self-interaction, and we assume no pole below threshold.

These assumptions are reasonable: firstly, they apply to neutral pions in the case of degenerate quark masses. Secondly, adding bound-states below threshold is usually a triviality. Lastly, adding spin sometimes does not change generic features (e.g. as in the work [PT27] where the graviton polarizations do not change the generic fact that dual-model amplitudes need to have infinitely many trajectories.) One thing stands out, however: the gap assumption. Without it, the notion of *elastic* unitarity vanishes, and the consequence of this for the method of Chap. 3 is not known. In the future, to describe QCD and mixed $\pi - K$ systems or attack quantum gravity amplitudes, the formalism will have to be extended. Some extensions are trivial (flavor), some less (gap).

2.1.1 Partial waves, double-discontinuity

Let us discuss the partial wave expansion, which will lead us to discuss analyticity domains and the double-discontinuity. We follow the conventions of [65].

2.1.1.1 Partial wave expansion

The partial wave expansion is a useful tool. Akin to a Fourier expansion of the amplitude, its primary purpose is to diagonalize the unitarity equation. In d -dimensions, for identical scalar particles we define

$$T(s, t) = \sum_{J=0, J \text{ even}}^{\infty} n_J^{(d)} f_J(s) P_J^{(d)}(z), \quad z = 1 + \frac{2t}{s - 4m^2}, \quad (2.1.2)$$

$$S_J(s) \equiv 1 + i \frac{\sqrt{s - 4m^2}}{\sqrt{s}} f_J(s),$$

where $z \equiv \cos(\theta)$ is the cosine of the scattering angle, and $P_J^{(d)}$ are the Gegenbauer polynomials

$$P_J^{(d)}(z) = {}_2F_1(-J, J + d - 3, \frac{d-2}{2}, \frac{1-z}{2}). \quad (2.1.3)$$

We also have the constants

$$n_J^{(d)} = \frac{(4\pi)^{\frac{d}{2}} (d + 2J - 3) \Gamma(d + J - 3)}{\pi \Gamma\left(\frac{d-2}{2}\right) \Gamma(J + 1)}. \quad (2.1.4)$$

In $d = 4$, the Gegenbauer polynomials are Legendre polynomials and the constant is just the usual $16\pi(2J + 1)$. The Gegenbauer polynomials satisfy orthogonality relations

$$\int P_J^{(d)}(z) P_{J'}^{(d)}(z) (z^2 - 1)^{d-4/2} dz = \frac{\delta_{J,J'}}{\mathcal{N}_d n_J^{(d)}}, \quad \mathcal{N}_d = \frac{(16\pi)^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}. \quad (2.1.5)$$

This expansion represents a change of basis from momentum eigenstates to angular-momentum eigenstates. In this basis, the unitarity relation Eq. (1.1.5) becomes diagonal:

$$\text{Elastic unitarity:} \quad |S_J(s)| = 1 \quad s \in [4m^2; 16m^2], \quad (2.1.6)$$

$$|S_J(s)| \leq 1 \quad s \in [16m^2; \infty], \quad (2.1.7)$$

Finally, orthogonality of the Gegenbauer polynomials yields the partial wave projection:

$$f_J(s) = \frac{\mathcal{N}_d}{2} \int_{-1}^1 (1-z^2)^{\frac{d-4}{2}} P_J^{(d)}(z) T(s, t(z)) dz, \quad t(z) = -\frac{s-4m^2}{2}(1-z). \quad (2.1.8)$$

In terms of f_J , unitarity reads

$$\text{Im } f_J(s) = \frac{(s-4m^2)^{(d-3)/2}}{\sqrt{s}} |f_J(s)|^2 \quad (2.1.9)$$

Convergence of the partial wave expansion Let us consider the amplitude at fixed s as a function of complex z . The physical domain is $z \in [-1; 1]$, but it is known from axiomatic QFT implies that for $s \geq 4m^2$ the amplitude is analytic in z inside a larger domain, the (*small*) *Lehmann ellipse* with foci at ± 1 , whose boundary is set by the first t -channel threshold $t = 4m^2$. The ellipse is characterized by $z = \cosh \alpha$ with

$$\cosh \alpha_0 = z|_{t=4m^2} = 1 + \frac{8m^2}{s-4m^2}. \quad (2.1.10)$$

This is proven in Lehman's paper [66] using the Jost-Lehman-Dyson machinery. Let us here assume analyticity and check consistency.

We follow Gribov's exposition here [62]. The partial wave expansion converges in the physical regime $s > 4m^2$, $t \leq 0$. It should cease to converge as t hits the first singularity, here $t = 4m^2$. For $t > 0$, as $s \rightarrow 4m^2$, we have $z \rightarrow \infty$ so that, loosely, $P_J^{(d)}(z) \sim_{J \rightarrow \infty} z^J$. For the partial wave sum $\sum n_J^{(d)} f_J(s) \frac{t^J}{(s-4m^2)^J}$ to converge for $t < 4m^2$ and diverge at $t = 4m^2$, using that $n_J^{(d)}$ is nothing approximately just a mildly growing function, a polynomial in J : $n_J^{(d)} \sim J^{d-3}$, the partial waves should decay with J like

$$f_J(s) \sim \frac{(s-4m^2)^J}{(4m^2)^J}, \quad s \rightarrow 4m^2, J \rightarrow \infty \quad (2.1.11)$$

but not faster for it needs to diverge at $t = 4m^2$.

The actual calculation from axiomatic QFT implies a more precise fall-off:

$$f_J(s) \sim C(s) e^{-J\alpha_0(s)} \quad (J \rightarrow \infty). \quad (2.1.12)$$

which near the threshold is identical to (2.1.11). One can then check that such a fall-off is consistent with the analyticity domain above, it is done as follows.

For complex z away from the cut $[-1, 1]$, the large- J behavior of Legendre polynomials is controlled by $\chi(z)$:

$$P_J^{(d)}(z) \sim \chi(z)^{J+1/2}, \quad (2.1.13)$$

Then the J -th term in the partial-wave series behaves as $e^{-J\alpha_0} |\chi(z)|^J$, hence the series converges absolutely whenever

$$|\chi(z)| < e^{\alpha_0(s)}. \quad (2.1.14)$$

Since level sets $|\chi(z)| = \text{const.}$ are precisely ellipses with foci ± 1 , Eq. (2.1.14) identifies the domain of analyticity of $T(s, z)$ obtained from the partial-wave expansion as the interior of an ellipse in the complex z -plane. This is the (*small*) *Lehmann ellipse* of above. For identical scalars the nearest crossed-channel singularity occurs at the t -channel threshold $t = 4m^2$, corresponding to

$$z_1(s) \equiv z|_{t=4m^2} = 1 + \frac{8m^2}{s-4m^2} > 1. \quad (2.1.15)$$

The small Lehmann ellipse is thus fixed by requiring that its rightmost point equals $z_1(s)$. Writing the ellipse as $z = \cosh(\alpha_0 + i\phi)$, this condition gives

$$\cosh \alpha_0(s) = z_1(s) = 1 + \frac{8m^2}{s - 4m^2}. \quad (2.1.16)$$

Ellipses. A convenient parameterization of ellipses with foci at ± 1 is provided by the Zhukovsky map

$$z = \frac{1}{2}(\chi + \chi^{-1}), \quad \chi(z) = z + \sqrt{z^2 - 1}, \quad (2.1.17)$$

where the branch of the square root is chosen so that $|\chi(z)| > 1$ for $z > 1$. For $\chi = re^{i\phi}$ with fixed $r > 1$ one finds

$$z(\phi) = \frac{1}{2}(re^{i\phi} + r^{-1}e^{-i\phi}) = \frac{1}{2}[(r + r^{-1})\cos\phi + i(r - r^{-1})\sin\phi], \quad (2.1.18)$$

which is an ellipse centered at the origin with semi-axes $a = \frac{1}{2}(r + r^{-1})$ and $b = \frac{1}{2}(r - r^{-1})$ and foci at ± 1 (since $a^2 - b^2 = 1$). Setting $r = e^\alpha$ gives $a = \cosh\alpha$ and $b = \sinh\alpha$, and the ellipse can be written as $z = \cosh(\alpha + i\phi)$ thanks to $\cosh(a + b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$.

The convergence of the absorptive part of the amplitude

$$T_s(s, t) := \text{Disc}_s T(s, t) = \frac{1}{2i}(T(s + I0, t) - T(s - I0, t)) = \frac{1}{2i} \text{Im } T(s, t) \quad (2.1.19)$$

is however guaranteed in a larger domain [67, 68]. Indeed, Gribov points out that if it converged in the same region, elastic unitarity for partial wave coefficients f_J Eq. (2.1.9) would be spoiled near the two-particle threshold, as it would read $(s - 4m^2)^J = \frac{\sqrt{s - 4m^2}^{d-3}}{\sqrt{s}}(s - 4m^2)^{2J}$ for all J which is impossible. Hence, the divergence for $\text{Im } A$ must extend beyond $t = 4m^2$, at an s -dependent location. The divergence occurs at the Landau or Karplus curve, and its location can be seen by inspecting the large- J growth of the Gegenbauer polynomials above in Eq. (2.1.13) The previous convergence argument for the partial wave expansion of the amplitude for $t \leq 4m^2$ reads

$$f_J(s) \sim e^{-J\alpha_0}, \quad \cosh(\alpha_0) = 1 + \frac{8m^2}{s - 4m^2} \quad (2.1.20)$$

Using that $\text{arccosh}(z) = \log(z + \sqrt{z^2 - 1})$, at large J we do recover (2.1.11) for $s \rightarrow 4m^2$. Then, using unitarity, we have that $\text{Im } f_J \sim e^{-2J\alpha_0}$, and thus the partial wave expansion of $\text{Im } T(s, t)$ ceases to converge when the exponential growth of the Legendre saturates the decay of the partial-wave coefficients, i.e. when

$$\alpha = 2\alpha_0 \quad (2.1.21)$$

This equation yields the Landau curve:

$$t = \frac{16sm^2}{s - 4m^2} = 16m^2 + \frac{64m^4}{s - 4m^2} \quad (2.1.22a)$$

The other Landau curve is found by crossing to be

$$t = \frac{4sm^2}{s - 16m^2} = 4m^2 + \frac{64m^4}{s - 16m^2} \quad (2.1.22b)$$

Just like $t = 4m^2$ marks the appearance of a threshold in the t -channel for the scattering amplitude $T(s, t)$, (2.1.22a) shows the emergence of t -channel discontinuity for the discontinuity of the amplitude $T_s(s, t)$:

$$\rho(s, t) \equiv \text{Disc}_s \text{Disc}_t T(s, t) \quad (2.1.23)$$

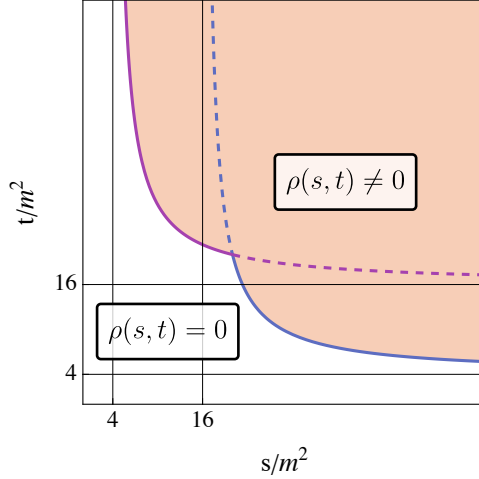


Figure 2.1: Support of double-discontinuity (orange region). The curves, purple and blue correspond to the LC Eq. (2.1.22) a) and b), respectively. The full envelope is determined by the union of both curves.

We thus encounter for the first time the *double-discontinuity* of the scattering amplitude, a tool of central importance for the results of Chap.3. Its domain of definition is depicted in Fig. 2.1.

Interestingly, the curves Eq. (2.1.22) have a very clear microscopic interpretation: they correspond to the actual locus of Landau singularities of the following ϕ^4 graphs, which we refer to as “Aks graphs” in this manuscript and in the papers. This implies that no process can generate a double-discontinuity before these graphs. This also shows a very interesting connection between perturbative, Feynman graph analysis and non-perturbative UV completion.



Figure 2.2: Aks graphs corresponding to Eq. (2.1.22)

In Chap. 3 we use an extension of this reasoning for the inelastic double-discontinuity ρ_{MP} . We reverse the logic above (axiomatic domain $\leadsto$ graphs) and use a conjecture on ϕ^4 graphs in [69] to define a domain for genuine inelastic processes with more than two-particle cuts in both channels.

2.1.2 Froissart-Gribov projection

Another useful tool is the Froissart-Gribov projection. It expresses the s -channel partial waves as an integral over the *crossed-channel* t and u cuts. This representation is the natural starting point for (i) analytic continuation in complex spin (Regge theory) and (ii) large-spin expansions. The basic input is an extended region of analyticity in the complex z -plane, which allows one to rewrite the partial-wave projection as a contour integral around the cut $z \in [-1, 1]$:

$$f_J(s) = \mathcal{N}_d \oint_{[-1,1]} \frac{dz}{2\pi i} (z^2 - 1)^{\frac{d-4}{2}} Q_J^{(d)}(z) T(s, t(z)) . \quad (2.1.24)$$

Here $Q_J^{(d)}(z)$ is the second independent solution of the d -dimensional Gegenbauer/Legendre Casimir equation, with its cut on $[-1, 1]$; explicitly it reads [65]

$$Q_J^{(d)}(z) = c_J^{(d)} z^{-J-d+3} {}_2F_1\left(\frac{J+d-3}{2}, \frac{J+d-2}{2}; J + \frac{d-1}{2}; \frac{1}{z^2}\right), \quad (2.1.25)$$

for an appropriate normalization constant $c_J^{(d)} = \frac{2^{-1-J} \sqrt{\pi} z^{3-d-J} \Gamma(\frac{-2+d}{2}) \Gamma(1+J)}{\Gamma(\frac{1}{2}(-1+d)+J)}$. Blowing up the contour in (2.1.24) gives integrals along the t - and u -channel cuts if the integral converges at infinity. For identical scalars, crossing symmetry removes the odd-spin sector and yields (for even J)

$$f_J(s) = \frac{2\mathcal{N}_d}{\pi} \int_{z_1}^{\infty} dz (z^2 - 1)^{\frac{d-4}{2}} Q_J^{(d)}(z) T_t(s, t(z)), \quad \text{Re } J > J_0(s), \quad (2.1.26)$$

where $J_0(s)$ is the Regge intercept controlling the large- $|t|$ growth, and

$$z_1 \equiv z|_{t=4m^2} = 1 + \frac{8m^2}{s - 4m^2}. \quad (2.1.27)$$

The function $T_t(s, t)$ denotes the discontinuity across the t -channel cut, defined in way similar to $T_s(s, t)$ in Eq. (2.1.19).

The Froissart–Gribov form (2.1.26) is well-suited for analytic continuation in J (Carlson-type uniqueness), making Regge poles manifest as singularities of $f_J(s)$ in the complex J -plane. It also makes transparent how the lowest crossed-channel thresholds control the large- J behavior and therefore the systematic large-spin expansion used throughout the bootstrap approach. For instance, at large J , $Q_J^{(d)} \sim \chi(z)^{-J-1}$ where $\chi(z) = z + \sqrt{z^2 - 1}$, and (2.1.26) is dominated by the end-point $z \rightarrow z_1$ $f_J(s) \sim \chi(z_1)^{-J-1}$.

2.2 Axioms

Most bootstrap methods find a way to “solve ACU”, but what exactly is solved differ and distinguishes methods. In the text I try to be precise on this. I also summarize the essential differences in Sec. 7.1.

2.2.1 Analyticity

Analyticity is the mathematical trace of locality/causality: in a local relativistic QFT, commutators vanish at space-like separation, and this constraint can be converted into analyticity domains for momentum-space correlators and scattering amplitudes. A useful physical entry point is the “signal model” discussion in [70] (see also Gribov’s lectures [71], ch. 2 for a more direct link to scattering amplitudes and subexponentiality). In this thesis, we will mostly use analyticity through its concrete implications, some mentioned in Sec. 2.1.1.1, and through a non-proven strong version of analyticity: Lightest Particle Maximal Analyticity.

[1] [Piotr; Cite Sommer \[72\]](#)

Lehmann and Lehmann–Martin ellipses. We have discussed the Lehmann ellipses above in the context of the convergence of the partial wave expansion. The minimal rigorous analyticity domains are the fixed- s analyticity in the complex z -plane inside the small Lehmann ellipse, summarized in our notation by Eq. (2.1.10) and reviewed in Sec. 2.1.1.1;

a proof based on the Jost–Lehmann–Dyson representation is due to Lehmann [66]. Unitarity enlarges the analyticity domain of the absorptive part to the *Lehmann–Martin ellipse*, as used implicitly in the discussion leading to the Landau/Karplus locus and the support of the double-discontinuity in Sec. 2.1.1.1; see Martin [67, 68] and [62], as well as the modern reference [65].

Besides complex analyticity, the amplitude satisfies *real analyticity*:

$$T(s^*, t^*) = T(s, t)^*, \quad (2.2.1)$$

which follows from hermiticity (and, in practice, from the fact that the amplitude is real below the first cut and admits Schwarz reflection across the real axis). In particular, for $s < 4m^2$ and $t < 4m^2$ T is real.

Assumptions used later: extended/maximal analyticity and polynomial boundedness. The rigorous results above provide nontrivial but limited analyticity domains. For the constructive bootstrap in Chap. 3, referred to as *maximal* analyticity, meaning that singularities are only those required by unitarity (cuts and poles) and their crossings. A concrete implementation is the Mandelstam representation used in Chap. 3. This assumption is not proven in general, and part of the discussion in this thesis concerns minimal extensions of axiomatic domains that would suffice for the dispersive bootstrap logic. A slightly stronger form that we assume is “Mandelstam analyticity” which on top assumes the existence of the double-dispersive Mandelstam representation, used in 3.

A further axiom we will use, especially in Chap. 5, is a high-energy growth bound at fixed t : polynomial boundedness,

$$|T(s, t)| \lesssim |s|^{N(t)}, \quad |s| \rightarrow \infty, \quad (2.2.2)$$

for some integer $N(t)$.

Meromorphy In the dual-model chapter on meromorphic amplitudes, Chap. 5, the analyticity reduces to the amplitude having only simple poles as singularities, with polynomial residues as an extra imprint of locality.

2.2.2 Crossing

Crossing symmetry is the statement that in and out states can be exchanged at the cost of transforming particles to anti-particles. It is proven to hold for $2 \rightarrow 2$ processes in [73, 74]. This is a highly technical subject, and in this work we just use its implication:

$$T(s, t) = T(t, s) \quad (2.2.3)$$

In Chap. II we present some results linked to dual models, which illustrate one non-trivial aspect of crossing symmetry, a situation where the infinite exchange of resonances in one channel reproduces the singularities of the cross-channel.

2.2.3 Unitarity

The statement of unitarity is easily phrased in terms of partial-wave unitarity, Eqs. (2.1.6), (2.1.7). In terms of the $2 \rightarrow 2$ amplitude, it can be also written as the non-linear relation Eq. (1.2.1). Elastic unitarity can be analytically continued in J , which is the content of the Mandelstam equation explain in Sec. 3.

The constraints of analyticity crossing and unitarity just explain underlie results like the Froissart–Martin bound, and they play a central role in our use of dispersive and Regge-inspired arguments.

2.3 Scattering implies production (Aks's theorem)

This section briefly recalls the content of Aks's theorem [45]

Theorem (Aks: scattering implies production). *Assume $d \geq 3$ and the analytic/crossing hypotheses above, including extended analyticity in a region large enough to establish (2.1.22) and (2.3.1). If the connected $2 \rightarrow 2$ amplitude is nontrivial, $T_{2 \rightarrow 2} \neq 0$, then not all inelastic amplitudes can vanish: there exists some $n > 2$ with $T_{2 \rightarrow n} \neq 0$.*

In Sec. 7.1.3, we propose a new take on the theorem based on an estimate argument provided recently in [PT29].

The idea is fairly simple and essentially stems from crossing symmetry. Consider a process such as the graph a) of Fig. 2.2. This graph has a 2-particle cut in the s -channel, hence no particle production. However, by $s \leftrightarrow t$ crossing, the graph b) should also be present and particle production is introduced this way. Aks's theorem makes this intuitive picture rigorous and non-perturbative.

Note: the argument fails in two-dimensions where kinematics is trivial and $t = 0$: particles on a line (1+1d) can only bounce back and forth, and $s \leftrightarrow t$ crossing does not exist to force this condition upon us. This allows 2d integrable S-matrices to exist, where it is well-known that while e.g. $3 \rightarrow 3$ can occur, $2 \rightarrow 4$ are zero, etc. A standard introduction is [75], giving rise to S-matrices that are pure phases $|S(s, u = 4m^2 - s)| = 1$ for all s .

Even though this material (and much more) is reviewed and covered in [65] Aks's paper actually contains a nice introduction to properties of the Mandelstam representation and analyticity properties of the amplitude.

Aks provides two proofs, one using Mandelstam representation, and one using a mild extension of analyticity, which he motivates *ad absurdum* by supposing no particle production. The Mandelstam representation proceeds by assuming that the amplitude is elastic, hence elastic unitarity as defined in Eq. (3.1.17) should hold for all $s \geq 4m^2$, but then by crossing we would not encounter the expected crossed Landau singularity, unless the amplitude vanished, and extend to all s by iterating the Landau singularities.

Mahoux–Martin positivity region A modern version of the proof is given in [65] and uses a variant, known as Mahoux-Martin positivity region, first described in [76]. Under this assumption, one can indeed show that ρ is *strictly positive* in a finite region above the leading Landau curve and up to what happens to be the next Landau curve:

$$\rho(s, t) > 0 \quad \text{for} \quad 4m^2 < s < 16m^2, \quad t_1(s) < t \leq \frac{4m^2(3s + 4m^2)^2}{(s - 4m^2)^2}. \quad (2.3.1)$$

which allows to conclude again, through *reductio ad absurdum*.

In the second part of his paper, Aks then relaxes the Mandelstam representation assumption and uses some method to extend analyticity region introduced in [77] to cover enough of s, t space to run his argument again.

Inelasticity screening. Let us anticipate on the discussion in Sec. 7.1.3. While Aks's theorem tells that *some* amount of particle production must be there, crucially it does not tell *how much*. In [PT29], we made three comments related to that aspect. Firstly, we empirically observed that adding inelasticity in the ϱ -bootstrap makes the bounds *a bit less extremal*. Secondly, we observed the converse statement: using our Atkinson-Mandelstam approach, we can screen inelasticity out in low-spin partial waves, and this makes bounds obtained this way *a little more extremal*. By screening inelasticity, we mean making a

certain number of low-lying partial waves S_J , $J = 0, \dots, J_{\max}$ purely elastic: $|S_J(s)| = 1$ for all $s \geq 4m^2$.

The basic idea, which we quantified in the paper, is that such screening can be neatly realized by a multi-particle double-discontinuity in the UV. What we show in [PT29] is that the cost for screening all partial waves for $J = 0, \dots, J_{\max}$ is an exponential growth of the double-discontinuity of the amplitude in the UV with $J_{\max}$, see Eq. (7.1.5). This implies that screening all inelasticity in this way would result in a singular amplitude, with double-discontinuity of infinite norm for any L_p norm.

Chapter 3

Solving unitarity: the Mandelstam equations

In this chapter, we describe the results of Tourkine and Zhiboedov [PT18, PT22] and Gumus, Leflot, Tourkine, and Zhiboedov [PT26, PT29]. These works were done in collaboration with Alexander Zhiboedov from CERN, and for the last two also with junior collaborators from LAPTh, Damien Leflot (PhD student) and Mehmet Asım Gümüş (post-doctoral researcher).

They aim to provide rigorous constructions of S-matrices satisfying the bootstrap axioms within a formulation initially suggested by Mandelstam [6] and for which solutions were proven to exist by Atkinson [7–10]. This was introduced as the Atkinson-Mandelstam bootstrap in the introduction. Our works provide realistic constructions with asymptotic unitarization, crossing symmetry and analyticity, and particle-production channels. This is a yet unfolding research program, with several important steps lying ahead, but already nontrivial results and proofs-of-concept achieved.

The main results presented in this chapter can be summarized as follows.

- **Tourkine and Zhiboedov [PT18] (2d warm-up).** In $d = 2$ the Atkinson-Mandelstam problem reduces to a one-variable nonlinear integral equation, providing a clean test-bed. It produces consistent amplitudes, where particle-production can be added. A naive fixed-point iteration converges only in a limited domain, while a Newton-Raphson scheme both accelerates convergence and enlarges the basin. The 2d-CDD ambiguity (see [4]) becomes concrete: for the same inelasticity input, multiple elastic fixed points exist and different seeds converge to different solutions, with fractal basins of attraction. Iterations fail near extremal/optimal solutions because the relevant Jacobians become singular.
- **Tourkine and Zhiboedov [PT22] (dispersive iteration in $d = 3, 4$).** We implemented the subtracted Mandelstam representation and Mandelstam equation (with an S -wave completion) as a practical bootstrap definition of a crossing-symmetric amplitude. The fixed-point iteration of the equations produces explicit solutions in $d = 3, 4$ with two-particle unitarity built in, and with inelasticity directly measurable in partial waves. We observe nontrivial logarithmic Reggeization in the spectral data and clear coupling dependence of the resulting amplitudes. Particle production shows up consistently with the Aks expectation in $d > 2$. When projected onto low-energy observables and compared to results from ρ -bootstrap, these constructive solutions lie very close to extremal boundaries. Iterations however fail to converge away from weak-mild coupling regime.

- **Gumus, Leflot, Tourkine, and Zhiboedov [PT26] (non-convex/ML solver, controlled toy model).** We introduced a gradient-based solver (neural network parameterization + gradient descent) to overcome the limited reach of iterative method. In this work we test it in a simplified system (“toy-model”) of the Atkinson-Mandelstam bootstrap equations with *vanishing double discontinuity*. This allows an efficient exploration of the allowed low-energy data and yields explicit UV completions to arbitrarily high energies, making asymptotic logarithmic effects directly visible. We also developed an analogue version of the ϱ -bootstrap to treat the same toy-model problem. The bounds match, but the asymptotic UV completion provided by the neural network approach is far superior. We conjectured that there exist a certain uniqueness of solutions with regular threshold boundary condition near some lower boundary of allowed space.
- **Gumus, Leflot, Tourkine, and Zhiboedov [PT29] (4d sub-sector with production).** We revisit the analysis of [PT22] with non-vanishing double-discontinuity with the machine-learning technology of [PT26]. The optimizer was pushed to the complete four-dimensional setting, though with still a single subtraction. We include and with particle production. In the paper, we focus on two aspects. Firstly, we produce UV complete amplitude in the “two-particle-recursively-reducible” class of amplitudes introduced in [PT29]. We manage to reach couplings 10 times larger than with the iterative method. Then we also introduce a new conceptual/technical tool: we make some amount of particle-production a variational degree of freedom, which allows to explore the space of amplitudes. In particular, we study a phenomenon we call “Aks screening” by which we remove inelasticity in low-lying partial waves. This allows to make contact with ϱ -bootstrap results and their apparent elasticity.
- **Aks tension / screening viewpoint (this chapter and sec.7.1).** In [PT29] we suggest and study concrete interpretation why SDPB optimizers often look numerically elastic in resolved partial waves: production can be pushed into corners of kinematics and/or into spins not captured by a given ansatz. The AM language provides a concrete diagnostic framework for locating where inelasticity is “hiding” (a practical notion of Aks screening). This is made possible in part through the use of a “variational” multi-particle degree of freedom to screen-away inelasticity in fixed partial-waves, which is a new element in this bootstrap.

Let us now, for the sake of presenting the main equations being solved, introduce a bit of formalism. The purpose of this section is mostly to introduce the equations (3.1.11), (3.1.12).

3.1 Mandelstam’s equation

In a seminal paper in 1958 [6] Mandelstam introduced a double-dispersive representation for massive scattering amplitudes, together with unitarity, and the suggestion for an iterative scheme to solve these unitarity equations. After a decade of unsuccessful attempts at solving it, David Atkinson in 1968-1970 formalized the iteration and proved in a series of theorems [7–10] that solutions exist rigorously.⁸ We review briefly the theorems below in 3.1.3.

⁸In [23], Chew and Mandelstam trivialized the unitarity equations by ignoring multi-particle production channels, the so-called “strip approximation” (by observing the 3-pion vertex does not exist, and hence the extension from $9m^2$ to $16m^2$ of the MP threshold helps to make the approximation less bad), solving just the leading partial wave in each isospin channel for $\pi - \pi$. Within the N/D representation [78, 79] yields non-linear equation, which they approach by iterative solutions. Overall, this approach fails for pion scattering (see the general account of Weinberg in [15]) because pions interact weakly at low

The most salient feature of this bootstrap is that the Mandelstam equation imposes unitarity for all spins J all at once. This is different from the ϱ -bootstrap that imposes unitarity spin by spin in its default implementation.⁹

3.1.1 Double-dispersive representation and unitarity

In the case of identical scalars, the Mandelstam representation is given by a double-dispersion relation:

$$\begin{aligned} T(s, t) &= c_0 + B(s, t) + B(s, u) + B(t, u), \\ B(s, t) &= \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\rho(s')}{s' - s} + \int_{4m^2}^{\infty} \frac{ds' dt'}{\pi^2} \frac{\rho(s', t')}{(s' - s)(t' - t)} \end{aligned} \quad (3.1.1)$$

where c_0 is just a constant. In this representation, $\rho(s, t)$ is the double-discontinuity of the amplitude, while the single discontinuity receives contributions from both $\rho(s)$ and $\rho(s, t)$, see Eq. (3.1.15).

For such a representation to make sense, the spectral functions $\rho(s), \rho(s, t)$ need to decay at infinity. If they do not, one can add *subtraction* terms that replace for instance

$$\begin{aligned} \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\rho(s')}{s' - s} &\mapsto (s - s_0)^N \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{\rho(s')}{(s' - s)(s' - s_0)^N}, \\ \int_{4m^2}^{\infty} \frac{ds' dt'}{\pi^2} \frac{\rho(s', t')}{(s' - s)(t' - t)} &\mapsto (s - s_0)^N (t - t_0)^N \int_{4m^2}^{\infty} \frac{ds' dt'}{\pi^2} \frac{\rho(s', t')}{(s' - s)(t' - t)(s' - s_0)^N (t' - t_0)^N} \end{aligned} \quad (3.1.2)$$

for N subtractions and, likewise in s, t for the $\iint ds' dt'$ term, see e.g. [5]. With N subtraction, the amplitude is supplemented with a free polynomial of degree $N - 1$ symmetric in $s - t - u$. Thus, with at least one subtraction, the constant c_0 is free, otherwise it is fixed by the spectral functions in terms of some dispersion relations.

Remark. *The representation of the ϱ -bootstrap trivially satisfies the Mandelstam representation with one subtraction, as the amplitudes decays to a constant at infinity in any direction of the complex plane. See the discussion around Eq. (4.1.3).*

In general, the Froissart bound would suggest two subtractions are needed. However, due to the fact that the amplitude at positive t might have Regge growth $T(s, t) \sim (-s)^{\alpha(t)}$, one might need more. This relates to the open question of whether Regge trajectories grow forever or not. If they do, and $\alpha(t) \sim t$ asymptotically, it is clear that a cutoff is necessary to define this representation since for any fixed number of subtractions N , there would exist a t large enough that beats it otherwise.

This representation assumes and makes explicit maximal analyticity: the only singularities of the amplitude are those dictated by unitarity: this amplitude is a fully analytic function in s, t, u away from the cuts $s, t, u \geq 4m^2$. Since the double-discontinuity vanishes below the leading LCs, the effective lower bound for the integral in Eq. (3.1.1) is not $4m^2$ where $\rho(s, t)$ vanishes, but instead the leading Landau curves shown in Fig. 2.1.

Let us quickly skim through the derivation of the Mandelstam equation, which is the unitarity equation written for the double-discontinuity. We refer to [65] for more details. The original Aks paper is also a nice reference [45] on this formalism.

energies. Before our works, to our knowledge, there was only a single, inconclusive attempt at a numerical solution [80].

⁹Subtracted positivity constraints introduced in [30, 42, 43] do impose some amount of unitarity for all spins.

In the text we always at least one subtraction, we set the crossing symmetric point $s_0 = t_0 = 4/3m^2 (= u_0)$ as the base for the subtractions, the crossing symmetric point, which defines a non-perturbative quartic coupling:

$$T(s_0, t_0, u_0) = c_0 \quad (3.1.3)$$

This constant is also the first term in the low energy Taylor expansion of the amplitude around the crossing symmetric point:

$$T(s, t, u) = c_0 + c_2((s - s_0)^2 + (t - t_0)^2 + (u - u_0)^2) + \dots \quad (3.1.4)$$

where we have defined another non-perturbative low energy quantity, c_2 , an analogue of derivative quartic coupling, which we use later in the text to map the space of consistent amplitudes.

Elastic unitarity and the s -discontinuity. Consider the unitarity relation Eq. (1.2.1). In the elastic strip $4m^2 < s < 16m^2$, the particle production processes cannot take place hence the equation closes on $2 \rightarrow 2$ states and yields the nonlinear elastic unitarity relation

$$T_s(s, z) = \frac{(s - 4m^2)^{(d-3)/2}}{8(4\pi)^{d-2}\sqrt{s}} \int_{-1}^1 dz' \int_{-1}^1 dz'' \mathcal{P}_d(z, z', z'') T^{(+)}(s, z') T^{(-)}(s, z''), \quad (3.1.5)$$

where $T^{(\pm)}(s, z) \equiv \lim_{\epsilon \rightarrow 0} T(s \pm i\epsilon, z)$. The kernel $\mathcal{P}_d$ comes from the intermediate two-particle phase space and contains Heaviside Θ function of z (Dirac delta in $d = 3$); hence $\mathcal{P}_d$ is *not* analytic as a function of z and (3.1.5) is not suitable for analytic continuation in t as such.

Cauchy representation and the Mandelstam kernel. Inside the small Lehmann ellipse, $T^{(\pm)}(s, z')$ is analytic in z' , so for $-1 < z' < 1$ we can write

$$T^{(\pm)}(s, z') = \oint_{[-1, 1]} \frac{d\eta'}{2\pi i} \frac{T^{(\pm)}(s, \eta')}{\eta' - z'}, \quad (3.1.6)$$

and similarly for z'' . Inserting (3.1.6) into (3.1.5) and interchanging integrals defines the *Mandelstam kernel*

$$K_d(z, \eta', \eta'') \equiv \int_{-1}^1 dz' \int_{-1}^1 dz'' \frac{\mathcal{P}_d(z, z', z'')}{(\eta' - z')(\eta'' - z'')}, \quad (3.1.7)$$

and allows to rewrite unitarity as

$$T_s(s, z) = \frac{(s - 4m^2)^{(d-3)/2}}{8(4\pi)^{d-2}\sqrt{s}} \oint \frac{d\eta'}{2\pi i} \oint \frac{d\eta''}{2\pi i} T^{(+)}(s, \eta') T^{(-)}(s, \eta'') K_d(z, \eta', \eta''). \quad (3.1.8)$$

For $|\eta'|, |\eta''| > 1$ the denominators in (3.1.7) do not vanish on $z', z'' \in [-1, 1]$, so the transform (3.1.7) continues $\mathcal{P}_d$ into an analytic function of z in a neighborhood of the physical segment $z \in [-1, 1]$. The only obstruction is a cut in z starting at the two-particle branch point

$$\eta_+(\eta', \eta'') \equiv \eta' \eta'' + \sqrt{\eta'^2 - 1} \sqrt{\eta''^2 - 1}, \quad (3.1.9)$$

so that $K_d(\cdot, \eta', \eta'')$ is analytic for $z < \eta_+(\eta', \eta'')$.

Double discontinuity and Mandelstam's equation. We have now a version of unitarity that has been continued in t , and we can finally write the Mandelstam equation on the double discontinuity (which required going to a regime where $t \geq 4m^2$). Since (3.1.8) is analytic in z around $[-1, 1]$, we can analytically continue in t and take Disc_t ; the discontinuity acts only on the kernel:

$$\rho(s, t) = \frac{(s - 4m^2)^{(d-3)/2}}{4\pi^2(4\pi)^{d-2}\sqrt{s}} \iint_{z_1}^{\infty} d\eta' d\eta'' T^{(+)}(s, \eta') T^{(-)}(s, \eta'') \text{Disc}_t K_d(z, \eta', \eta''). \quad (3.1.10)$$

This is the Mandelstam equation: an integral over t -channel discontinuities weighted by $\text{Disc}_z K_d$, whose support is restricted by the condition $z > \eta_+(\eta', \eta'')$. **It enforces elastic unitarity analytically continued in the complex J plane for all J such that $\text{Re } J > 0$.**

Beyond the elastic strip. This equation was derived in the elastic strip $s \in [4m^2; 16m^2]$. After the first multi-particle threshold, unitarity adds positive $2 \rightarrow n$ contributions to $\text{Disc}_s T$. Crossing-symmetry allows to decompose neatly these contributions into a crossed-elastic term and a remaining multi-particle contribution:

$$\rho(s, t) = \rho_{\text{el}}(s, t) + \rho_{\text{el}}(t, s) + \rho_{\text{MP}}(s, t), \quad (3.1.11)$$

where $\rho_{\text{el}}(s, t)$ is defined by the right-hand side of the Mandelstam equation:

$$\boxed{\rho_{\text{el}}(s, t) \equiv \frac{(s - 4m^2)^{(d-3)/2}}{4\pi^2(4\pi)^{d-2}\sqrt{s}} \iint_{z_1}^{\infty} d\eta' d\eta'' T^{(+)}(s, \eta') T^{(-)}(s, \eta'') \text{Disc}_t K_d(z, \eta', \eta''),} \quad (3.1.12)$$

while ρ_{MP} encodes additional multi-particle input. A priori, $\rho_{\text{el}}(s, t)$ needs to obey (3.1.12) only in the elastic strip. Since $\rho_{\text{MP}}(s, t)$ is *a priori* free to take arbitrary values for $s, t \geq 16m^2$, we can if we want define ρ_{el} from (3.1.12) for all values of s, t above the leading-LCs (2.1.22).

Assumption 1. In [PT29], we define ρ_{el} as above for all s, t . This becomes a non-trivial assumption because we furthermore impose that ρ_{MP} is not defined exactly for $s, t \geq 16m^2$, but rather after an inelastic LC of the planar cross-box graph (left graph in Fig. 3.5) worked-out in [69].

The situation is depicted in Fig. 7.1, where it is clear that there is a non-empty space between the envelope of ρ_{MP} and the lines $s = 16m^2$ and $t = 16m^2$.

This assumption implies a stronger analytic structure, with a refinement of the Mandelstam analyticity. It was motivated below Fig. 2.2 using graph theory arguments from [69].

3.1.1.1 A lemma: non-positivity of Mandelstam kernel.

The support restriction in Eq. (3.1.10) yields a compact, hyperbola-shaped domain, depicted in Fig. 3.1. In [PT29] we made the point that the quadratic form defined by the r.-h.-s. of Eq. (3.1.10) is not sign-definite. This is trivially seen by taking functions f_i compactly supported around a pair of points η_i depicted in the figure. With these pairs, the map $\iint \text{Disc } K f_2 f_2$ vanishes, hence the Cauchy-Schwartz inequality $(\iint \text{Disc } K f_1 f_2)^2 \leq \iint \text{Disc } K f_1 f_1 \iint \text{Disc } K f_2 f_2$ can simply not hold.

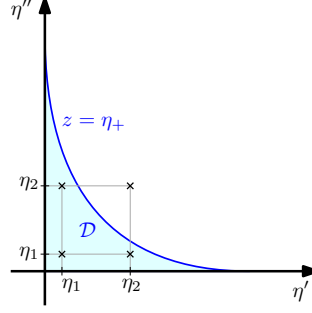


Figure 3.1: Typical integration domain of Mandelstam equation (blue region). Taking test functions supported around the points η_i allows to see that Cauchy-Schwartz cannot hold.

3.1.2 The Atkinson-Mandelstam bootstrap

Let us summarize the equations that define the Atkinson-Mandelstam bootstrap. They can be written for any number of subtractions. In [PT22, PT26, PT29] we studied the once-subtracted case but in the near future we will extend this to higher number of subtractions to treat the full 4d problem. We indicate in blue the three variables not fixed by the algorithm a priori: the subtraction constant c_0 and the genuine multi-particle data $\rho_{\text{MP}}(s, t), \eta(s, t)$.

Atkinson–Mandelstam bootstrap: double-discontinuity

Subtracted Mandelstam representation.

$$T(s, t) = c_0 + B(s, t) + B(s, u) + B(t, u), \quad s + t + u = 4m^2, \quad (3.1.13)$$

where $B(s, t)$ is defined with any number of subtractions, with $s_0 = t_0 = u_0 = \frac{4m^2}{3}$ and $c_0 \equiv T(s_0, t_0)$. The unknown spectral data are the *single* spectral density $\rho(s)$ and the *double* spectral density $\rho(s, t)$.

Double discontinuity and its decomposition.

$$\rho(s, t) = \rho_{\text{el}}(s, t) + \rho_{\text{el}}(t, s) + \rho_{\text{MP}}(s, t), \quad \rho_{\text{MP}}(s, t) = \rho_{\text{MP}}(t, s). \quad (3.1.14)$$

t -channel discontinuity.

$$T_t(s, t) = \rho(t) + \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{(s - s_0)}{(s' - s)(s' - s_0)} \rho(s', t) + \int_{4m^2}^{\infty} \frac{du'}{\pi} \frac{(u - u_0)}{(u' - u)(u' - u_0)} \rho(u', t). \quad (3.1.15)$$

Mandelstam equation (elastic double discontinuity). Use

$$z \equiv 1 + \frac{2t}{s - 4m^2}, \quad t(\eta) \equiv \frac{s - 4m^2}{2}(\eta - 1), \quad z_1 \equiv 1 + \frac{8m^2}{s - 4m^2}, \quad \eta_{\pm} \equiv \eta' \eta'' \pm \sqrt{\eta'^2 - 1} \sqrt{\eta''^2 - 1}. \quad (3.1.16)$$

Then $\rho_{\text{el}}(s, t)$ is defined by

$$\rho_{\text{el}}(s, t) = \frac{\sqrt{s - 4m^2}}{(4\pi)^2 \sqrt{s}} \int_{z_1}^{\infty} d\eta' \int_{z_1}^{\infty} d\eta'' \frac{\Theta(z - \eta_+) T_t(s + i0, t(\eta')) T_t(s - i0, t(\eta''))}{\sqrt{(z - \eta_-)(z - \eta_+)}}. \quad (3.1.17)$$

We further use the content of the assumption 1, so that Eq. (3.1.17) holds for all s, t above the LCs and ρ_{MP} is nonzero above the planar cross-box LC.

The contents of this first box were discussed above. The second box relates to a technical aspect discussed in [PT22, PT29], which relate to spin-0 unitarization, on which

I only give a few details below.

Atkinson–Mandelstam S -wave completion

Spin-0 unitarity (with general S-wave inelasticity input $\bar{\eta}_{\text{MP}}$).

$$f_0(s) \equiv \frac{1}{16\pi} \int_0^{2\pi} d\theta \, T\left(s, -\frac{s-4m^2}{2}(1 - \cos \theta)\right), \quad (3.1.18)$$

$$\mathrm{Im} f_0(s) = \sqrt{\frac{s-4m^2}{s}} |f_0(s)|^2 + \sqrt{\frac{s}{s-4m^2}} \eta_0(s), \quad \eta_0(s) \equiv 1 - |S_0(s)|^2. \quad (3.1.19)$$

The total S-wave inelasticity is decomposed as

$$\eta_0(s) = \eta_{2\text{PRR}}(s) + \bar{\eta}_{\text{MP}}(s), \quad (3.1.20)$$

$$\eta_{2\text{PRR}}(s) = \frac{1}{4\pi^2} \sqrt{\frac{s-4m^2}{s}} \int_{z_1}^{\infty} dz \tanh^{-1}\left(\frac{1}{z}\right) \rho_{\text{el}}(t(z), s), \quad (3.1.21)$$

$$\bar{\eta}_{\text{MP}}(s) = \eta_{\text{MP}}(s) + \frac{1}{4\pi^2} \sqrt{\frac{s-4m^2}{s}} \int_{z_1}^{\infty} dz \tanh^{-1}\left(\frac{1}{z}\right) \rho_{\text{MP}}(s, t(z)). \quad (3.1.22)$$

Inelastic unitarity for higher spins (above the 4-particle threshold).

$$|S_J(s)| \leq 1, \quad \forall s > 16m^2, \quad \forall J > 0. \quad (3.1.23)$$

In the papers [PT22, PT29] we solved the full system of equations, while in [PT26] we solved a toy-model introduced in [PT22] where the double-discontinuity is set to zero.

What these bootstrap equations compute The boxed system (3.1.13)–(3.1.23) should be read as a *constructive definition* of a nonperturbative $2 \rightarrow 2$ amplitude. These equations solve unitarity in a crossing-symmetric manner, together with exact analyticity of the amplitude. Pick a subtraction constant (here c_0) and multi-particle data $\eta_{\text{MP}}(s), \rho_{\text{MP}}(s, t)$ (possibly fixed to 0). The remaining spectral data—the single spectral density $\rho(s)$ and the elastic double discontinuity $\rho_{\text{el}}(s, t)$ —are then fixed by the nonlinear constraints of two-particle unitarity (the Mandelstam equation (3.1.17)) together with crossing built into the Mandelstam representation. Pictorially, the functional equation is represented in Fig. 3.2.

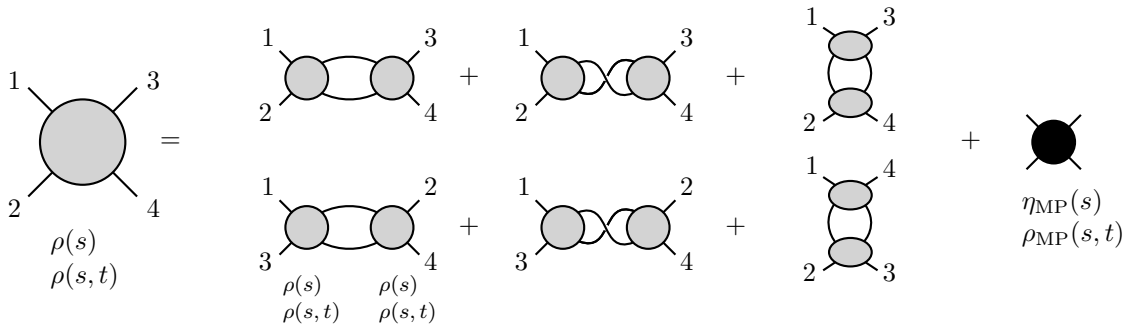


Figure 3.2: Pictorial representation of the AM bootstrap equations.

In this way, it is apparent that the quadratic nature of (3.1.17) is literally a sewing operation, together with an extra insertion of the multi-particle data.

Uniqueness The question of the uniqueness of the solutions to these equations when given all the input data $(c_0, \eta_{\text{MP}}, \rho_{\text{MP}})$ is important and discussed most recently in [PT29].

In short, in this paper we made the point that solutions cannot be unique. However, we conjectured that a sub-sector that we can treat with our current technique, defined by the requirement of regular threshold boundary conditions and positive c_0 , is unique and corresponds to deformations of a class of amplitudes which we call 2-particle-recursively-reducible amplitudes introduced below.

3.1.2.1 2PRR sub-sector: $\rho_{\text{MP}}, \eta_{\text{MP}} \equiv 0$.

A particularly sharp limit is obtained by setting the genuine multi-particle data $\rho_{\text{MP}}(s, t) = \eta_{\text{MP}} = 0$. The AM bootstrap equations then define a closed system of equations, with only external input the subtraction constant c_0 , that satisfy ACU in the AM sense.

Considering Fig. 3.2, we see that the solution to these equations is made of a specific class of planar graphs: the *two-particle-recursively-reducible* (2PRR) graphs. These are graphs that can be reduced to a single effective vertex by repeatedly cutting two internal lines; equivalently, they are built recursively by gluing two smaller graphs across a two-particle cut, together with all crossing images, as in Fig. 3.2.

G is 2PRR $\Leftrightarrow \exists$ 2PRR graphs G_1, G_2 , and
a 2-particle cut through G such that:

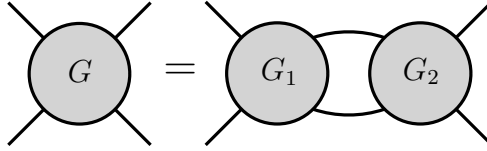


Figure 3.3: Fig. 1 of [PT29]: definition of the 2PRR class as planar graphs reducible by repeated two-particle cuts (and their crossing images).

Seen iteratively, in this subsector, the unitarity equation glues two discontinuities $T_t T_t$ across an on-shell two-particle state with crossing, generating the next layer of 2PRR contributions. Examples of 2PRR graphs are shown in Fig. 3.4. Crucially, turning off the

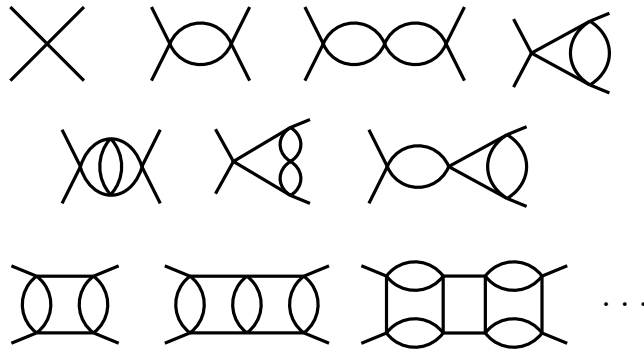


Figure 3.4: Fig. 1 of [PT29]: definition of the 2PRR class as planar graphs reducible by repeated two-particle cuts (and their crossing images).

multi-particle data $\rho_{\text{MP}}(s, t), \eta_{\text{MP}}$ does not shut inelasticity off. Most 2PRR graphs have multi-particle cuts in one channel, as can be seen in the examples. In particular, the two Aks graphs of Fig. 2.2 are 2PRR graphs and do contribute to inelasticity. In Tourkine and Zhiboedov [PT22], we even compared the partial waves of 2PRR amplitudes to perturbative expansion and observed a good match near the threshold to these 3-loop graphs.

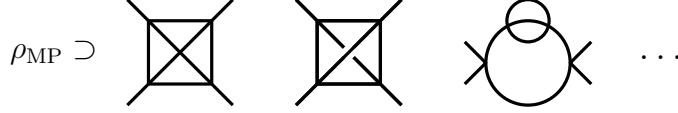


Figure 3.5: Fig. 2 of [PT29]: interpretation of $\rho_{\text{MP}}(s, t)$ as the multi-particle sector, and the statement that the 2PRR model corresponds to setting $\rho_{\text{MP}}(s, t) = 0$.

Interestingly, this class of graphs has been studied in [81]. In a sense, it seems that our numerical procedure defines a non-perturbative resummed amplitude whose microscopic definition correspond to this graph sum. We elaborated on this in more details in [PT29]. These amplitudes are parameterized from a single degree of freedom: c_0 , and we conjectured that they are located close to the extremal lower-boundary of the allowed space as described in [29, 30].

Finally, one can wonder: can these amplitudes be realized in some actual scalar Lagrangian theory? It is obvious that it cannot simply be ϕ^4 ; firstly the theory cannot exist in $d = 4$, but also it has a lot more graphs. The 2PRR graphs are planar, which firstly suggest a sort of large N limit. But there are fewer 2PRR graphs than planar graphs, see for instance the “planar cross-box” in Fig. 3.5 is a planar graph but not a 2PRR graph. Therefore, there must exist a further flavor selection of vertices. At the moment, we have not identified such a theory (see further discussion elements in [PT29]).

$\rho_{\text{MP}}(s, t)$ as an extra interaction. Turning on $\rho_{\text{MP}}(s, t)$ goes beyond the 2PRR closure. The key point is that ρ_{MP} is *not* produced by repeated two-particle sewing: it captures physics that is “multi-particle in all channels” and, in perturbation theory, it is precisely what is contributed by two-particle-irreducible topologies [PT22, PT29], as in Fig. 3.5.

From the point of view of the bootstrap equations, ρ_{MP} therefore acts like an additional, nonlocal interaction: it injects multi-particle information into the dispersive data, while (3.1.17) continues to resum the two-particle-reducible skeleton around it.

This separation between two-body and genuine multi-particle inelasticity is a very powerful refinement, made possible only through the use of this whole AM formalism. The split of ρ in Eq. (3.1.11) in particular is instrumental. Overall, the formalism allows to capture some inelasticity from first principle (the 2PRR skeleton), and leaves the remaining inelasticity very cleanly defined.

In the papers, we use two strategies to treat that remaining part.

- *Iterative Atkinson–Mandelstam solvers* take $(\lambda, \eta_{\text{MP}}(s), \rho_{\text{MP}}(s, t))$ as *fixed* external inputs and iterate the unitarity/dispersive map to generate the corresponding amplitude [PT18, PT22]. This is the logic of **scattering from production**.
- *Neural-optimizer solvers* [PT26, PT29] treat the same constraints as a loss minimization problem. This makes it possible to turn the multi-particle inputs $\rho_{\text{MP}}(s, t)$ as a **new variational degree of freedom** which can be used during the gradient descent in order to explore the solution space. We discuss some details of the ML implementation in Sec. 3.2.2.1.

3.1.3 Quick discussion on Atkinson’s theorems

Let us quickly review Atkinson’s theorems. In [7–10] D. Atkinson proved that the equations above admitted solutions, taking inspiration from the idea of Mandelstam iterations and specifically [82].¹⁰ The main difficulty in the proofs lie in the fact that he needed to

¹⁰Iterative fixed-point methods were in the air for a long time [83].

tailor specific decay conditions to control all asymptotic tails in the integrals. A good introduction to these proofs are the lecture notes by Atkinson [84].

The proofs basically go as follows. Rewrite the Mandelstam equations as a non-linear mapping of the form

$$\vec{\rho} = \Phi [\vec{\rho}] \quad (3.1.24)$$

where

$$\Phi [\vec{\rho}] \sim \iint (K \rho^2) + \vec{\rho}_{\text{MP}}, \quad (3.1.25)$$

where the vector notation is short for the collection $\rho(s), \rho(s, t)$ and likewise for $\vec{\rho}_{\text{MP}}$ which is short for $\eta_{\text{MP}}(s), \rho_{\text{MP}}(s, t)$.

Define a suitable Banach space of functions on which the nonlinear map acts. In his papers Atkinson chooses Hölder-continuous functions.¹¹ Then show that the map is continuous. The non-trivial step is to show the map sends a given ball into itself. If yes, one invokes the Leray–Schauder fixed-point principle to show that at least one solution exist. This is just the infinite dimensional analogue of the Brouwer fixed point theorem. In papers ii, iii, iv, Atkinson further uses the contractibility of the map Φ to show that the iteration defined by $\vec{\rho}_{n+1} = \Phi [\vec{\rho}_n]$ converges to a unique solution in some neighborhood. Most of the time, the proofs conclude to the existence of solution near the origin, for small amplitudes. In addition, Atkinson often needs to assume some specific forms of cutoffs and decay conditions that are glued by hand on the functions. In some cases, he can prove that not only elastic unitarity is satisfied, but also inelastic unitarity.

Comparison to our approach. Our implementation is slightly more general than the one used by Atkinson in his proofs.

Firstly, at the level of the equations themselves, we add the S-wave backreaction defined in our second box above. Another difference is that we define ρ_{el} throughout the whole $s, t \in [4m^2, +\infty[$ domain. We have to check inelastic unitarity by hand afterwards.

In the appendix of [PT22] we produced a proof of Atkinson contractibility for the toy model Atkinson-Mandelstam problem where one sets to zero and ignore the double-discontinuity. In this proof we saw that contractibility can be proven in our framework for $3 < d < 4$ with difficulties arising in integer d , related to the necessity of technical cutoff procedure introduced by Atkinson.

Atkinson’s theorems.

Theorem 1 (Atkinson i: neutral pions, no subtractions [7]). *Consider the unsubtracted Mandelstam representation for the neutral channel, with double spectral function $\rho(s, t) = \rho(t, s)$. Fix an “inelastic input” $\rho_{\text{MP}}(s, t)$ in a bounded Hölder class S_B , supported away from the elastic strips at $s, t > 16m^2$. If $B \in \mathbb{R}_+$ is sufficiently small (and ρ_{MP} obeys additional positivity constraints used to control the inelastic region), then there exists at least one solution $\rho(s, t)$ of the crossing–unitarity fixed-point equation, hence an amplitude $T(s, t)$ such that:*

- *T has full crossing symmetry and satisfies the unsubtracted Mandelstam representation;*
- *elastic unitarity holds for $4 \leq s \leq 16$, and the inelastic inequalities hold for $s \geq 16$.*

The proof uses a continuous map of a compact convex Hölder set and the Leray–Schauder fixed-point principle.

¹¹One cannot simply use continuous functions because of the principal value integrals that enter dispersion relations, that can turn continuous functions into non-continuous functions.

Theorem 2 (Atkinson ii: charged pions, no subtractions, local uniqueness [8]). *Extension of Theorem 1 to the isospin-coupled pion system. With additional positivity constraints on v , the inelastic unitarity is enforced as well. Proof of local uniqueness with contracting map.*

Theorem 3 (Atkinson iii: finitely many subtractions [9]). *Extension to add subtractions. Local uniqueness.*

Theorem 4 (Atkinson iv: nearly constant cross-sections [10]). *Proof of existence of solutions with nearly constant asymptotic total cross sections*

$$\sigma(s) \sim (\log s)^{-2-\epsilon}, \quad \epsilon > 0 \quad (3.1.26)$$

as $s \rightarrow \infty$.

The proof introduces better norms, and allows for non-trivial growth behavior. It would be interesting to see if we can borrow some techniques from this work in our numerical implementation.

3.2 Results in [PT18, PT22, PT26, PT29]: Numerical solutions

Section 3.1 gives a concrete “bootstrap” definition of a crossing-symmetric $2 \rightarrow 2$ amplitude through the Mandelstam representation and the Mandelstam equation, supplemented by an S -wave completion.

Below, we build explicit solutions and then inspect what they do (particle production, Landau support, high-energy behavior, proximity to numerical bootstrap boundaries, ...). Precise algorithms and further details are given in the original papers [PT18, PT22, PT26, PT29]; here I only keep the main takeaways.

Three outcomes.

1. **Atkinson–Mandelstam iterations become practical.** Formal fixed-point iterations produce actual solutions in $d = 2, 3, 4$. We do observe that convergence of fixed-point method fails beyond weak-coupling, confirming Atkinson’s picture.
2. **Non-convex optimization works in function space.** Neural optimizers can solve the AM systems without the narrow convergence domains of fixed-point maps, and produce numerically UV-completed amplitudes with particle production.
3. **Production can be hidden at low spin.** Explicit solutions make the “Aks tension” operational: low-spin inelasticity can be screened by the multi-particle sector, and this screening correlates sharply with ϱ -bootstrap extremality.

A short discussion on ML methods is given in 3.2.2.1. A compact comparison between the ϱ -bootstrap and the AM approach is given in section 7.1 in the discussion chapter.

3.2.1 Iterative solvers

The first numerical strategy we followed in [PT18, PT22] was the iterative strategy. Start from a seed, and turn the map Φ above as an iteration device. Pictorially, this amounts to turning Fig. 3.2 into the iterative picture described in Fig. 3.6.

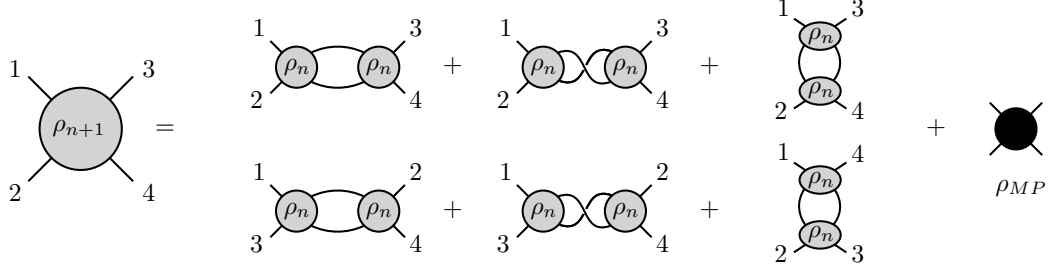


Figure 3.6: Diagrammatic interpretation of the AM iteration and its basic inputs $(\lambda, \eta_{\text{MP}}(s), \rho_{\text{MP}}(s, t))$. See Tourkine and Zhiboedov [PT18, PT22].

3.2.1.1 $d = 2$: a clean benchmark [PT18].

In 2d, the AM bootstrap simplifies drastically due to the kinematic fact that there is no momentum transfer on the line (one spatial dimension), hence $t = 0$, leaving only s as kinematic invariant and $u = 4m^2 - s, t = 0$. Unitarity+dispersion relation then becomes a simple quadratic equation of the form $f(x) = \int K(x, x') f(x')^2 dx'$. Most of the numerical techniques mentioned above can thus be tested in this simpler context. The paper [PT18] was our entry point in the Atkinson-Mandelstam problem.

At the same time, 2d is also a subtle place where to investigate particle production since it is the only place where S-matrices with zero-particle production exist, corresponding to integrable models. One can nevertheless add particle-production by hand, but it turns out that in 2d, the problem of writing an S-matrix given particle production is actually *solved*. Given an inelasticity profile function, explicit S-matrices with that exact inelasticity can be written [4], see Eq. (2.10) of [PT18]. This gives a perfect place where to test the Atkinson-style iterations. In Tourkine and Zhiboedov [PT18] we found:

- A straightforward fixed-point iteration converges in a finite region of parameter space.
- Newton-Raphson’s method extends the convergence domain and speeds up the iteration.
- The CDD ambiguity becomes very concrete: for the same “production” input, several elastic solutions can exist, and different initializations converge to different fixed points. The basins of attraction form a fractal pattern.
- The convergence range of either methods remains strictly smaller than the extremal amplitudes [4], the reason being that on the extremal amplitude the Jacobians of the iterative (fixed-point, Newton) methods are singular, making the iterations diverge.

3.2.1.2 $d = 3, 4$: dispersive iterations of two-particle unitarity, Tourkine and Zhiboedov [PT22].

In Tourkine and Zhiboedov [PT22] we implemented a higher-dimensional version based on the subtracted Mandelstam representation, the Mandelstam equation (3.1.17), and the S -wave condition (3.1.19). Operationally, the iteration is driven by the set of inputs described above: the subtraction constant, and a parameterization of multi-particle production data, and iterates to a fixed point.

Main takeaways.

- The method produces explicit *crossing-symmetric* amplitudes in $d = 3, 4$ with *two-particle unitarity built in* and with inelasticity measurable in higher partial waves.
- We observe non-trivial logarithmic Reggeization for the spectral functions and coupling dependence for the amplitudes.
- Particle production shows up somewhere in spin, consistent with the general expectation that scattering implies production in $d > 2$ [45].
- When projected onto low-energy data, the constructed solutions sit very close to the extremal boundaries obtained by SDPB-based bootstraps [29, 30].

Interestingly, the Reggeization leads to a logarithmic decay at fixed angle predicted by the one-loop running of the $\lambda\phi^4$ coupling, as commented on in the paper. We discuss in sec. 7.1 further details about how the iteration compares to the primal ϱ -bootstrap construction of [29, 30].

3.2.2 Non-convex optimization solvers

One lesson from the previous studies is that fixed-point maps are useful but fragile: the basin of attraction can be small, and the iteration fails at strong coupling. Another strategy is then to use a gradient descent to solve the equation. To do this, we used machine learning algorithms in [PT26, PT29]. In these two papers, my student Damien Leflot was the main driving and coding force behind the numerical implementation of the machine-learning part. All the coding was made in python using pytorch. At the level of the code, the method we used forked from the implementation of a similar idea in [85], and ideas discussed with Michael Kagan from SLAC/CERN (Kagan, Tourkine unpublished). The main idea is to parameterize the spectral functions by means of neural-networks, and minimize via gradient descent a *loss function* made of $\|\vec{\rho} - \Phi[\vec{\rho}]\|$ sampled over a grid of values to solve the equation (3.1.25). The process is called *unsupervised*, because at no point any external data is used to “learn” anything. Surprisingly, such a method, rather simple in appearance, actually works.

3.2.2.1 Advantages and inconvenients of machine-learning

For more details on the use of machine learning methods, see the discussion in the papers. For the purposes of the present discussion, let us just say that they are powerful for a few reasons.

- Firstly, the neural-networks (NNs) which we are very flexible tools that can parameterize efficiently the functions over several orders of magnitude. One trick we use in particular is to take as entries both s and $\log(s)$. This generate a network with several subparts, roughly of the form shown in Fig. 3.7. In general, NNs are known to be able to fit any function through the universal approximation theorem [86].
- The algorithm of *backpropagation* allows to compute gradients *at no extra computational cost* at each step via automatic-differentiation [87].
- Many libraries exist that implement these methods efficiently. We use a version of gradient descent known as Adam [88].
- ML is an empirically observed good way to solve gradient descent without being stuck in local minima.¹²

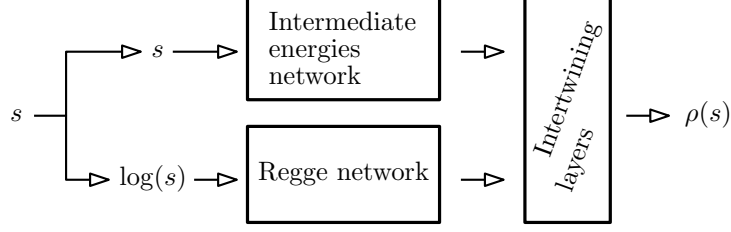


Figure 3.7: Fig. 1 of [PT26] (2412.09610): neural-network parameterization of different energy scales. Demonstrates flexibility of the NN parameterization.

One significant inconvenient of the approach is the well-known “black-box” aspect of machine-learning. Our NNs have $O(\text{one million})$ parameters, and although modern GPUs deal with that complexity easily, it is difficult to know what combination of parameters of the gradient descent will yield convergence, or even extract any information from the NN itself after training.¹³

As a result, it is difficult to interpret why a given code stops yielding results. For instance, in our most recent result [PT29], we observed that we cannot describe amplitude past a certain coupling, $c_0/(32\pi) \simeq 1.4$. The results from primal (and dual) ϱ -bootstrap suggest instead a maximal value of 2.66. Currently, we cannot know if the true bound once all constraints, including particle-production, are currently dealt with as in the AM bootstrap, or simply if our algorithm needs to be improved.¹⁴

Two interesting options are discussed in the conclusion section for (i) more interpretable and physically grounded approach and (ii) a non-machine-learning option using semi-definite relaxation to solve AM bootstrap.

Technical digression on loss functions. Another related aspects pertains to multi-objective optimization. The AM equations are not just one single equation sampled over a grid, they are a sum of several equations. In addition, we sometimes want to include constraints, for instance to force elastic sub-sectors. Then the loss function we minimize becomes a sum of several sub-losses, schematically

$$\mathcal{L} := \mathcal{L}_{AM_1} + \mathcal{L}_{AM_2} + |||S_2| - 1|| \quad (3.2.1)$$

where $\mathcal{L}_{AM_i}$ represent schematically the equations for AM bootstrap, simplified a though there were only two, and an extra constraint that sets the $J = 2$ partial wave purely elastic.

In this sum, there is a non-trivial interaction between the terms. In particular, it might be that, within a given grid precision, one cannot satisfy the first two losses better than 10^{-4} , but the last one could be much better. Then, the algorithm will probably try to satisfy the first two losses and ignore the third term, leaving the constraint $|S_2| = 1$ unsatisfied. If one then adds a large weight factor $w \sim 10^5$ in from of the constraint, maybe the algorithm will try to satisfy that constraint and mostly ignore the AM equations.

This discussion illustrates a complex problem in the NN approach: how to design the losses. Some weight factors have to be added by hand sometimes, and there is some arbitrariness in how to do so. Combined with the fact that runs can take several hours, half-days sometimes, and that the results are difficult to interpret as we discussed above, this approach does present a significant numerical complexity and development time.

¹²One mathematical description of this is the ideal of Neural tangent kernel methods [89].

¹³*Training* means doing the descent.

¹⁴For now we decided to postpone answering this question till we have implemented the full problem with 2 subtractions, compared to the not-fully-generic-yet one-subtraction case we studied in this paper.

Fortunately, since it is immediate to check if unitarity is satisfied in our solutions by mere inspection of partial waves, it is at least trivial to check the validity of the solutions.

3.2.2.2 A controlled toy model: zero double discontinuity.

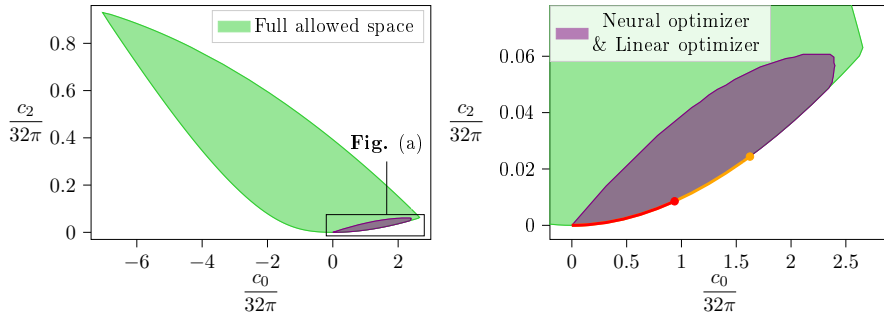
In Gumus, Leflot, Tourkine, and Zhiboedov [PT26] we studied the simplified system where the double discontinuity is set to zero. This removes genuine multi-channel cuts while keeping a dispersive structure. A neural-network parameterization combined with gradient descent can solve the nonlinear constraints efficiently and map the allowed space of low-energy data.

In this paper, we introduced a technique to mimic the extremization of the ρ -bootstrap in the AM language, which we have not yet used in the full problem which we studied in [PT29]. The trick is simply to penalize the loss function with a target. Take for instance the generic loss as in Eq. (3.2.1) and change it as

$$\mathcal{L} \rightsquigarrow \mathcal{L} + (c_2^{\text{target}} - c_2)^2 \quad (3.2.2)$$

In this way, as the algorithm solves the AM equations by minimizing the positive functional $\mathcal{L}$, it also tries to minimize the second term and set $c_2 = c_2^{\text{target}}$. In this way we can explore the interior of the space and its boundary efficiently.¹⁵

In the overlap regime, the resulting bounds and extremal solutions agree extremely well with the ϱ -bootstrap using SDPB and provide clean UV completions to all energies ($s \rightarrow 10^{100} m^2$!), necessary to observe the logarithmic decays predicted in [PT22].



[3] Figure 3.8: Fig. 2 of [PT26]. Space of toy-model amplitudes in a 2d slice of the space of non-perturbative amplitudes. Green area: full space of 4d amplitudes realizable by primal/dual ϱ -bootstrap methods, taken from [90]. Purple area: the space of toy-model amplitudes found in our paper, by NN + ϱ -bootstrap. **Piotr; For the toy-model,** Both methods give the same space. The x -axis is the subtraction constant (renormalized by 32π), the y axis is the Taylor coefficient c_2 defined in Eq. (3.1.4). In red, resp. orange, we show the reach of fixed-point, resp. Newton method.

The few salient points are:

- We used four methods: primal (constructive) / dual (exclusionary) in both NN and ϱ -bootstrap. They all give the same space of amplitudes.
- The method explicitly demonstrates that the gradient descent outperforms the iterative methods (orange and red lines in the figure)

¹⁵The definition of the boundary requires extra care and we describe in details in the paper how we do this.

- The results demonstrate a small non-decoupling theorem: outside of the purple region, the S -wave cannot be unitarized independently of the higher partial-waves.¹⁶

Details on the structure of the amplitudes are given in the paper (presence of resonances, threshold boundary conditions, etc.)

3.2.2.3 Full problem with one subtraction [PT29].

In [PT29] the optimizer is pushed to a genuinely four-dimensional setting with particle production, focusing on the two-particle-recursively-reducible (2PRR) sector. The output is a family of explicit, strongly coupled amplitudes that satisfy ACU over a wide kinematic range and exhibit a large particle-production. Note that these solutions unitarize not by the exchange of an infinite tower of Regge resonances, but by a different UV mechanism with logarithmic tail for the scattering amplitude, close in spirit to the asymptotic freedom of the wrong sign $\lambda\phi^4$ theory.

Space of solutions With that method, we can reach couplings 10 times larger than with iterations, though still limited to around 50% of the maximal coupling $c_0/(32\pi) = 2.66$ of [5], see the black curve in Fig. 3.9 below.

Remarks: the comparison to the space obtained from ϱ -bootstrap should be taken with these two caveats. Firstly, it is not known if the whole space can be covered once particle production and non-perturbative Reggeization is included. Secondly, our results are only obtained for one subtractions. This is a delicate intermediate case before the full two subtractions and it is not guaranteed at all it can cover the full space. Therefore, any serious discussion on the question can only be made once the full two subtractions case will have been done in the near future.

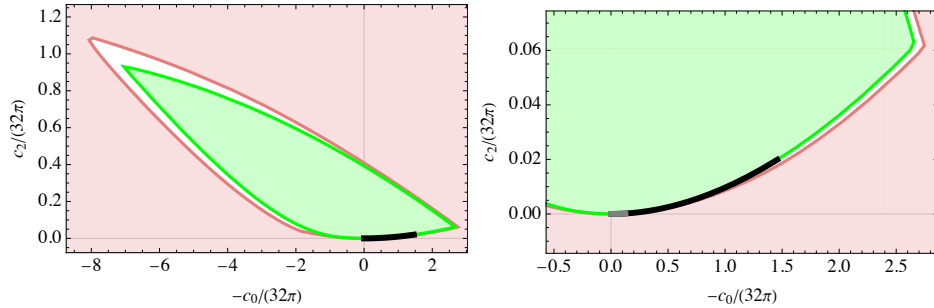


Figure 3.9: Space of S-matrices found using ϱ -bootstrap primal and dual methods, reproduced from [90]. Green: allowed space from primal method, red: rigorously excluded from dual. Thick black: amplitude constructed in this work. Thick gray: solutions constructed with iterations.

Beyond “having a solution”, the interesting physics is in the regimes that can be probed directly on the resulting amplitude: correct Landau-curve support in the double discontinuity, logarithmic decay at fixed angle, and a nonperturbative Regge features. In both these works, it was important to have ϱ -bootstrap results in order to be able to compare and diagnose on both sides.

¹⁶This is a non-trivial point. It is seen in two steps. Firstly, note that setting to zero the double-discontinuity implies via Froissart–Gribov that unitarity in all partial waves S_J $J \geq 2$ is violated. This implies that in the complement of the purple zone are amplitudes where all partial waves are (conjecturally, from ϱ -bootstrap) unitary, but where S_0 alone could not have been unitarized on its own, otherwise it would have belonged to the purple region.

Aks screening, [PT29] Another category of results we got relates to making ρ_{MP} a dynamical, variational degree of freedom. Using it, we could implement what we call “Aks screening”, and use ρ_{MP} to screen-out inelasticity in low partial waves, up to $J = 0, 2$, at the cost of generating a large and oscillatory ρ_{MP} in the UV. This allow to make contact with the results from the ϱ -bootstrap, for which semi-definite optimization produces systematically amplitudes which try to saturate all constraints, so imposing $|S_J(s)| \leq 1$ for some J and s quasi systematically yields instead $|S_J(s)| = 1$. In our paper, we show that this process is compatible with Aks theorem, as long as it is not *all* J , and provide some estimates on this phenomenon. We discuss this more in Sec. 7.1.3.

At the level of the bounds, the consistent trend that was observed in both [PT22] and [PT29] is that **removing inelasticity makes bounds more extremal**.

3.3 Future directions

In the discussion chapter, we provide two sections dedicated to the Aks tension and comparison between ϱ -bootstrap and AM bootstrap. We also collect future directions. For now, let us briefly conclude by listing a few future directions that seem most useful:

- **Implement two subtractions.** Aim: amplitudes with towers of Regge resonances, non-perturbative behavior of Regge trajectories, Froissart growth.
- **Hybrid bootstrap.** Develop a hybrid bootstrap using ϱ -bootstrap as seeds for AM bootstrap gradient descent to generate speed-up.
- **Study the role of ρ_{MP} in unitarization** through its variational formulation. What amplitudes maximize particle production(anti-Aks-screening)? Minimize it?
- **Beyond the identical massive scalar case.** Adding flavor (pions, kaons), massless limits, higher dimensions, etc.

Chapter 4

S-matrix bounds across dimensions: kinks, threshold singularities, and locality

After having discussed the AM-bootstrap, we now come to results pertaining to the direct use of the ϱ -bootstrap. This chapter summarizes the main results of [PT28], where we applied ϱ -bootstrap technology as initially introduced in [3–5] and later SDPB-based implementations e.g. [29, 30] to construct primal bounds on low energy observables on identical massive scalar amplitudes in higher dimensions $d = 3, \dots, 12$ including half-integers. These results are to my knowledge the first charts of S-matrix bounds in higher dimensions. They yielded a clean and robust signal of drastic changes marked by numerically stable kinks, in the way these methods capture UV completions around $d = 5$ and $d = 7$, summarized in Fig. 4.1 in this thesis. In the paper, we were able to tie the families of solutions before and after the kink with locality near the threshold and a phenomenon of IR subtractions which occurs in higher dimensions.

These works were done with two postdoctoral researchers of LAPTh, Mehmet Asım Gümüş and Simon Metayer. Mehmet Gümüş shared his knowledge of the semi-definite techniques which he mastered from Andrea Guerrieri and Joan Elias Miro, and Simon Metayer was the main driving force of the numerical implementation. In particular, Simon Metayer developed a powerful mathematica wrapper for SDPB [91] which allowed us to generate large and flexible database of integrals and run SDPB at large ansatz size and constraint sizes in $O(20)$ dimensions.

One further motivation for this work was to acquire first-hand experience on the ϱ -bootstrap methods, to build a better intuition of how UV completion works in these methods.

A physical discussion on what theories could correspond to such S-matrices is provided in the paper and below in Sec. 8.2.1.

4.1 Setup

Let me explain briefly the setup.

4.1.1 Setup and observables

We follow the conventions of Chapter 2, in particular the partial-wave expansion (2.1.2) and the unitarity conditions (2.1.6)–(2.1.7). We set $m = 1$, so $s + t + u = 4$, and study crossing-symmetric, maximally analytic amplitudes with a mass gap.

The observables that we bound below are low-energy data at the crossing-symmetric point $s = t = u = 4/3$,

$$\bar{c}_0 := \frac{\mathcal{N}_d}{2} A\left(\frac{4}{3}, \frac{4}{3}\right), \quad \bar{c}_2 := \frac{\mathcal{N}_d}{8} \frac{\partial^2}{\partial s^2} A\left(s, \frac{4}{3}\right) \Big|_{s=4/3}, \quad (4.1.1)$$

where $\mathcal{N}_d \sim \Gamma(d/2)^{-1}$ removes a trivial factorial growth with d so that these observables are now of order 1 in any dimension. These observables are measured in mass units of the mass m .

The essence of the ρ -bootstrap is at first sight similar to the AM bootstrap. It starts from a crossing-symmetric ansatz that assumes maximal analyticity in the sense that it defines an amplitude function $T(s, t, u)$ analytic in the cut s, t, u complex plane with cuts starting at $s, t, u = 4m^2$.

The way unitarity is imposed differs: it is imposed spin-by-spin as convex semi-definite constraints on an energy grid $s \in \{s_1, \dots, s_{N_{\text{grid}}}\}$. Instead of then solving unitarity, we optimize a coupling. For the question of maximum $\bar{c}_0$, we solve the semi-definite problem:

$$\begin{aligned} &\text{maximize} && c_0 \\ &\text{over} && \alpha_{a,b,c} \in \mathbb{R} \\ &\text{subject to} && c_0 \equiv T(s_0, t_0, u_0), \\ &&& U_J(s_n) \succeq 0 \quad \forall J \in \{0, 2, \dots, J_{\text{max}}\}, \forall n \in \{1, \dots, N_s\}, \end{aligned} \quad (4.1.2)$$

where the optimization parameters are the ansatz parameters $\alpha_{a,b,c}$ on which the amplitude depends crucially *linearly*: $T(s, t, u) = \sum \alpha_{a,b,c} \rho(s)^a \rho(t)^b \rho(u)^c$, where ρ is a function that conformally maps the cut plane to the unit disk

$$\rho(s) = \frac{\sqrt{\sigma - 4} - \sqrt{s - 4}}{\sqrt{\sigma - 4} + \sqrt{s - 4}} \quad (4.1.3)$$

This function depends on a parameter σ that can be adjusted. Finally

$$U_J(s) = \begin{pmatrix} 1 - \frac{1}{2}\phi_d^2(s) \operatorname{Im} f_J(s) & \phi_d(s) \operatorname{Re} f_J(s) \\ \phi_d(s) \operatorname{Re} f_J(s) & 2 \operatorname{Im} f_J(s) \end{pmatrix} \quad (4.1.4)$$

with $\phi_d^2 = \sqrt{s - 4}^{d-3} / \sqrt{s}$ so that

$$U_J(s) \succeq 0 \Leftrightarrow |S_J|^2 \leq 1 \quad (4.1.5)$$

See Sec. 7.1 for a compact summary of the ρ approach and summary of differences with AM bootstrap.

Comment on numerics The numerically challenging part here is to compute the partial-waves for the ansatz. In practice, we do this numerically. We impose unitarity on a grid of points $\{s_1, \dots, s_{n_{\text{grid}}}\}$, for a number of spins $J = 0, \text{dots}, J_{\text{max}}$. This means that we store an amount of order $n_{\text{grid}} \times J_{\text{max}}$ of integral for each ansatz. For the results of the paper, we do 20 different ansatz sizes, for d spanning all integers and half-integers between 3 and 10, plus 11 and 12, giving rise to millions of integrals to compute and store.

One advantage of the wrapper developed by Simon Metayer [91] is an efficient implementation of a database system using SQL, allowing for concurrent reading and writing, tailored for intensive parallel computing.

4.1.2 Threshold expansion: where d changes the rules

The way the dimension enters in these bootstraps (ϱ -bootstrap or AM) is mostly analytic. In addition, the Froissart-bound in d dimension is $\sigma_{\text{tot}} \leq \log(s)^{d-2}$ which does not change the number of subtractions needed to describe growing amplitudes in higher dimensions.

As we learned, the most important point of this higher dimension bootstrap lies in the threshold behavior near the two-particle threshold. From the elastic relation for partial waves (see (2.1.9) in Chapter 2), one finds that the spin-zero partial wave can diverge as bad as

$$f_0(s) \sim \frac{1}{(s-4)^{\frac{d-3}{2}}}, \quad (4.1.6)$$

up to logarithms in odd integer d [65]. Once $\frac{d-3}{2} \geq 1$ (i.e. $d \geq 5$), non-integrable threshold singularities are allowed by unitarity. The extremal solutions that we find use this freedom. This non-integrability furthermore allows a new phenomenon of negativity for low-energy constants, slightly counter-intuitive from the 4d classic positivity expectations [92].

In the paper, we separated threshold singularities into two classes:

$$\frac{1}{(s-4)^n} \iff \begin{cases} n \leq 1 & \text{“mild” threshold behavior,} \\ n > 1 & \text{“strong” threshold behavior.} \end{cases} \quad (4.1.7)$$

The bootstrap kinks discussed below are, in a very concrete sense, transitions between solutions that activate or not the strong threshold behavior.

Remark: ansatz terms. The most non-trivial part of the work [PT28] was to tailor an ansatz that would yield that threshold behavior, once projected into the $J=0$ sector. This turned out to be a non-trivial task if one seeks to avoid messing up the analytic structure of the amplitude. All details are given in the paper.

4.1.3 IR subtractions and what replaces Wilson-coefficient positivity

In low enough d (notably $d=4$), standard fixed- t dispersion relations turn low-energy coefficients into positive moments of $\text{Im } A$, yielding the familiar positivity bounds on EFT data [92, 93]. For $d \geq 5$, the threshold behaviors allowed by (4.1.6) spoil the usual derivation unless one introduces explicit infrared subtractions. The clean qualitative statement is:

When strong threshold singularities are allowed, coefficients c_{2n} with $2n \leq d-4$ become genuine IR subtraction constants and are not sign-definite.

This “threshold-induced negativity” is purely an IR effect. At the same time, differentiating sufficiently many times restores positivity, but now for *new combinations* of EFT data. A basic example is $d=6$:

$$c_4^+ \equiv c_2 - \frac{128}{9}c_4 \geq 0, \quad (4.1.8)$$

The precise relations are given in the appendix of the paper. This makes the link between higher- d threshold analyticity and the modern positivity program sharp: the *right* positivity statements depend on the allowed IR singularities [94, 95].

4.2 Results: two branches and the kinks

The headline numerical result is that extremal of $(\bar{c}_0, \bar{c}_2)$ as functions of d organize into smooth branches separated by sharp kinks, see fig. 1 of [PT28] reproduced in Fig. 4.1 below. The kinks are stable under cutoff increases and correlate with qualitative changes in the threshold structure of the extremal amplitudes. Here I show an updated version of the plot with a data point at $d = 12$, new compared to the v1 of the paper.

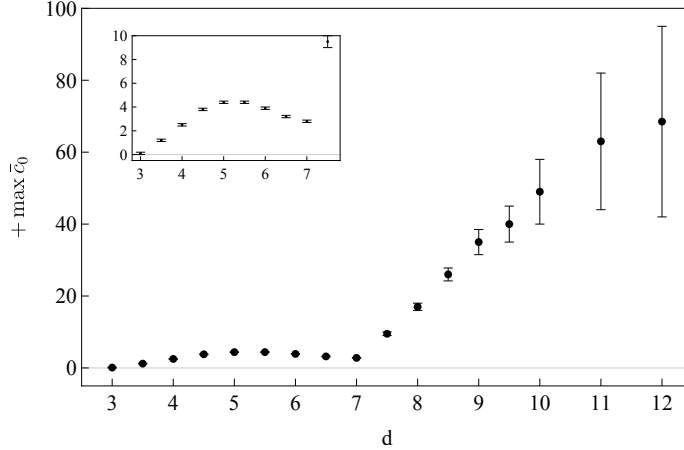


Figure 4.1: Dimensional extremal bounds on $\max(\bar{c}_0)$. The point at $d = 12$ is added compared to current version of [PT28]. Error bars were added with a conservative extrapolation procedure described below, and the dot is just the middle point.

A useful way to read the data is:

- **The $d = 5$ kink (min branch):** this is where the standard dispersive positivity of c_2 ceases to apply once threshold singularities are allowed. Numerically, $\bar{c}_2^{\min}$ becomes negative precisely in that region, matching the IR-subtraction logic of §4.1.3.
- **The $d = 7$ kink (max branch):** this coincides with the onset of *strong* threshold terms in the extremal solutions, i.e. with activating singularities with $n > 1$ in (4.1.7). In other words, the optimizer changes *which* IR lever it uses to satisfy ACU while pushing $\bar{c}_0$ or $\bar{c}_2$ upward.
- The bounds do not undergo a dramatic change past $d = 11$: although the $d = 12$ bounds required a heavier numerics (much larger ansatz size), they appear to smoothly line up on the post-kink branch. It was not obvious that supersymmetry was in any way related to the question, but it is a useful information to have nevertheless.

Amplitudes and Regge trajectories The amplitude we obtain that give rise to the bounds appear to have a very rich Regge resonances structure, which will be studied in another publication.

Comment on error bars. At the dimension increases, the numerics becomes harder and harder, and require more and more parameters in the ansatz to give reliable estimates. Our error bars are designed as follows (see Appendix C in [PT28]). In “low” d , we observe that a linear fit with $1/N^2$ provides a good description of the convergence. We use that fit in higher d to estimate the upper bound. The lower bound is the value achieved at

the ansatz size we worked on. Crucially, not all the points in the plots are on the same footing. Up to $d = 10$, we have the same $N \sim 200$ parameters. This number of parameters is nowhere near sufficient to describe $d = 11$ or $d = 12$ where we had to reach $N = 450$ and $N = 800$ to obtain the bounds.

4.3 Brief discussion

Threshold structure, locality. The two families of extremal solutions seem to represent something about the nature of the UV completions captured by the method. We identified one precise indicator of these families of amplitudes: after the kink, the strong threshold behavior turns on systematically (see table in the paper).

What could be the meaning of this? A trivial possibility would have been that this is related to the threshold being integrable or not. Remark however that the definition strong/mild of the threshold is *not* correlated to the threshold being integrable or not: the edge case $1/(s - 4)^1$ is indeed already not integrable, but belongs to the mild class from the perspective of the kink. So it seems that the branches are connected to a deeper problem. In the paper, we made the case that it could be tied with locality, as follows.

In local QFTs with a mass gap, singularities are mild. At worst, they are poles. In certain, these singularities can be resummed, and in known cases they correspond to mild singularities. For instance in $d = 3$, a divergent $1/\log(s - 4)$ behavior is known to arise [96]. In $d = 4$, empirical examples were constructed with the ϱ -bootstrap with $1/\sqrt{s - 4}$ behavior. By contrast, the strong-threshold branch we see at higher d involves singularities such as double poles, and in general $1/(s - 4)^n$ with $n > 1$. These are not forbidden by elastic unitarity, but they are unusual from the pure local QFT standpoint.

There are two conservative readings:

- **Genuine dynamics:** strong threshold behavior could reflect a non-perturbative resummation effect that is still compatible with locality, just more extreme in higher d .
- **Missing IR structure:** our ansatz does not dynamically generate new light degrees of freedom (extra poles, massless cuts). If consistent UV completions in higher d require such ingredients, as suggested by string universality ideas, the bootstrap may be mimicking them in the IR by such non-localities.

Independently of this speculation, what appears clear in this analysis is that *threshold analyticity is an organizing principle across d , and it controls which low-energy positivity statements survive*. The strong-threshold branch is a natural place to look for a sharper diagnostic of when “4d-like” local massive scattering ceases to exist, or becomes impossible without extra degrees of freedom.

We discuss some further points in the discussion chapter.

Part II

Regge theory, dual models

Chapter 5

Regge theory and dual models

In Chapters 3, 4, we have described results related to the non-perturbative bootstrap. At the cost of important numerical efforts, numerical scattering amplitudes and bounds were obtained and discussed. In this chapter, we turn to a simplifying assumption that allows to use analytical techniques in the bootstrap: that of “narrow resonances”, which is to say, stable resonances, which is achieved at weak-coupling (tree-level).

In this chapter, I describe the works [PT27, PT20, PT23, PT24], done in collaboration with my previous PhD student Felipe Figueroa and LAPTh postdoctoral researchers Christopher Eckner and Simon Metayer.

In QCD, hadrons typically appear as resonances: poles of scattering amplitudes on unphysical sheets, with masses and widths shaped by confinement. As we discussed in the introduction, in the 60s, physicists trying to understand the nuclear force were confused by the profusion of particles seen in collision experiments, departing significantly from the QED picture, and were looking for an organizing principle. The cleanest structure was that mesons appear to line up on approximately linear Regge trajectories when plotted in the Chew–Frautschi plane [97],

$$J \simeq \alpha_0 + \alpha' M^2. \quad (5.0.1)$$

Regge theory [98] ties the origin of these trajectories to the analytic continuation of partial-waves $f_J(t)$ from integer, quantized angular momentum J to complex J :¹⁷ a Regge pole of $f_J(t)$ as a function of complex J at $J = \alpha(t)$ controls the high-energy behavior

$$\boxed{f_J(s) = \frac{\beta(t)}{J - \alpha(t)} \quad \Longleftrightarrow \quad A(s, t) \sim_{s \rightarrow \infty} \beta(t) s^{\alpha(t)}. \quad (5.0.2)}$$

The rise of $\alpha(t)$ is reflected in the spectrum by how J increases as a function of M^2 . Experimentally, the same families often admit daughter trajectories: roughly parallel lines associated with higher excitations at fixed internal quantum numbers, see e.g. [99].

A concrete area where Regge ideas can be implemented very concretely is the very area that gave rise to the Veneziano model, that of *dual models*, which we describe in this chapter. Dual models are tree-level, crossing-symmetric amplitudes whose only singularities are simple poles, interpreted as an infinite tower of massive higher-spin resonances with “narrow width” (meaning exactly stable at weak coupling). Historically they were introduced to model the strong interactions and led to the Veneziano amplitude and then string theory. What keeps them useful today in the context of the bootstrap is that they remain an unusually explicit corner of the S-matrix: the full amplitude is encoded in a

¹⁷Of course analytic continuation is not unique, but the Froissart-Gribov projection and Carlsson theorem arguments allow to select a unique one relevant physically.

spectrum and residues, yet it still realizes nontrivial consistency conditions (crossing and positivity) in a way that can be tested and bootstrapped.

Dual models are also the natural language to capture aspects of large- N gauge theories, where resonances are narrow at leading order in $1/N$ [100, 101]. Very concretely, the large- N S-matrix in confining gauge theories is a meromorphic amplitude and should be describable by a dual-model amplitude. But constructing generic dual amplitudes is hard in practice: the defining infinite sums over poles converge slowly, and naive truncations spoil crossing and make numerical searches difficult.¹⁸ In [PT27, PT20, PT23, PT24] we revisited several aspects of this quest for dual models. The results are summarized as follows.

Coon amplitude [PT20]. In this paper, with my student of the time Felipe we revisited a deformation of the Veneziano amplitude called the Coon amplitude. We analyzed the unitarity conditions for ghost decoupling, and emphasized that it is not meromorphic in the way people thought it was. The paper triggered some domain of activity around dual models.

Regge bootstrap [PT23]. In this paper, we revisited a dynamical bootstrap scheme for dual models where crossing is applied at the level of the Regge pole expansion. We studied the phenomenon of bending of Regge trajectories, useful to make contact with what happens in real QCD.

Number of trajectories [PT27, PT24]. In these two papers we proved some no-go theorem: dual models amplitudes **Piotr; cannot** be too simple and need to have infinitely many trajectories. This statement was not present in the literature before. The question originally came from our Regge bootstrap (asking how simple can dual models be), but interestingly it made contact with modern literature on large- N gauge theories in bootstrap [55–58], where objects with a single trajectory were argued to play a role in extremizing certain observables. In [PT27], we further study specifically gravitational theories in $d = 4$, refining existing statements about stringiness of UV completions of weakly coupled gravity [58, 59, 70, 107] to include statements about the necessity of having daughter trajectories as well.

5.1 Basics of Regge theory, dual models

Most definitions and introductory technical material can be found the in the papers. Other useful entry points on Regge theory already mentioned are the standard textbook treatment [60] and the lectures by Gribov [108].

5.1.1 Definitions, properties

Meromorphic amplitudes admit a pole expansion, distinct from the partial wave expansion. For the scattering of external scalars, we have

$$A(s, t) = \sum_{n=0}^{\infty} \sum_{J=0}^{J_{\max}(n)} \frac{c_{n,J} P_J^{(d)} \left(1 + \frac{2t}{m_{n,J}^2 - 4m_0^2} \right)}{s - m_{n,J}^2}, \quad (5.1.1)$$

¹⁸As recently discussed in [102–105], crossing-symmetric dispersion relation in part cure this issue. It would be nice to revisit some problems touched upon in this thesis in this language, especially with the technology of [106].

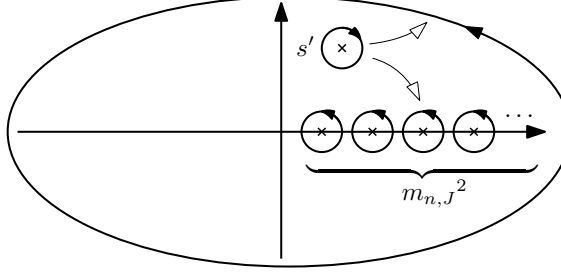


Figure 5.1: Dispersion relation for meromorphic amplitudes.

where n are the *levels* and J the spins of exchanged particles at level n . If the external states are spinning particles, one needs to decompose the exchanged states according to representations of the little group in d dimensions [109, 110]. In [PT20, PT23] we studied scalar amplitudes, and in [PT27] we studied graviton amplitudes in $d = 4$.

The representation (5.1.1) can be demonstrated as follows by a dispersion relation. For simplicity let us assume we deal with a *color-ordered* amplitude $A(s, t)$, stripped from color factors, with only $s \leftrightarrow t$ crossing, so we do not have u -channel poles.¹⁹ Fix t , and write the following dispersion relation as in Fig. 5.1:

$$A(s, t) = \frac{1}{2\pi i} \oint \frac{A(s', t)}{s' - s} ds' = - \sum_{n, J} \frac{\text{Res } A(s', t)|_{s'=m_{n, J}^2}}{m_{n, J}^2 - s} + A_\infty(t), \quad (5.1.2)$$

where $A_\infty(t)$ is the contour at infinity, $m_{n, J}^2$ are squared masses of resonances at level n and spin J . If the amplitude decays for a value of t , the contour at infinity drops and one gets

$$A(s, t) = \sum_{n, J} \frac{\text{Res } A(s', t)|_{s'=m_{n, J}^2}}{s - m_{n, J}^2}, \quad (5.1.3)$$

For the scattering of external scalars, the residue function can further be decomposed on Gegenbauer polynomials. When a spin- J state is exchanged at $s = m_{n, J}^2$, the residue at the pole as a function of t is given by a Gegenbauer polynomial evaluated at $\cos \theta_{n, J} = 1 + \frac{2t}{m_{n, J}^2 - 4m_0^2}$:

$$\text{Res } A(s', t)|_{s'=m_{n, J}^2} = \sum_{J=0}^{J_{\max}(n)} c_{n, J} P_J \left(1 + \frac{2t}{m_{n, J}^2 - 4m_0^2} \right). \quad (5.1.4)$$

Note that the exchange of finitely many spins at a given level imposes that $\text{Res } A(s, t)$ is a polynomial in t (= of finite degree) $J_{\max}(n)$. Thus:

$$\boxed{\text{Locality of exchanges} \leftrightarrow \text{polynomial residues.}} \quad (5.1.5)$$

The series converges up to when t hits the first pole. For s, t smaller than this first singularity, crossing $A(s, t) = A(t, s)$ yields

$$A(s, t) = \sum_{n=0}^{\infty} \sum_{J=0}^{J_{\max}(n)} \frac{c_{n, J} P_J^{(d)} \left(1 + \frac{2t}{m_{n, J}^2 - 4m_0^2} \right)}{s - m_{n, J}^2} = \sum_{n=0}^{\infty} \sum_{J=0}^{J_{\max}(n)} \frac{c_{n, J} P_J^{(d)} \left(1 + \frac{2s}{m_{n, J}^2 - 4m_0^2} \right)}{t - m_{n, J}^2}, \quad (5.1.6)$$

The sum of resonances in one channel reproduces poles in the other. This property is known as *duality* and came from the vanishing of the contour integral. Historically, it was put forward in [12].

An immediate consequence is that duality requires the exchange of an infinite tower. Otherwise the scattering amplitude would behave as $A(s, t) \sim_{s \rightarrow \infty} s^{J_{\max}}$, where $J_{\max}$ is the highest spin exchanged, thus unable to reproduce the crossed-channel pole and growing too fast for unitarity to hold if $J_{\max} > 2$.

¹⁹A non-color ordered amplitude could be for instance a crossing-symmetrized version of Eq. (1.1.1) $A(s, t, u) := A(s, t) + A(t, u) + A(u, s)$, or a closed-string version such as the Virasoro-Shapiro amplitude.

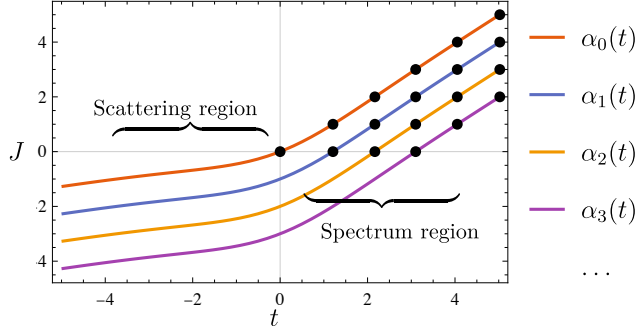


Figure 5.2: Regge trajectories.

Tree-level unitarity constraints For meromorphic amplitudes, “unitarity” is a tree-level statement that reduces to positivity: the decoupling of negative norm states (ghost-freedom) means positivity of partial-wave coefficients in the residue polynomials,

$$\text{Unitarity} \implies c_{n,J} \geq 0. \quad (5.1.7)$$

Since these coefficients are modulus-squared of couplings for elementary three-point interactions between the external states and the higher-spin field $c_{n,J} \sim \lambda_{n,J}^2$, this positivity also relates to hermiticity of the Lagrangian. For a few more details, see appendix of [107].

The Veneziano amplitude defined in Eq. (1.1.1) [11] is the prime example of such a dual model. In general in string theory, positivity follows from the no-ghost theorem. The question of positivity of the coefficients of the Veneziano amplitude (and others) was revisited recently using direct methods in [111, 112]. After the Veneziano amplitude was discovered, a whole industry developed, aiming to understand the space of dual models to describe phenomenological amplitudes, such as the pion amplitude. The Lovelace–Shapiro model [113, 114] for pion scattering is a well-known example.²⁰ Models given in terms of infinite sums of shifted Veneziano amplitude known objects as satellite models were developed [116, 117], which are the subject of a little study below for closed strings in Sec. 5.2.3.

5.1.2 Linearity of Regge trajectories

A central phenomenological tension is the shape of Regge trajectories in the scattering region ($t < 0$), see Fig. 5.2. The Veneziano amplitude has linear trajectories and the extreme “stringy” softness associated with exponential suppression at fixed angle [118, 119] (the Gross–Mende decay). By contrast, in QFT one expects power-law behavior [120] (up to logs), so trajectories should flatten for $t < 0$, as in Fig. 5.2. This raises the question: is it possible to engineer a dual model with phenomenological features such as bent Regge trajectories? The works [PT20, PT23] over two aspects.

[5]

The Coon amplitude that we studied in [PT20] provides an interesting data point. It is a q -deformation of Veneziano with genuinely non-linear, logarithmic trajectories, and an accumulation point in the spectrum, interpolating between a field-theory-like limit and the Veneziano limit. One of the results of [PT20] is however that in order to maintain locality of higher-spin exchanges, a non-meromorphic factor must be added, which generates a cut past the accumulation point which then resembles a ionization threshold, with a continuum

²⁰Which we now know today to be inconsistent [115].

of states past the accumulation point. This illustrates a general warning: bending can come with analytic subtleties, and the Coon model makes that cost (a cut) explicit.

In [PT23] we then managed to build numerically a simple tunable model that interpolates between string-like and QFT-like behavior by changing the effective slope across the scattering/spectrum regions, so that the effective slope differs for $t > 0$ and $t < 0$.

Before closing this section, let us recall a few older points related to bending. An argument due to Mandelstam [121] suggests that the Regge function $\alpha(t)$ should be linear at weak coupling. The argument assumes that $\alpha(t)$ is analytic and admits a once-subtracted dispersion relation

$$\alpha(t) = \alpha' t + \alpha(0) + \frac{(t - t_0)^2}{\pi} \int dt' \frac{\text{Im } \alpha(t')}{(t' - t)(t' - t_0)^2}. \quad (5.1.8)$$

Then, since meromorphic amplitudes describe stable particles, an imaginary part in α would give a width, we should thus have $\text{Im } \alpha \equiv 0$, and the conclusion is that $\alpha(t)$ must be exactly linear. This relies on assumptions on $\alpha(t)$ that are not derived from crossing or unitarity, and it should be viewed as a plausible but not fully controlled argument.

Another direction is to engineer bending by summing Veneziano blocks with rational slopes [122, 123], while maintaining an exact string spectrum. These constructions provide a counter-example to Mandelstam's argument above.

And lastly, in general, all the constructions mentioned so far pass only tests of $2 \rightarrow 2$ positivity. Higher-point constraints and factorization are crucial in order to select actual physically meaningful theories, and some related aspects are discussed below.

5.2 Results

5.2.1 The Coon amplitude [PT20]

Soon after Veneziano proposed his amplitude, Coon proposed a one-parameter deformation [124–128]. This is a q -deformation of Veneziano built from q -products (equivalently, q -Gamma functions). In [PT20] we revisited its definition and its unitarity properties. Two points matter for this thesis:

- we clarified that the correct definition includes a non-meromorphic prefactor that was missed in the older literature;
- despite this exotic analytic structure, one can still sharply map out when the amplitude is ghost-free.

5.2.1.1 Definition

For the sake of definiteness, I write the explicit form of the amplitude below. A convenient definition is the explicit infinite-product representation

$$A_q(s, t) = -q^{\frac{\log \sigma}{\log q} \frac{\log \tau}{\log q}} \prod_{n=0}^{\infty} \frac{(\sigma \tau - q^n)(1 - q^{n+1})}{(\sigma - q^n)(\tau - q^n)}, \quad (5.2.1)$$

where

$$\sigma = 1 + (s - m^2)(q - 1), \quad \tau = 1 + (t - m^2)(q - 1). \quad (5.2.2)$$

Notice the prefactor $q^{\frac{\log \sigma}{\log q} \frac{\log \tau}{\log q}}$: it is manifestly non-meromorphic. I will come back to this point after briefly describing the spectrum. In this definition, it is important that

$$q \leq 1. \quad (5.2.3)$$

There exist a formulation for $q > 1$, but the amplitude is immediately non-unitary.

The amplitude interpolates between a simple rational function and Veneziano:

$$\lim_{q \rightarrow 0} A_q(s, t) = \frac{1}{s - m^2 + 1} \frac{1}{t - m^2 + 1}, \quad (5.2.4)$$

$$\lim_{q \rightarrow 1} A_q(s, t) = -\frac{\Gamma(-s + m^2)\Gamma(-t + m^2)}{\Gamma(-s - t + 2m^2)} \equiv V(s, t). \quad (5.2.5)$$

Spectrum, trajectories The s -channel poles, depicted in Fig. 5.3, sit at

$$m_n^2 = m^2 + \frac{q^n - 1}{q - 1}, \quad (5.2.6)$$

and accumulate at a finite value

$$s_\star = m^2 + \frac{1}{1 - q}. \quad (5.2.7)$$

where the branch cut of the prefactor in Eq. (5.2.1) starts.

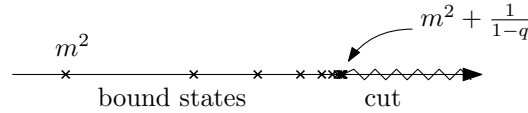


Figure 5.3: Coon spectrum: an infinite tower of resonances with an accumulation point at finite m_n^2 .

In the Regge limit one finds a genuinely non-linear trajectory,

$$j(t) = \frac{\log(1 + (t - m^2)(q - 1))}{\log q}, \quad (5.2.8)$$

which diverges at $s_\star$ and straightens as $q \rightarrow 1$ to reproduce Veneziano's exactly linear trajectories.

5.2.1.2 Results

Tree-level unitarity constraints In [PT20] we studied tree-level unitarity of the Coon amplitude in general dimension d by scanning the partial-wave coefficients and mapping the ghost-free region in (q, m^2, d) . The resulting picture is simple: there are two regimes separated by a critical curve $q_\infty(m^2)$ such that (i) for $q \leq q_\infty(m^2)$ the amplitude is ghost-free for all d , while (ii) for $q > q_\infty(m^2)$ there exists a critical dimension $d_c(q, m^2)$ below which ghosts appear. An explicit expression for the separator is $q_\infty(m^2) = \frac{m^2 - 3 + \sqrt{9 + 2m^2 + m^4}}{2m^2}$. Beyond this analytic boundary (and excluding a small- q region ruled out analytically), the remaining ghost-free volume is determined numerically and can be summarized by envelope curves controlling the large- d limit together with numerical determinations of d_c at fixed (q, m^2) , as illustrated in Fig. 2 of [PT20]. A useful benchmark is the unitary model at $m^2 = \frac{1}{3}$ near $q \rightarrow 1$, for which the critical dimension approaches $d_c \simeq 6.3$.

Linearity, meromorphy, unitarity For $0 < q < 1$ the amplitude admits the representation Eq. (5.2.1). The subtle point is the overall factor

$$q^{\frac{\log \sigma \log \tau}{\log q \log q}} = \exp\left(\frac{\log \sigma \log \tau}{\log q}\right). \quad (5.2.9)$$

Because it is built out of logarithms, this prefactor is not meromorphic in the kinematic variables: it carries branch-cut structure that coincides with the essential singularity that is the accumulation of poles.

At the same time, the prefactor is essential from the standpoint of locality. As we emphasized in [PT20], if one drops it (one then recovers Coon’s strictly meromorphic object), the residues at the poles are no longer polynomials in the scattering angle: they become infinite-product functions. This is a sharp signal of non-locality. By contrast, keeping the prefactor restores polynomial residues. Interestingly, although this choice drastically changes the analytic/locality properties, it affects the sign pattern of residues only mildly.

5.2.1.3 Main takeaways

- The Coon amplitude provides a controlled non-linear Regge trajectory while preserving duality.
- Polynomial residues force a non-meromorphic prefactor; dropping it hides a real non-locality issue.
- Unitarity carves out a sharp allowed region in (q, m^2, d) with a clear two-regime structure.

5.2.2 Regge bootstrap [PT23]

Dual models are simpler than full non-perturbative amplitudes, but nevertheless hard to construct from first principles, especially when you depart from the Veneziano-like integer-separated, linear spectrum. The idea we are trying to implement here is: **given a spectrum**, can one **construct an amplitude that satisfies crossing and unitarity**?

The slow **Piotr; speed depends on values of s, t would be nice to add quick characterization, in Veneziano $R_n(t) \sim n^{t-m_0^2}$** convergence of the resonance sum Eq. (5.1.1) makes [6] crossing symmetry numerically non-constraining in naive truncations, and trying to solve by hand a truncated crossing $\sum_{n=0}^{\infty} \rightsquigarrow \sum_{n=0}^N$ as:

$$\sum_{n=0}^N \sum_{J=0}^{J_{\max}(n)} \frac{c_{n,J} P_J^{(d)}\left(1 + \frac{2t}{m_n^2 - 4m_0^2}\right)}{s - m_n^2} \simeq \sum_{n=0}^N \sum_{J=0}^{J_{\max}(n)} \frac{c_{n,J} P_J^{(d)}\left(1 + \frac{2s}{m_n^2 - 4m_0^2}\right)}{t - m_n^2} \quad (5.2.10)$$

does not give meaningful results.

In [PT23] we introduced a numerical bootstrap for meromorphic dual amplitudes that works directly at the level of Regge data and packages infinite amount of resonances into finite number of Regge-trajectory parameters.

Bootstrap strategy; linear programming. The key idea is to apply crossing not on the resonance sum, but on the Regge-pole sum. The inputs are the trajectory functions and the output is an amplitude that has that set of trajectories. In this primal construction, crossing holds in a finite portion of the s, t space and satisfying it is the difficult part.

The leading Regge trajectory $J = \alpha_0(t)$ gives the high-energy behavior $A(s, t) \sim \beta_0(t) s^{\alpha_0(t)}$, but dual models typically have several trajectories (in fact, infinitely many

[PT27, PT24]). The amplitude can therefore be approximated better and better by summing more trajectories. We define the truncated amplitude, schematically:

$$A_N(s, t) \simeq \sum_{i=0}^{N_{\text{traj}}} \beta_i(t) s^{\alpha_i(t)}. \quad (5.2.11)$$

In this way, each trajectory i describes an infinite sequence of poles according to

$$c_{n, n-i} \sim \beta_i(m_n^2), \quad m_n^2 = \alpha^{-1}(n) \quad (5.2.12)$$

Crossing is then enforced in regions where the Regge expansion is accurate **Piotr; add that [7] the whole point of the paper was to figure out these regions empirically** via a tolerance procedure,

$$|A_N(s, t) - A_N(t, s)| \leq \epsilon. \quad (5.2.13)$$

This formulation repackages infinitely many couplings on a given trajectory into a known spectrum i.e. set of trajectories $\alpha_i(t)$ and unknown Regge residue function $\beta_i(t)$ to be determined.

Via (5.2.12), positivity is easy to impose on the whole trajectory by looking for positive functions β_i . We then solve crossing by (5.2.13). We use a Veneziano-inspired polynomial ansatz for the $\beta_i(t)$, so that the constraints can be handled with linear programming (we used Mosek [129] in this work, a commercial numerical linear solver with academic licenses). To build solutions, we take simultaneously a large ansatz size and small tolerance.

Super-unitarity In order to enforce positivity, we apply a stronger version of unitarity that we discovered empirically that the Veneziano amplitude appears to satisfy which we called *super unitarity*. In the Veneziano amplitude, after factoring out a universal piece, the residues admit an expansion with non-negative coefficients, which is a stronger property than vanilla positivity and implies in particular that the coefficients $c_{n, n-i}$ are positive also at real n . Imposing it keeps the bootstrap as a linear programming problem, allowing to use linear programming methods. This approach would deserve to be revisited using semi-definite methods.

5.2.2.1 Results

Let me present a few salient results from that analysis.

Veneziano as a dip. We calibrated the bootstrap on the Veneziano amplitude itself, with linear trajectories $\alpha_i(t) = t - i$. As the tolerance ϵ to crossing is decreased and the number of trajectories is increased, the allowed region shrinks around Veneziano. Operationally, Veneziano sits at the bottom of a sharp dip in the minimal achievable tolerance $\epsilon_{\min}$, see Fig. 5.4. This **dip detection** is the practical diagnostic that the algorithm locks onto a distinguished amplitude.

In simple words, this means that crossing is satisfied much easier by the Veneziano amplitude than by any neighboring would-be amplitudes. The intensity of that “increased crossing”, reflected by the spiky dip in tolerance, makes us believe that our algorithm truly detects physical amplitudes.

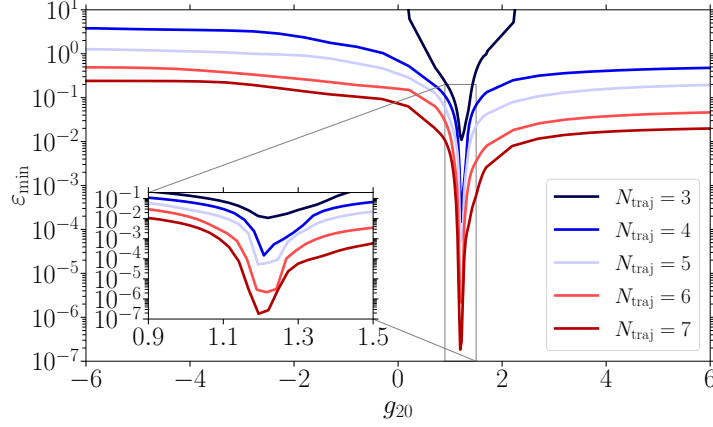


Figure 5.4: Linear trajectories: the crossing tolerance $\epsilon_{\min}$ develops a pronounced dip at the Veneziano point as the number of Regge trajectories N_{traj} is increased [PT23], showing that crossing can be much better satisfied for the Veneziano spectrum with the actual Veneziano amplitude.

Main result: toy-model bending and QCD-motivated deformations. We then deform the trajectories to mimic a key qualitative QFT expectation: trajectories should flatten in the scattering region $t < 0$. Using a smooth two-slope model of the form described in [PT23], represented in Fig. 5.2 we still find clear dips (see fig 4 of the paper): the bootstrap continues to identify preferred amplitudes even with non-linear trajectories. We interpret them as deformations of the Veneziano amplitude with bent trajectories.

This is the main result of this paper: a proof of concept that bent trajectory dual models can be built. Having a model of bending at hand, it would be interesting to now compare it to other approaches that realize bending, such as the satellite sums of [122, 123], and enlarge the ansatz for the Regge residues so as to describe a larger space of amplitudes and explore connections with actual QCD.

Computing Regge residues. Another outcome of the work is that we explained in details how Regge residues of the Veneziano amplitude can be constructed. This then lead to a formula (Guilleminot and Tourkine [130], see also M1 internship report of B. Guilleminot) for the residues of the Veneziano amplitude which I produce here for completeness. The Regge residues can be expressed as sums of generalized Bernoulli polynomials:

$$p_k(s) = (-1)^k 2^{2k} \sum_{n=0}^{\lfloor k/2 \rfloor} \frac{\left(s - k + \frac{d-1}{2} + n\right)_{\lfloor k/2 \rfloor - n}}{2^{4n} n! (k - 2n)!} s^{2n} B_{k-2n}^{(s+1)}\left(\frac{s}{2}\right). \quad (5.2.14)$$

which allows to compute the Regge residues $\beta_k(s)$ for the k -th trajectory of the Veneziano amplitude via

$$\beta_k(s) = \beta_0(s) \frac{\left(s - k + \frac{d-1}{2} + \lfloor k/2 \rfloor\right)_{\lfloor (k+1)/2 \rfloor}}{s^k} p_k(s), \quad \beta_0(s) = \frac{\pi^{\frac{3-d}{2}} s^s}{2^{2s+2d-4} \Gamma\left(s + \frac{d-1}{2}\right)}. \quad (5.2.15)$$

To the best of my knowledge, this explicit formula for the residues of the Veneziano amplitude does not exist in the literature.

Limitation: Asymptotic summation The approach described here is, in theory, limited by the fact that the Regge pole expansion is expected to be only an asymptotic

[8] seriesPiotr; details / Slava asks why . In the case of the Veneziano amplitude, it very clearly is. After summing $O(20-30)$ terms, for typical values of s, t , the approximation ceases to improve upon adding more Regge trajectories.

Comment on Mandelstam’s finite-energy sum rules. Mandelstam [131] suggested to apply crossing by means of integrated relations known as finite energy sum rules. In [PT23] we explained why this approach cannot work in the present context: one cannot close a contour while staying in a region where the Regge expansion is a good description of the amplitude. In practice crossing must be applied point-wise, and the crux is to find kinematic regions where the Regge pole expansion is accurate.

A question raised by this work is whether a truncated amplitude with a finite number of Regge trajectories could ever satisfy crossing truly. The next section answers this in the negative: under standard boundedness assumptions, dual models need infinitely many trajectories [PT27, PT24].

5.2.3 Infinitely many trajectories and the no-go of [PT27, PT24]

Let us now describe another series of works, mostly of analytical nature, relating to Regge trajectories in dual models. A typical dual-model spectrum contains infinitely many states organized into several Regge trajectories. In [PT27, PT24] we show that this is not optional: finitely many trajectories are incompatible with standard bootstrap assumptions.

In the CFT bootstrap, one gets more leverage and similar rigorous statements on Regge trajectories can be made away from perturbative approximations such as weak coupling as in QFT and dual models, see most notably [132, 133].

In [PT24] we proved a no-go theorem for dual amplitudes with scalar external states: in unitary meromorphic amplitudes, finitely many trajectories are inconsistent. In [PT27] we sharpened the logic and extended it to weakly-coupled graviton scattering in $d = 4$, making explicit the role of polynomial boundedness.

Before we start, let us contrast a single-trajectory spectrum with a genuine string spectrum. A spectrum with exactly one spin per mass level looks like a classical rotating string tuned to obey $J \propto M^2$. Sub-leading Regge trajectories, by contrast, encode additional vibrational modes: the ripples on the string’s worldsheet. In usual string theory they build up the large number of spinning states at fixed mass level. The no-go theorem can be read as a sharp statement that this “classical” limit is not compatible with crossing and boundedness in a meromorphic S-matrix. A useful motto would be: strings have to vibrate.²¹

5.2.3.1 Context

Besides our own conceptual motivation (extent of physical validity of approximating amplitudes by finite sums of trajectories in the Regge bootstrap), the question of finitely many trajectories came up recently in the literature. The authors of [55] used semi-definite EFT bootstrap methods to study large- N gauge theories. This was revisited in follow-ups by the same authors and different groups [56, 57]. I do not aim to review the full extent of these results here; I only extract two points. First, they observed a remarkable agreement with the physical $N = 3$ QCD meson spectrum on the ρ trajectory in $\pi\text{--}\pi$ scattering at

²¹Interestingly there is a special case where this motto fails to a quantum anomaly in the worldsheet angular momentum operator: the case of twisted/chiral strings and their null strings cousins, described in Sec. 6.2. These strings are frozen to their supergravity subsector. However, in that sense they have no Regge trajectory at all so they don’t provide a counter-example to the actual no-go’s.

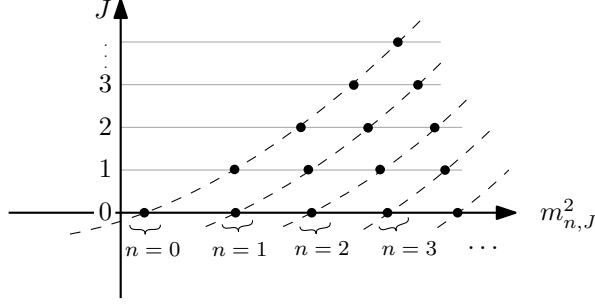


Figure 5.5: A typical dual-model spectrum contains infinitely many states organized into several Regge trajectories. Finitely many trajectories are incompatible with standard bootstrap assumptions [PT27, PT24].

large N [56]. Second, extremal functionals indicate clearly that the spectrum is discrete and Regge-like, but generically *corresponding to a single Regge trajectory*.

Our no-go results leave a priori no doubt that exact single-trajectory (or finite-trajectory) meromorphic amplitudes cannot exist under the usual assumptions, but they do not by themselves explain why the numerics can look so close to a single trajectory. For this reason, [PT27] supplemented the theorem with a “primal” bootstrap study of meromorphic graviton amplitudes, using the stringy ansatz proposed in [134]. We observed the appearance of single-trajectory-like objects in extremal amplitudes in primal construction, shedding some light on the aforementioned tension.

5.2.3.2 Proof of the no-go theorem

The essence of the proof is a tension between crossing and polynomial boundedness. For simplicity, I explain the improved proof of [PT27] in the context of scalar amplitudes considered in [PT24].

We start from a crossing-symmetric amplitude $A(s, t) = A(t, s)$ with no subtractions, assumed to be made of exactly a single trajectory, i.e. the spin sum in Eq. (5.1.1) reduces to the highest spin $J = n$ at level n :

$$A(s, t) = \sum_n \frac{c_n P_n \left(1 + \frac{2t}{m_n^2 - 4m^2} \right)}{s - m_n^2}. \quad (5.2.16)$$

By crossing, the sum over s -channel poles must reproduce the singularities in the t channel. The series converges for $t < m_0^2$ and diverges at $t = m_0^2$, the location of the first singularity.

In what follows, we assume a standard **growing** spectrum, $m_n^2 \rightarrow_{n \rightarrow \infty} \infty$.

At large n , the relevant object is the Legendre polynomial evaluated at $z_n(t) = 1 + \mathcal{O}(1/m_n^2)$. As we discussed, its asymptotics depends on how m_n^2 grows compared to n^2 . If $m_n^2 \gg n^2$, then $P_n(z_n) \rightarrow 1$ and the series cannot generate a localized singularity at $t = m_0^2$. If $m_n^2 \sim n^2$ one instead obtains a smooth Bessel-like limit. The only way to make the sum diverge precisely at $t = m_0^2$ is the “slow growth” regime $m_n^2 \ll n^2$ asymptotically, where

$$P_n \left(1 + \frac{2t}{m_n^2} \right) \sim \exp \left(2n \sqrt{\frac{t}{m_n^2}} \right) \quad (n \rightarrow \infty). \quad (5.2.17)$$

Requiring convergence for $t < m_0^2$ but divergence at $t = m_0^2$ forces the coefficients to

saturate a borderline decay [PT27, PT24],

$$c_n \sim \frac{1}{m_n^2} \exp\left(-2n\sqrt{\frac{m_0^2}{m_n^2}}\right), \quad (5.2.18)$$

up to sub-exponential factors. This step uses only crossing.

Now fix $t = t_0$ and assume that the amplitude is polynomially bounded in s . Then there exists an integer N such that $A(s, t_0)/s^N \rightarrow_{s \rightarrow \infty} 0$. If that is the case, the following integral converges,

$$I^{(N)}(t_0) = \int_{m^2}^{\infty} ds \frac{\text{Disc}_s A(s, t_0)}{s^{N+1}}. \quad (5.2.19)$$

If $t_0 > m_0^2$, one can evaluate the integral explicitly and find

$$I^{(N)}(t_0) \sim \sum_n \frac{\exp\left(2n\left(\sqrt{\frac{t_0}{m_n^2}} - \sqrt{\frac{m_0^2}{m_n^2}}\right)\right)}{(m_n^2)^N}, \quad (5.2.20)$$

which diverges for any N . Therefore crossing implies that the amplitude must violate polynomial boundedness for $t \geq m_0^2$.

This shows that a single-trajectory meromorphic amplitude cannot satisfy simultaneously crossing and polynomial boundedness. The same logic extends to any finite number of trajectories [PT27]. Note also that the proof does not use unitarity: crossing and boundedness already generate the contradiction. In the presence of infinitely many trajectories, this obstruction can be evaded by cancellations between trajectories.

Polynomial boundedness at $t > 0$. Note that our argument is in tension with a strong form of polynomial boundedness: polynomial boundedness past the first (or n -th) pole, in particular polynomial boundedness at positive t .

Importantly, the Veneziano amplitude explicitly passes these tests: the integral Eq. (5.2.19) does converge at $t > m_0^2$ or past any other singularity, if subtracted enough times.

Older proofs Some older arguments against single-trajectory meromorphic amplitudes already exist in the 60s–70s literature, reviewed in [PT24]. The point of the proof above is that one can isolate a minimal obstruction that only uses crossing and polynomial boundedness.

Dual functional proof. To provide another layer of proof, in the first paper [PT24], we also provided a rigorous dual exclusion bootstrap of the amplitude proposed in [55]. Their proposed spectrum, behaving as $m_n^2 \sim -0.33 + n^{1.25}$ can be completely excluded by means of a positive functional acting on the full crossing relation $A(s, t) = A(t, s)$ showing that no set of positive parameters $c_{n,n}$ could solve the bootstrap equation, discarding rigorously the existence of that fitted spectrum as a valid meromorphic amplitude spectrum, thus ruling out such an amplitude.

Other theories The proof presented here for unsubtracted scalar amplitudes apply identically to the case of subtractions, or even external spinning states like we did in $d = 4$ for external gravitons [PT27].

5.2.3.3 Implications for weakly coupled UV completions

The no-go above is a statement about *meromorphic* amplitudes. Physically, it aims to apply to weakly coupled UV completions where the tree-level amplitude is dominated by narrow massive higher-spin exchange (as in large- N gauge theories or weakly coupled strings).

In the context of gravity, it is long known since Camanho, Edelstein, Maldacena, and Zhiboedov [70] (CEMZ) that causality constraints force the exchange of an infinite tower of higher spins to regulate graviton scattering amplitudes in gravitational EFTs at weak coupling. A follow-up on this result was the universality of the Veneziano asymptotics result by [107], see also details in [134]. This result was refined in [134] where it was observed that universality would only follow if extra requirements on the zeros of the amplitudes were assumed.

Our result is a refinement on top of CEMZ: *assuming* a weak-coupling UV completion by a meromorphic amplitude, we see that it is necessary that this meromorphic amplitude is fully stringy, with all tower of subleading trajectories.

Of course, the study of gravitational EFTs, especially in 4d where graviton amplitudes are IR divergent, is a whole subject of its own, with a large activity within the larger field of the positivity program [135].

Of course, these results say nothing about moving to strongly coupled UV completions, and since, as pointed out long ago [136], parametric control over realistic solutions in string theory probably require strong coupling. It would thus be important to see how these results can be extended beyond tree-level, using non-perturbative bootstrap techniques for gravitational scattering, as pioneered by [42, 43]

5.3 Brief conclusion

This chapter used Regge theory as a practical language to study a very explicit corner of the S-matrix: meromorphic dual models. We reviewed the basic Regge/duality dictionary, used the Coon amplitude as an example where bending is possible but comes with non-meromorphy, and described a Regge bootstrap that turns crossing into a constructive numerical constraint.

The main conceptual output is the no-go theorem of [PT27, PT24]: under standard boundedness assumptions, a unitary meromorphic amplitude cannot have only finitely many Regge trajectories. For gravity, this tightens the connection between weakly coupled UV completions and string theory: combined with causality constraints [70] and Regge arguments [107], it adds to the notion of “string universality” in the weak-coupling regime.

Part III

Other contributions

Chapter 6

Other contributions

6.1 Eikonal amplitudes on the Celestial sphere

In another line of work, since 2021, in collaboration with Tim Adamo and younger collaborators of his group, Andrea Cristofoli, Wei Bu and Bin Zhu, we explored various aspects related to high-energy gravitational scattering and eikonal resummation. Results have been published in [PT19, PT21, PT25]. The collaboration is still going on to this day, including discussions with my former collaborator Eduardo Casali.

Eikonal resummation is a long-known scheme that describes semi-classical scattering in quantum mechanics and QFT [137, 138] through the exponentiation of the impact parameter amplitude, the eikonal phase. For a spin- $J = 1, 2$ exchange, at fixed $t < 0$ and large $s \gg -t$, we generically have

$$M_{\text{eik}}(s, t) = \frac{s^{J-1}}{t} \int d^2b e^{iq \cdot b} (e^{i\chi(s, b)} - 1) \quad (6.1.1)$$

where $\chi(s, b)$ is the eikonal phase, given by the Fourier transform of the tree-level amplitude.

In [PT19], we generalized a calculation of 't Hooft [139] relating eikonal scattering to scattering through a shockwave to arbitrary spacetimes, including boosted Kerr spacetimes, as proxys for semi-classical amplitudes with spin. In [PT21], we investigated certain spinning shockwave spacetime solutions and proved that most of them do not respect basic causality requirements.

These papers showed also some interesting facts about the structure of the eikonal amplitude, that actually allowed to perform new non-trivial calculations in nontrivial spacetimes. Most notably, because the eikonal amplitude is a two-dimensional integration d^2b in four spacetime dimensions (the transverse space is $d - 2$ -dimensional in general), the integral can be holomorphically factorized as $\int d^2b \sim \int db \int d\bar{b}$, allowing to formulae similar to the stringy Kawai-Lewellen-Tye relations [140] which express closed-string amplitudes as products of open string amplitudes, on which I had previously extensively worked on [PT4, PT11, PT16, PT15, PT17]. Note that without that string-inspired technique, it would have been absolutely impossible to compute the boosted Kerr eikonal amplitude [PT19, Eq. 6.19]. This correspondence is intriguing. It seems to relate to an attempt at “stringifying” eikonal scattering by Verlinde and Verlinde [141]. It would be interesting to delve into this question further and see if this is an artifact linked to the fact that transverse space in $d = 4$ has the same dimension as the string worldsheet or if there is more to it.

In [PT25], we investigated another aspect of trans-Planckian scattering: the role of the exponentiation in generating a well-defined analytic structure of Mellin transforms of

flat-space scattering amplitudes, also known as “Celestial amplitudes” [142]. Celestial amplitudes are S-matrix elements expressed in boost eigenspace basis instead of momentum eigenstates. The two-to-two amplitude simplifies and can essentially reduce to a function of two variables, an angle on the celestial sphere and a boost weight β

$$\mathcal{A}(z, \beta) = \int_0^\infty d\omega \omega^{\beta-1} A(-\omega^2, -z\omega^2) \quad (6.1.2)$$

Of course the gravitational scattering amplitudes are IR divergent as usual and this should be interpreted with caution if looking at a gravity amplitude. The authors of [142] emphasized that such a transform, because it mixes IR and UV in the energy integral, should be a particularly well suited place to study quantum gravity amplitudes. They insisted in particular that black-hole formation in the UV was the way how this integral is regulated. In [PT25], we made the point that eikonal exponentiation also provides a mechanism to generate well defined analytic in β convergent expressions. This study thus constitutes a step in the study of ACU in this boost eigenstate basis. It would be interesting to see how much non-perturbative information can be leveraged in this language, in the spirit of [143].

6.2 Ambitwistor strings, the double copy, monodromies.

My PhD thesis revolved around loop amplitudes in string theory and supergravity, and UV divergences in the field theory limit of string theory. After it, I worked for several years on scattering amplitudes, using string theory as a tool to understand better perturbative QFT. In particular, I worked on the color-kinematics duality and associated double-copy of gauge theory to gravity at loop level in string theory [144]. I also worked on twistor strings, in particular on model invented by Mason and Skinner [145] to explain the Cachazo-He-Yuan formulae (CHY) [146, 147].

In this manuscript, I have decided to be brief on these questions, as I do not work actively on the problem at the time of writing. I will mention briefly my main contributions and mention some open questions left in this field.

The general idea behind twistor theory, invented by Roger Penrose, is to reformulate the physical laws in a way that make on-shellness manifest. Twistor space is the space of null rays of space-time: a null ray corresponds to a point in twistor space via the *incidence relations*.²² This motto became fully operational with Witten’s twistor string construction in 2004 [150] to describe gauge theory tree-level scattering amplitudes as integrals over holomorphic maps in twistor space. That string theory however could not easily be extended to loop level, due in particular to the presence of unphysical states running in loops.

In 2013, CHY proposed remarkable new formulae that expressed the tree-level S-matrix of scalar, gauge and gravity theories at any number of points. These S-matrices were generalized to a whole zoo of theories, which are all deep down related by a version of the double-copy between open and closed strings [140].

6.2.1 Scattering equations at loop-level

The Ambitwistor string was a formalism much closer to ordinary string theory, and it allowed to solve the loop problem of twistor strings. Adamo Casali and Skinner wrote down one-loop scattering amplitudes for Ambitwistor strings [151], see also Ohmori’s formulation [152]. These amplitudes however unfortunately relied on a set of yet-unsolved

²²See [148, 149] for introductory texts on twistor theory.

transcendental equations *the loop-level scattering equations*. In [PT6], right after my PhD and together with E. Casali who was a PhD student at the time, we proposed the only numerical approach to this problem, showing that in some infra-red limit, one recovers expect one-loop integrand in supergravity.

Then, my main contributions in this topic were to elaborate a construction made with the Oxford group to make these amplitudes usable, by bypassing the need to solve the scattering equation, instead reducing them as tree-level worldsheets. I met them as I was giving a seminar there (I was a postdoc in Cambridge at the time), and we started collaborating efficiently, as I brought them my string perturbation theory expertise. This gave rise to a series of publications for one and two-loop scattering amplitudes [PT7, PT8, PT10]. These works provided a simple and elegant solution on how to use (Ambi)twistor string amplitudes to loop level.

6.2.2 String quantization and a curious ambiguity

I also had several contributions on another question: the link between these (Ambi)twistor strings and ordinary strings. I mentioned already that the formalism was much closer to vanilla string theory than Witten's original twistor string. My main contribution with Casali (and then Herfray) was to show that Ambitwistor strings are nothing but null strings, and study their quantization. We made the connection in [PT9, PT13, PT14], using ideas from Siegel [153] and used it to study the quantization of these objects, together with their tensionful version called chiral strings in [154] and twisted strings in our papers. The main point that we contributed to clarify is that there is a quantization ambiguity for the worldsheet string theory. There exist a complex-worldsheet prescription that leads to a complex CFT where the normal ordering ambiguity is transferred from the mass operator to the angular momentum, yielding the string to freeze onto a specific level, with exactly two left (right) movers and no right (left) movers in the physical Hilbert space.

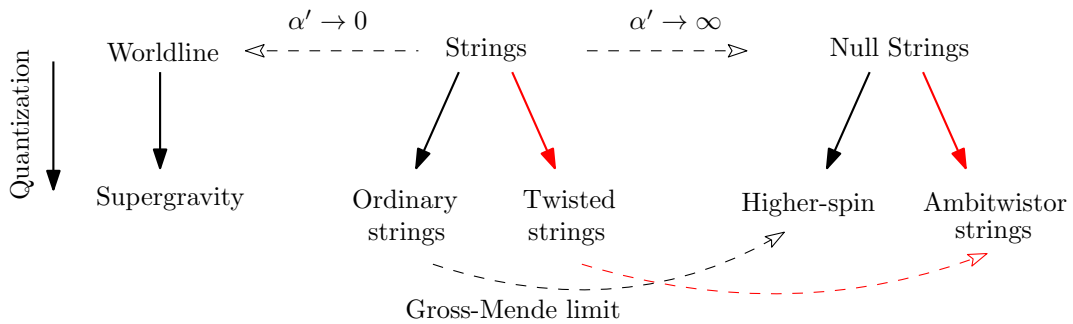


Table 6.1: Picture of relationship (or rather, absence thereof) between string theory and Ambitwistor strings.

Today, I would say that the link with string theory is mostly deciphered. There is room for a deeper connection between both formalisms, but in surface their differences are relations are clear, as I will explain below briefly. For the loop amplitudes, the situation is more complex. The reference [155] then allowed to fully clarify the picture by showing how twisted string integrals are computed.

The tensionful version gives interesting perspectives to be related to an old construction of Witten [156] that I would like to revisit one day, seeing maybe the analytic continuation as a gauging of time reversal symmetry as in [157].

6.2.3 Monodromies and integrands in string and field theory

Then in [PT16, PT17], we studied a question pertaining to the generalization of the tree-level KLT construction to loop-level, and the associated notion of color-kinematics duality [144]. We showed that naturally in string theory the structure of open string integrand should lead to a proliferation of contact terms, linked to how contours must be closed on the string worldsheet, based on my initial construction [PT11]. These contact terms are consistent with the construction of in [158, 159] where the vanilla double-copy used in previous papers by the UCLA group to compute 3 and 4 loop supergravity amplitudes seem to fail at five-loops. The question of how the double-copy works at loop level is still an open question. String theory is most likely the correct way to understand the problem, but it is a very hard question, which involve high-loop string theory, a notoriously complex problem.

Part IV

Conlusion

Chapter 7

Discussion

In this manuscript, I presented two main parts, revolving around how non-perturbative UV completion occur for the S-matrix at the level of the 2-to-2 matrix elements. I presented several new techniques of analytical and numerical, some of them aiming to tackle some long standing questions in the field. Several of these techniques were inspired by works of the 60s (for both parts I and II), revisited with modern tools and concepts.

Part I: constructive ACU amplitudes. Chapter 3 developed a practical version of the Atkinson–Mandelstam program. In this bootstrap, the unknowns are spectral data (single and double discontinuities). Crossing is built in at the level of the representation, and unitarity becomes a non-linear equation to be solved (the Mandelstam equation). This provides direct access to objects that are otherwise hard to diagnose from convex bounds: double discontinuities, Landau-curve support, and explicit particle production. This language makes the split between the various contributions, elastic, inelastic coming from 2-body and genuinely inelastic fully transparent, allowing a very precise control over the amplitude.

Part II: Regge theory, dual models, and UV completion. Chapter 5 uses Regge/dual language to ask what analytic structure is compatible with a weakly coupled UV completion, and what the spectrum must look like under standard boundedness assumptions. In particular, we saw with the Regge bootstrap how crossing can be made constraining for dual models, allowing to deform stringy straight trajectory into bent trajectories, expected in QFT. Then we demonstrated no-go theorems that show that dual models need to have infinitely many trajectories and commented on the ripples on the string.

A common tension Interestingly, these two unrelated stories also shared a common issue. In both cases, existing semi-definite methods are in tension with physical expectations: particle production in part I and daughter trajectories in part II. Let us discuss them in more details now.

7.1 Non-perturbative bootstraps: Atkinson-Mandelstam vs ϱ -bootstrap and Aks tension

7.1.1 Summary of both formalisms

The ϱ -bootstrap: convex bounds on low-energy data

What is optimized. One chooses an ansatz space of crossing-symmetric, maximally analytic amplitudes $A(s, t)$ and optimizes a low-energy observable, such as $\bar{c}_0$ or $\bar{c}_2$ at the crossing-symmetric point.

How unitarity enters. One imposes the unitarity disk condition $|S_J(s)| \leq 1$ on a grid

of energies and up to a spin cutoff, and rewrites it as semi-definite positivity constraints. This convexity is the main practical advantage: SDPB can map allowed regions within constraints and find extrema very reliably.

What it is good for.

- Fast scans over parameters (mass gaps, spacetime dimension, symmetry choices).
- Bounds on low-energy data, and robust numerical trends (e.g. kinks with dimension).
- A concrete family of extremal amplitudes (within the chosen ansatz class) with strong IR features like Regge resonances.

What it hides. By default the method closes on $2 \rightarrow 2$ data. Multi-particle physics is not parameterized explicitly, and UV completion is only reached over transient ranges of energies in the cases I know of. In particular, extremal points often look “fully elastic” in the resolved partial waves ($|S_J| \simeq 1$), raising the question of where multi-particle production is encoded in the truncation/ansatz.

The AM bootstrap: constructive completion in spectral variables

What is solved for. AM starts from a dispersive representation in which crossing is manifest and analyticity is encoded in spectral data: a subtraction constant, a single spectral density, and a double spectral density (the double discontinuity). In this language, the separation between the 2-body physics and genuinely multi-particle input is explicit; see §3.1 for the concrete equations used in this thesis.

How unitarity enters. Unitarity is a *non-linear* integral equation (the Mandelstam equation) plus an additional scalar ($J = 0$) condition. Numerically, this is a non-convex problem: one uses fixed-point iteration, Newton-type methods, or function-space optimizers. The Mandelstam equation solves all spins J , $\text{Re } J > 0$ at once.

Caveat: in practice, the Mandelstam equation is solved on a grid and up to a numerical precision tolerance, which induces in theory a spin cut-off.

What it is good for.

- Particle production is built in: one can track where inelasticity enters and how it is distributed in energy and spin.
- Analytic structure is clean: support of the double discontinuity, Landau curves, and the distinction between “forced by two-particle unitarity” and “supplied as multi-particle data”.
- It produces *actual* amplitudes that can be inspected to all energies, spin or impact parameter.

Main caveat. The problem is hard numerically, and in its current stage it cannot yet explore the whole space of solutions.

7.1.2 General comparison

Even if the basic idea (start from $A+C$ and solve U) is therefore the same, the *logic* of the bootstraps is totally different. In the ϱ -bootstrap, a quantity is extremized, in the AM bootstrap, an amplitude is built. Therefore, by default, the AM bootstrap does not generate extremal amplitudes, and solutions could be anywhere inside the allowed space. In the toy-model study [PT26] we could construct amplitudes filling the whole space, as

explained in Sec. 3.2.2.2. The fact that our results (as seen in Fig. 3.9) land close to the boundary is therefore a priori surprising. In fact, this appears to be related to a sort of uniqueness property of 2PRR amplitudes, which live close to the boundary.

Table 7.1 summarizes a few other salient points of comparison. A few comments are in order. Firstly, “double-discontinuity” and “Landau curves” appear as separate entries. By construction Landau curves are not included in the ϱ -bootstrap. Ref. [30] did actually the effect of a toy-model version of LCs, without seeing any chance on the observables. The double-discontinuity can however be accessed and computed. However, the reason for the $\times$ symbol in this line rho ϱ -bootstrap is a point I make below (see discussion above Sec. 7.1.4) that in the ϱ -bootstrap, the double-discontinuity does not converge, as a consequence of the Aks screening phenomenon and its relation to how semi-definite optimization works. Secondly, resonances are indicated as “?”. For the ϱ -bootstrap, regions

	ϱ	AM
Crossing-symmetry	✓	✓
Maximal analyticity	✓	✓
Double-discontinuity	✗	✓
Landau curves	✗	✓
Asymptotic control	hard	✓
Particle-production	?	✓
Resonances	✓	?
Speed / ease of use	✓	✗
Bounds	✓	hard

Table 7.1: Recap of the semi-definite ϱ -bootstrap and the Atkinson–Mandelstam approach.

with strong tower of resonances have been clearly identified [30, 34, 42, 43, 90, 160]. In the AM approach, we have identified resonances in the toy-model [PT26], not yet in the full problem. At face value, this is a consequence of the limited reach of convergence of the AM bootstrap with one-subtraction in the ML approach: in the black-thick line where we construct amplitudes in Fig. 3.9, both the ϱ -bootstrap and AM bootstrap agree that no resonances are present. I expect this problem to be solved once we include more subtraction (see Future direction chapter below).

Let us now comment more precisely on Aks screening, de-screening and the asymptotic control in these models.

7.1.3 Aks screening

Inelasticity screening. In [PT29], we called Aks screening the effect by which multi-particle production can be suppressed in a chosen set of partial waves by tuning the multi-particle input η_{MP} and ρ_{MP} and what are the consequences of doing so. In [PT29], we used ρ_{MP} as a tunable degree of freedom to at the same time solve the AM equations and set to zero the inelasticity in a few low-lying partial waves, at spins $J = 0, 2$. At higher spins, we encountered difficulties to make the algorithm converge but from our experience it seems clear that the difficulty was only technical, and not conceptual.

The screening happens in the following way. Inelasticities $\eta_J(s)$ are defined as

$$\eta_J(s) = 1 - |S_J(s)|^2 \quad (7.1.1)$$

When $\eta_J(s) > 0$, then we have particle production in spin J , and conversely $\eta_J(s) = 0$ means no particle production at spin J . Through Froissart-Gribov, these inelasticities

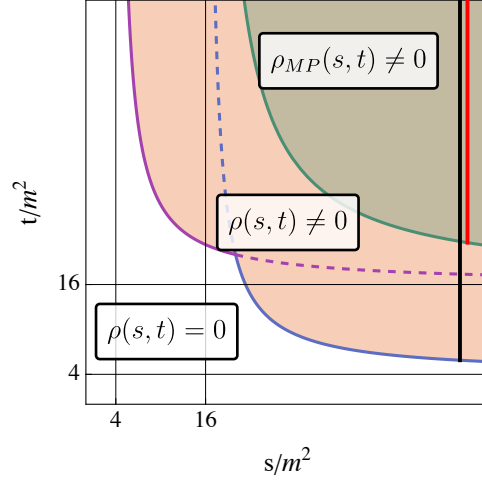


Figure 7.1: Aks screening contours: inelasticity from the black thick vertical line in Froissart-Gribov has to be canceled by integration of ρ_{MP} on the red line.

can be computed as the integral of the inelastic part of the double-discontinuity, and are basically given by

$$\forall J = 0, \dots, J_{\text{max}} : \quad \eta_J(s) \propto \int_4^\infty Q_J \left(1 + \frac{2t}{s - 4m^2} \right) (\rho_{\text{el}}(t, s) + \rho_{\text{MP}}(t, s)) dt, \quad (7.1.2)$$

along the contour given shown in Fig. 7.1. Aks screening is realized for a given J if, for all $s \geq 16m^2$, the integral or ρ_{MP} cancels exactly the integral of $\rho_{\text{el}}(t, s)$.²³ In the actual Mandelstam equations, ρ_{MP} backreacts on the elastic double-discontinuity in a fully non-linear manner, also involving crossing, as it take part in a system that (very) schematically read like:

$$\rho_{\text{el}}(s, t) = \iint (\rho_{\text{el}}(s, t') + \rho_{\text{el}}(t', s) + \rho_{\text{MP}}(s, t'))^2 \quad (7.1.3)$$

and the problem is of course unsolvable analytically. Nevertheless, our numerical results show that this can be done and in [PT29], we achieved screening while maintaining UV Reggeization of spins $J = 0, 2$.

The most interesting aspect of the numerical implementation, we see that as we increase the screening, we make the low-energy observables more extremal, see Table 1 on page 44 of [PT29].

Analytic argument and *rigid* Aks screening. The problem can be simplified and we can draw intuition from the following rigid case, presented in [PT29]. Ignore (i) back-reaction of ρ_{MP} on ρ_{el} and (ii) crossing constraints on ρ_{MP} . This is what was called *rigid* screening. The goal is thus not to solve the full coupled system, but to isolate a mechanism: screening low-spin inelasticities typically requires increasingly oscillatory and large multi-particle double-discontinuity.

Using Froissart–Gribov (2.1.26) at fixed $s \geq 16m^2$, rigid screening means that, given a $\rho_{\text{el}}(s, t)$ we search for a fixed $\rho_{\text{MP}}(s, t)$ such that the inelasticities $\eta_J(s)$ vanish:

$$\forall J = 0, \dots, J_{\text{max}} : \quad \int_4^\infty Q_J \left(1 + \frac{2t}{s - 4m^2} \right) (\rho_{\text{el}}(t, s) + \rho_{\text{MP}}(t, s)) dt = 0, \quad (7.1.4)$$

²³The scattering is already elastic for $s \in [4m^2, 16m^2]$ thanks to the Mandelstam equation being satisfied.

At fixed s , the elastic and multi-particle contributions start at different locations $t_{\text{el}}(s)$ and $t_{\text{MP}}(s)$ with $t_{\text{MP}} > t_{\text{el}}$, given by the Landau curves (2.1.22a) and (2.1.22b). The integration ranges are thus vertical slices in Fig. 7.1) starting at the Landau curves.

The main point which we prove in the paper is that forcing the equality (up to a sign) of the integrals of (fixed) ρ_{el} and (searched) ρ_{MP} makes the norm of ρ_{MP} diverge as more and more moments are set equal. The demonstration is trivial and relies on the fact that, by Mahoux-Martin positivity, ρ_{el} cannot vanish on the interval before the inelastic integral starts. If all integrals were equal for all J , by some completeness argument the functions ρ_{el} and ρ_{MP} must be identical, but this cannot happen since they do not have the same support. At finite J_{max} , this causes the norm of ρ_{MP} to grow exponentially. The final bound we find is

$$\|\rho_{\text{MP}}\| \gtrsim B(s)^{J_{\text{max}}} \quad \text{with} \quad B(s) \equiv \chi(2R(s) - 1) > 1, \quad (7.1.5)$$

where $\chi(z) = z + \sqrt{z^2 - 1}$, and $R(s) = \frac{\chi(z_{\text{MP}}(s))}{\chi(z_{\text{el}}(s))} > 1$. Conversely, these bounds imply that, at fixed $\|\rho_{\text{MP}}\|$, one can only screen a fixed amount of moments, controlled by

$$J_{\text{max}} \lesssim \frac{\log(\|\rho_{\text{MP}}(s, \cdot)\|)}{\log(B(s))} \quad (7.1.6)$$

From the perspective of $B(s)$ alone, the bound is thus stronger in the IR, where it becomes maximum at $s \simeq 32.2m^2$, where $B(s) \simeq 16$. Further information would have to come from more constraints on ρ_{MP} .

Non-convergence of double-discontinuity This discussion also gives a clear sense in which in the ϱ -bootstrap the double-discontinuity should not converge. The reason is tied to how convergence is achieved in the ϱ -bootstrap: it is a double scaling limit where the ansatz size and number of constraints are simultaneously sent to infinity. “Convergence” means reaching an empirical stable plateau for the target observable, where in particular adding a constraint $J_{\text{max}} \rightsquigarrow J_{\text{max}} + 1$ does not change the bound.

Empirically, what happens when a new constraint $|S_{J_{\text{max}}+1}(s)| \leq 1$ is added is that semi-definite optimization will produce instead a saturation of the constraint $|S_{J_{\text{max}}+1}(s)| = 1$, thereby screening one more spin. From our previous discussion, this should backreact in a non-trivial way on the double-discontinuity in the UV. Hence when adding a new constraint, $J_{\text{max}} \rightsquigarrow J_{\text{max}} + 1$, the double-discontinuity *has to* change if that spin is to be saturated by the semi-definite optimization as well. Therefore it seems unclear how a limit for the double-discontinuity would exist. Note that this does not prevent bounds from converging in the plateau sense described above, because the action happens in the UV, and has little to no effect on bounds on low-energy observables.

This observation is consistent with the empirical observation on ϱ -bootstrap amplitudes that the double-discontinuity obtained in the ϱ -bootstrap is anyway not a nicely behaved object with respect to convergence.

7.1.4 Adding inelasticity in the ρ bootstrap and de-screening

We have seen that inelasticity can thus be screened, producing more extremal bounds. This seems compatible with the picture that in the ϱ -bootstrap, where amplitudes are built through an extremization mechanism, the default results should be amplitudes with minimum inelasticity. In [PT29] we studied also the converse statement: we added inelasticity by hand in the ϱ -bootstrap. We call this *de-screening*.²⁴ It can be easily added in

²⁴The authors of [161] also studied inelasticity in the ϱ -bootstrap using a method similar to the one we employed.

the ϱ -bootstrap: simply change the matrix condition Eq. (4.1.2) which imposed $|S_J| \leq 1$ for another semi-definite condition which imposes $|S_J(s)|^2 \leq 1 - \eta_J(s)$ where $\eta_J(s)$ is the profile extracted from the AM-bootstrap results.

As one adds more and more inelasticity in the ϱ -bootstrap, the bounds become less extremal (see Tables 2 and 3 in the paper.)

This observation is consistent with what discussed above, as a sort of converse statement. This de-screening also provides another interesting picture at the level of the amplitudes. In Fig.7.2, we depict the forward limit of the imaginary part of the amplitude, at a given value of $c_0/(32\pi) = 1.25$. The variable $J_{\min}$ indicates the amount of spins where inelasticity was added, from no inelasticity $J_{\min} = 0$, to inelasticity up to $J = 10$. It is very interesting to see that this causes the results of the ϱ -bootstrap to align more and more with the behavior from the NN. This observation is new, and deserves to be studied in more details in the future. This figure also illustrate another interesting fact: past

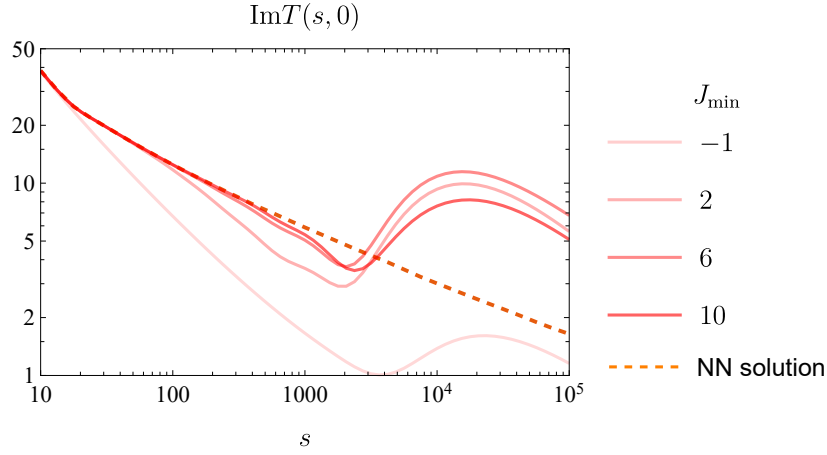


Figure 7.2: Fig. 31 of [PT28]. Asymptotic completion as inelasticity is de-screened in the ϱ -bootstrap and $J_{\min}$ increases.

$s \sim 1000m^2$, the amplitude develops oscillations, while the AM amplitude remains stable. This stability actually persists over orders of magnitude, as is shown in Fig. 7.3.

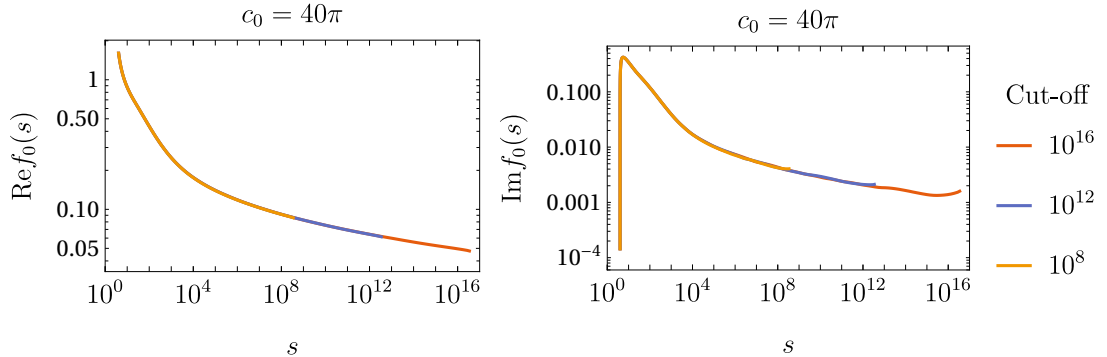


Figure 7.3: Fig. 17 of [PT28]. Stability of NN solution represent Reggeization and asymptotic numerical UV completion

In Fig. 7.4, we thus forget de-screening and simply display the amplitude at growing

ansatz sizes for relevant $J_{\max}$.

We see that convergence is improved at larger ansatz sizes, but it does not really settle and oscillations persist. From experience in both [PT29, PT28], this state of affairs is standard in the ϱ -bootstrap.

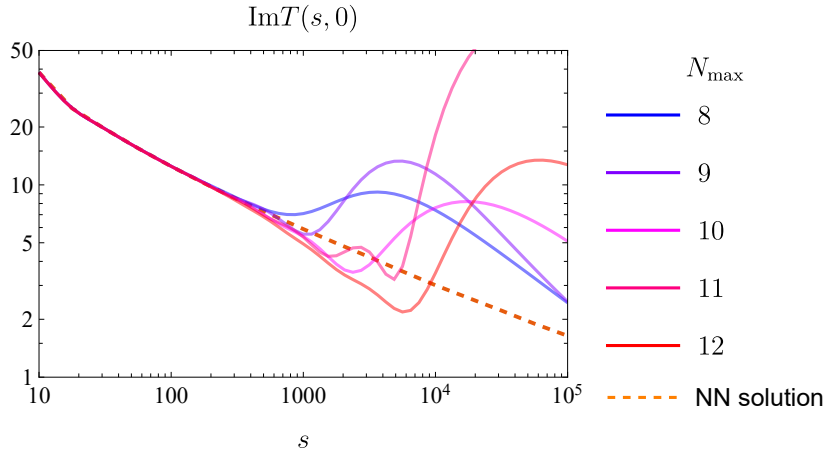


Figure 7.4: Fig. 30 of [PT28]. Asymptotic completion as inelasticity is de-screened in the ϱ -bootstrap and $J_{\min}$ increases.

7.2 Meromorphic amplitudes

We now come to another tension that was mentioned in the text. Recent dual bounds for large- N QCD have produced sharp numerical constraints on pion scattering and identified extremal amplitudes with a single leading trajectory [55–57]. Related ideas were pushed further in the gravitational context, in particular in [58, 59]. The results presented in the text [PT27, PT24] show that this interpretation must be in tension with polynomial boundedness which forces an infinite number of trajectories. The remaining question is thus: why does the numerics look so close to a single trajectory in these references?

The most obvious direction for a resolution, also mentioned in [57], lies in the following fact: *subleading trajectories can be arbitrarily small*. The no-go theorem does not provide a lower bound on the couplings of subleading trajectories. Crossing can require infinitely many trajectories while allowing their residues to be extremely suppressed over a wide energy range. In that situation, a finite-precision numerics might not be able to detect them and the spectrum looks effectively single-trajectory. This would require a detailed inspection of the class of extremal functionals obtained and their scaling limit, a probably numerically messy task. The authors of [59] mention a **threshold** value for the **spectrum** function of SDPB chosen to 10^{-5} . The results presented below indicate that in concrete cases, this value might be insufficient.

It might also be relevant that our no-go indicates that the conflict with polynomial boundedness occurs at *positive* t , after the first singularity. It would be useful to discuss this point in more details with the authors of the references mentioned above in the near future, to see if that constraint can be added and at what cost.

To study the question of how single-trajectory-like amplitudes emerge, in [PT27] we studied a primal bootstrap with a string-inspired ansatz. We found a tendency towards single-trajectory-like extremal solutions at finite ansatz size. As the truncation is increased, this effect become more prominent but the amplitude becomes singular. This provides a concrete mechanism for why a finite-dimensional numerical ansatz might dis-

play a sharp, apparently single-trajectory extremum even though the exact theory cannot have finitely many trajectories.

7.2.1 Primal bootstrap and the emergent sister trajectory

Let us summarize briefly the content of the numerical analysis of [PT27].

We built meromorphic graviton amplitudes using the “stringy ansatz” of Häring and Zhiboedov [134], using SDPB for linear programming, taking advantage of the high numerical precision that one can use in the software. This was greatly helped by the development by the use of the mathematica SDPB wrapper [91].

Remark (IR divergences.). *Let me here make the following important remark. In the paper, we studied graviton amplitudes in $d = 4$. Of course these do not exist nonperturbatively because of IR divergences. Our paper is supposed to be understood for instance in dimensional regularization $d = 4 + \epsilon$, in the same sense as [134, 162]. The mechanism we isolated on trajectories is sufficiently robust that it is clear that in higher dimensions the no-go would persist. We just wanted to check that no cancellation due to polarized states exchanged would not allow to evade the no-go obtained for scalars in our earlier paper [PT24] (they do not).*

Ansatz and primal constraints. Following [109, 134, 162], we parameterize the $2 \rightarrow 2$ MHV graviton amplitude by a manifestly crossing-symmetric meromorphic basis built out of Gamma functions, see Eq. (4.1) in [PT27].²⁵ By construction, the ansatz satisfies analyticity and crossing; the remaining nontrivial condition is tree-level unitarity (positivity). Since external states are gravitons, in $d = 4$ we need to analyze the residues of different representations of the couplings between gravitons and higher-spins. In $d = 4$ there are two such combinations, in higher dimensions the problem grows in complexity [110, 163, 164]. Overall we impose two conditions

$$c_{n,J}^{++} \geq 0, \quad c_{n,J}^{+-} \geq 0. \quad (7.2.1)$$

on finitely many poles (convergence happens fast, and actually at finite ansatz size as observed in [134]).

We then choose a given coupling which we maximize. For instance, in the Fig. 7.5 we maximized the first coupling on the graviton trajectory after the graviton coupling, $c_{1,4}^{++}$. We solve the resulting *linear* optimization problems with SDPB.

Emergence of a dominant sister trajectory. There are three robust features of all extremal solutions. The first one is the appearance of a numerically dominant *sister* trajectory, whose slope is half that of the leading one. Furthermore, increasing the ansatz size systematically results in a spectrum where states below that leading trajectory appear to decouple (black zone in Fig. 7.5). Lastly, states between the sister trajectory and the diagonal (above the sister) are left roughly untouched.

In Fig. 7.5 I show the result at the highest ansatz size, $N_{\max} = 30$, which we reached. The reader can also watch this animated gif [here](#) which shows the progressive suppression of states below the sister trajectory as the ansatz size is increased. Note that the number of parameters in the ansatz is $O(N_{\max}^2)$ here.

Therefore in this context “single-trajectory-like” appears to mean “large sister trajectory”. We have no explanation for this robust and curious fact.

²⁵Note that in the original Häring-Zhiboedov ansatz, there was a parameter redundancy which we solved explicitly in our paper, useful to reach higher ansatz sizes.

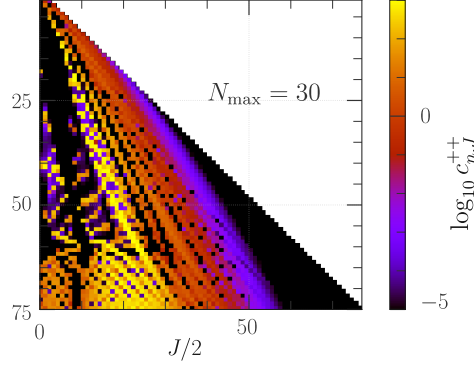


Figure 7.5: Primal extremal solution at $N_{\max} = 20$ obtained by maximizing a coupling $c_{1,4}^{++}$: emergence of a numerically dominant sister trajectory (yellow diagonal, left). From [PT27]. See the paper for more details, and [here](#) for an animated gif.

The $N_{\max} \rightarrow \infty$ limit is singular. As $N_{\max}$ increases, the apparent decoupling region below the sister is pushed to higher masses (the black zone progresses). If this pattern persisted smoothly, one could hope for a limiting amplitude where all states below the sister decouple. [PT27] shows evidence against this: extremal parameters of the ansatz grow with $N_{\max}$ and the amplitude itself becomes unbounded.

This suggests that the extremal solutions with single trajectories are artifacts of the truncated basis, even though bounds obtained at finite $N_{\max}$ can be well behaved.

This is interestingly similar to the exact same problem with the non-perturbative bootstrap, where extremal amplitudes also appear to be singular, of infinite norm.

Takeaway. Within the Haring–Zhiboedov meromorphic ansatz, extremizing couplings produces amplitudes that look single-trajectory-like over a wide window because a sister trajectory becomes numerically dominant and pushes the visibility of subleading trajectories to higher masses. However this “cleaning” of the spectrum is not compatible with a smooth large-basis limit: the extremal amplitudes themselves diverge with $N_{\max}$. This provides a concrete mechanism by which finite-dimensional numerical bootstraps can display sharp, apparently single-trajectory extremality without contradicting the analytic requirement of infinitely many trajectories in a genuine weakly coupled UV completion.

One specific mechanism was identified to try to remedy the absence of daughters in the dual large- N bounds: impose polynomial boundedness at positive t .

7.3 Conclusion on tensions

We discussed the tensions and how to tackle them. There are two general lessons.

The first one is that in both cases semi-definite optimization tends to saturates constraints. The boundary of space of allowed objects appears to have non-physical elements, suggesting that maybe the boundary of the space might not belong to the space of consistent S-matrices.

From the CFT bootstrap perspective, where theories famously live near or close to boundaries [16, 25, 165] this is an interesting fact.²⁶

Related phenomena were already observed in [32] for pion scattering, and later in graviton scattering as well [43] from the perspective of the ϱ -bootstrap alone.

²⁶This lesson fails in $d = 2$, where the ϱ -bootstrap yields spaces of amplitudes with exact solutions at its boundary given by integrable models [4, 166–169].

This suggests the second lesson: the need to develop additional tools to be able to efficiently explore the bulk to approach physical theories. The AM is one such proposal.

As a final counterpoint, it is worth noting that in the case of the flux-tube bootstrap [170], multi-particle can be treated from first principles using the (two-dimensional) $\mathcal{Q}$ -bootstrap [171], and does yield significant particle production which allows to zoom in the physical theory, in a manner similar to the CFT bootstrap.

Chapter 8

Future directions

Along the thesis we already suggested some future directions. Let me collect some and present a few more now.

8.1 Non-perturbative bootstrap

The AM bootstrap is an ongoing project that will continue to be developed, with many open and interesting questions to attack. For this reason, we need to reach a milestone where we can release and share with the community a reasonably easy-to-use architecture (code, package) so that hopefully a larger community could use these methods.

Let me start with some physics open questions.

8.1.1 Space of ACU amplitudes

Characterize 2PRR amplitudes The 2PRR amplitudes were conjectured to correspond to a microscopic model of graph resummation. What are the non-perturbative Regge properties of these amplitudes? Can they be related to a large- N limit of some sort? In [PT29], we noted that the large- N counting in the tensor models [172–174] naturally produces a large- N enhancement for a sub-class of 2PRR graphs. It would furthermore be interesting to see if we can related it to the fishnet study [81]. This model could be used to test basic predictions of Regge-Gribov field theory in a concrete and minimal though non-trivial setting.

Space of amplitudes with more subtractions. Despite some successes, such as the construction of numerically Reggeized amplitudes, with particle production, the project is still in an intermediate stage right now: with the current algorithm with one subtraction we seem to be only able to describe amplitudes with positive c_0 and regular boundary conditions, which basically pushes us on the lower boundary of the conjectured allowed space in Fig. 3.9.

In order to be able to explore the full space, we need to the implementation two subtractions, which is the expected number of subtraction from the Froissart-bound. Three might be easier, as they would allow to use both c_0 and c_2 as external inputs. This problem is not difficult, it is just a matter of doing it. No extra development should be needed.

With it, we could describe amplitudes on the left part of the space depicted in Fig. 3.9, at negative c_0 , where UV completion has been conjectured from ρ -bootstrap to occur via exchange of tower of resonances [30, 34, PT28, 90].

The first question that would pose then is: do the extremal bounds found with ϱ -bootstrap change?

We would then be able to study many questions related to the non-perturbative unitarization of these amplitudes, such as the behavior of non-perturbative Regge trajectories.

Froissart bound. Importantly, we should try to construct an amplitude with asymptotic Froissart growth. Our construction demonstrated that we can describe slow logarithmic effects (decays). It would thus be particularly interesting to target growing amplitudes. The question of the Froissart growth was recently investigated in [41] using ρ -bootstrap techniques. A remarkable picture for the parton structure emerged (reflective white annulus) at intermediate energies, which would be very interesting to test at asymptotically high energies.

Furthermore, the Froissart growth has long been hypothesized to violate unitarity due to high multiplicity processes [47, 175], leading to a prediction of a universal subleading correction [176] $\sigma_{\text{tot}}(s) \sim c \log(s)^2 - \tilde{c} \log(\log(s))$ which would be very interesting to test with our methods. We could already study $\log(\log(s))$ effects in the toy-model [PT22, PT26], but this required a grid extending to $10^{100} m^2$. Achieving such high cutoff for the full problem will require more development. An idea could be to use $\log(\log(s))$ as inputs in the NN, as in Fig. 3.7.

Study inelasticity and particle-production For all the questions mentioned, having ρ_{MP} as a variational degree of freedom would be instrumental in understanding how actual UV-completion is achieved.

Further questions related directly to particle-production could then be asked: what is the amplitude with maximal particle production? Does Aks screening at large spin follows the prediction from rigid screening Eq. (7.1.5). Besides Aks screening, are there other mechanisms to minimize particle production?

8.1.2 Improving AM bootstrap

The typical speed of one run for a Reggeized amplitude is currently of the order of one day when running on a GPU. This is for an amplitude with the finest grids, and for purposes of explorations, smaller grids yield faster results in orders of hours.

Nevertheless, it is clear that more development is needed in the approach to make it faster, more robust, and more efficient. Among many things to improve, Kolmogorov-Arnold networks [177] seem particularly interesting. They have the advantage that they are more “interpretable” and hopefully would require fewer parameters. They might even help to read directly the Regge structure of the spectral functions.

8.1.2.1 Hybrid scheme

More ambitiously, it would be really nice to be able to merge both ρ -and AM-bootstraps, as they both have strong advantages. A hybrid scheme would for instance use the solid UV completion provided by the ρ -bootstrap from threshold to intermediate energies, with tower of resonances, and use the AM bootstrap to unitarize this seed non-perturbatively to all energies.

A concrete idea would be to use the ρ -bootstrap data to hotstart the neural-network for the gradient descent. One would first *learn* what double-discontinuity could approximate best a few low-lying partial waves by a sort of numerical inverse projection. This would provide an initial ρ_{el} with hopefully enough structure so that the gradient descent could start from it and unitarize the UV. A flowchart would resemble the picture below in Fig.8.1.

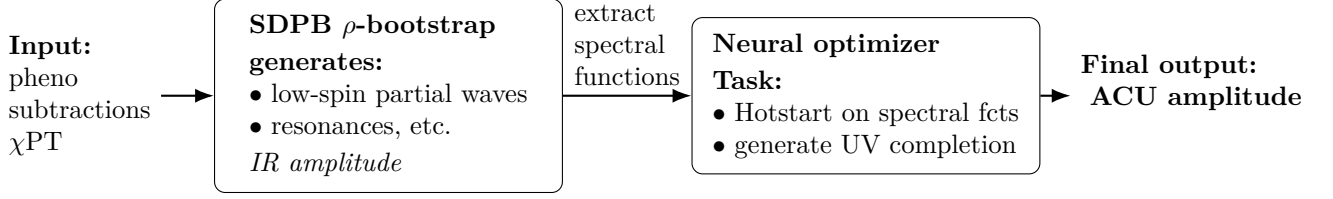


Figure 8.1: Hybrid flowchart

8.1.2.2 Semi-definite relaxation

Another interesting direction to improve the AM bootstrap and bypass the difficult of non-convex optimization is that of *semi-definite relaxation*. This way would give an alternative to direct nonlinear solvers is to seek a semi definite relaxation of the Mandelstam equation. This strategy was introduced by Shor [178], and an introductory level description can be found in the lectures [179, Chap. 3]. The basic idea is to linearize the quadratic nonlinearity by lifting to a higher-dimensional object. After discretizing the spectral functions over the kinematic domain $(s, t) \geq 4m^2$ (or expanding them on a finite basis), one replaces the vector of unknown samples/coefficients $x \sim (\rho(s_1, t_1), \dots, \rho(s_n, t_n))^{27}$ by a matrix variable

$$X = x x^\top, \quad (8.1.1)$$

Note that $X \succeq 0$ trivially. All quadratic expressions in x become linear in X through identities of the form $x_i x_j = X_{ij}$, so the bilinear integral operator $\iint K \rho^2$ becomes an affine function of X . For instance, in the discretized setting the Mandelstam equation (3.1.24) can be written as a set of constraints

$$x_i = \sum_{j,k} K_{i,jk} x_j x_k \quad \longrightarrow \quad x_i = \sum_{j,k} K_{i,jk} X_{jk}, \quad (8.1.2)$$

which is linear in (x, X) . To tie x and X together one adds convex constraints, for instance

$$\begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \succeq 0 \quad \implies \quad X \succeq x x^\top, \quad (8.1.3)$$

One then obtains a an optimization problem over (x, X) that can be attacked by SDP solvers.

The crucial point is that this is only a *relaxation*: the original nonlinear problem corresponds to the *rank-one* constraint

$$\text{rank}(X) = 1 \quad (8.1.4)$$

which is *non-convex* and is precisely what is being dropped.²⁸ Therefore, there is no guarantee *a priori* that the SDP solution will return $\text{rank}(X) = 1$, or even approximately rank-one.

In fact, it is generic for relaxations to admit higher-rank solutions that satisfy all lifted linear constraints while not being representable as $X = x x^\top$. In that case the SDP provides a consistent “moment matrix” for the quadratic constraints but does not reconstruct a genuine ρ solving the Mandelstam equation. Still, the relaxation can be useful. It may produce rigorous *infeasibility* results and could thus maybe be used in order to develop a

²⁷We ignore for simplicity the single spectral function $\rho(s)$ in this discussion.

²⁸Rank-1 is trivially non-convex: the linear combination of any two rank-1 matrices is generically rank-2.

dual AM bootstrap. As a matter of fact a similar trick was implemented in [30] for the dual using the Roy equations.

Another potential issue is the size of the matrix. A discretization on a grid of size N for one variable s and for t yields a vector of samples N^2 and hence a matrix of size $N^2 \times N^2$. I would hope to report on the feasibility of this point soon.

8.1.3 Application to other systems

Besides codes, and space of scalar theories, this bootstrap, by its constructive nature, is geared towards all sorts of applications, from QCD to hopefully gravity.

8.1.3.1 Meson amplitudes: $\pi - \pi$, $\pi - K$.

There is in theory a clear path for meson amplitudes, already studied a lot with modern ρ -bootstrap [32, 33, 35–40]. One interesting goal would be to treat the full (π, K, η) octet meson system. This would allow to determine $K\bar{K}$ inelasticity, otherwise put by hand in the bootstrap paper [35] taken from semi-phenomenological models in [180]. This computation would help to given independent elements on the recent remarkable prediction of a new tetra-quark state in that bootstrap study.

The analytic structure for the $\pi - K$ system was worked out in [181–183] and in theory is amenable to a Mandelstam representation.

Before $\pi - K$, the first step would of course be the pion amplitude, whose Mandelstam representation and equation was worked out as early as the original papers of Mandelstam. Furthermore, many techniques to study systems of multiple amplitudes were already studied in ρ -bootstrap literature [170, 171, 184–186], providing useful entry points and formalism details for this problem.

8.1.3.2 EFTs and massless systems

Can we use the AM bootstrap technology to describe massless scattering? A priori, we loose the power of elastic unitarity, since all cuts starting at $4m^2, 16m^2$ collapse to the origin and the elastic gap $s \in [4m^2, 16m^2]$ vanishes. However, one could still hypothesize a massless Mandelstam representation, keep the decomposition of the double-discontinuity as in Eq. (3.1.11), and in theory the approach should still allow to neatly separate iterated 2-body interactions from genuinely higher-point processes. One then seek to enforce *inelastic* unitarity, which as in the AM-bootstrap needs to be checked a posteriori. One might hope that the splitting would give some leverage by allowing to control from first principles some amount of inelasticity. Maybe one could even define a controlled 2PRR amplitude in massless theories or in gravity.

The obvious entry points here would be scattering of massless Goldstone bosons in $d = 4$, as in [160], and extension to gravity would require moving away from $d = 4$ where IR soft divergences make the graviton S-matrix ill-defined. Clear targets would be the super-graviton amplitudes in 9-10-11 dimensions studied in [42, 43].

8.2 Higher dimensions

Our work on higher-dimensional S-matrices open many interesting research questions, related to the bounds themselves, and the physical theories that could generate such kinks and families of UV completions. To attack the question of the bounds, it would be interesting to develop the AM bootstrap in higher dimensions.

8.2.1 What microscopic theories could realize these gapped S-matrices?

In general, one open question is: what theories could correspond to the gapped amplitudes we studied in higher dimensions? The smoothness of the bounds with d (before and after the kinks) suggests some rather well defined families of theories, but what could they be? In $d = 4$, the cleanest example of a gapped relativistic S-matrix is glueball scattering in a confining gauge theory, but in higher dimensions it is not clear. We discussed some open directions in the paper and I summarize them here. From a string-theoretic perspective, it is expected that UV completion beyond $d = 4$ involves non-local ingredients related to Little String theories (LST), and that beyond $d = 6$ gravity cannot even be fully decoupled [187, 188], with a possible borderline case in $d = 7$ [188]. It is intriguing that the onset of strong threshold behavior for the maximum branch occurs in dimensions where such expectations become relevant. Note that [189] did observe that in 6d LST, a QFT-like regime persists over several orders of magnitude of energies. Maybe this dynamical fact can “trick” the ϱ -bootstrap into believing that such constructions are genuinely QFT-like, and well approximated by our gapped ansatz. At any rate, in these scenarios no explicit construction using a deformation of stringy theories has been proposed to yield purely gapped S-matrices, but it does not seem unlikely that it can be done.

Another aspect about gravitational interactions in higher dimensions is that they are known to generate negativity in Wilson coefficients [163, 164, 190]. In our amplitudes we also observe negativity when the strong threshold behavior.

This, added to the apparent non-locality of the threshold behavior, asks the question: do these coincidences reflect a genuine connection between threshold singularities and the necessity of non-local UV structure, or are they merely accidental? It is a very interesting question, that requires further investigations, in particular of the Regge structure of the extremal amplitudes.

Another broad class of scenario for UV completion might take place, related to non-trivial UV fixed-point (asymptotic safety scenarios). There are hints that such constructions as possible, see for instance the very interesting lattice simulations of $5d$ $SU(2)$ Yang–Mills [191]).

8.2.2 Large d ?

The existence of the post-kink branches, which continue past $d = 11$ apparently smoothly also indicate the possibility that the post-kink family of S-matrices might be amenable to some large- d simplification, in the spirit of [192]. In the paper, we commented that the $d \rightarrow \infty$ and the spin sum do not commute and the scattering is actually dominated by large spins $d \sim J \gg 1$, while the fixed- J limit only probes the $\theta = 0$ or $\theta = \pi$ angles. It would be very interesting to push this question further and develop a large- d bootstrap.

8.3 Dual models

There are also many open questions related to the dual-model part of the thesis.

Regge bootstrap and open strings with massive endpoints Firstly, the Regge bootstrap technology could be revisited and improved using SPDB rather than Mosek. A particularly interesting model to study is that of open string with massive endpoints [193, 194]. Asymptotically, the spectrum is known to deviate from Veneziano in a well-controlled manner with $n^{1/4}$ corrections [195–197]. At minimum this deformation can be included in the bootstrap easily. To zoom in on the would-be “Veneziano amplitude of open strings

with massive endpoints” would however most certainly requiring a lot of extra analytic input, in particular regarding the spectrum at low n .

It would also be interesting to use the Regge bootstrap to discard spectra which grow faster than linear, for which $\alpha(m_n^2) = n \mapsto n^\gamma$ with $\gamma > 1$, in accord with expectations from [107].

8.3.1 Accumulation point amplitudes

We have seen that to maintain locality, the Coon amplitude needs a non-meromorphic factor. On the resulting cut, unitarity cannot be satisfied [PT20, 198]. Nevertheless the object presents solid and robust features, and it is natural to ask whether it can arise from more conventional physics.

A q -deformed string? Attempts at a q -deformed string worldsheet have been made (e.g. [199, 200]), but they have not given rise to a full worldsheet theory. To build a theory one needs not just the four-point function but the full N -point function. For the Coon amplitude with the non-meromorphic factor, no such construction has been proposed to date (see refs. in [PT20]). Simple Coon-like q -deformations of for instance Veneziano 5-point amplitude fail for trivial reasons: the kinematic invariants are modified in a way that does not respect momentum conservation at higher points. This constrains such scenarios.

Other dual models? Besides on our work, several more recent works tried to bootstrap dual models, finding new solutions and new constraints [112, 134, 198, 201–225]. As advocated long ago by Gross [226], it is essential in these endeavors to reach higher points. Interestingly, a few examples seem to pass these constraints, see [213, 224] in particular.

Accumulation in AdS and on the Coulomb branch. Based on a flux-tube analysis [227], Maldacena and Remmen observed that open-string scattering on a D-brane in AdS shares qualitative features with the Coon amplitude [201]. Semi-classical quantization gives an exponential density of states as the string extends in the bulk and gets redshifted. However, for the maximal-spin states, rotating string solutions obey a Coulombic growth $E_J - E_\infty \propto -1/J^2$, in crucial distinction with the Coon amplitude where the accumulation persists exponentially.

A related (and physically realized) accumulation point appears in scattering on the Coulomb branch of $\mathcal{N} = 4$ SYM [228]. These amplitudes exhibit an accumulation point of high-spin resonances at the two-particle threshold $s = 4m^2$, interpreted as an ionisation threshold of an attractive Coulomb problem, with hydrogenic spectrum structure

$$m_J = 2m - \frac{m \Omega(g)^2}{2(J+1)^2}. \quad (8.3.1)$$

This matches the AdS rotating-string behavior and highlights once again the specific sense in which Coon’s deformation is exotic: its exponential accumulation persists even along the maximal-spin states.

8.3.2 Towards string uniqueness at weak coupling

We have argued that the daughter trajectories represent the ripples on the string worldsheet and hence dual models have to be “stringy”. It would be interesting to go beyond this properly and rigorously prove that an actual worldsheet can be reconstructed.

Regarding the specific case of gravity, it would be interesting to study further the implications of our no-go regarding the uniqueness of string UV completion at weak coupling.

Right now, our no-go is just a statement on meromorphic amplitudes. It needs to be tied cleanly to [70], within the standards of rigor of the modern positivity program and study in particular graviton loop effects [135, 229–232]. IR divergences also need to be treated properly if one insist to stay in $d = 4$ spacetime dimensions.

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