Black Holes: Energetics and Thermodynamics

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ICRANet, Nice, 4-9 June 2012

Black holes as background solutions

Initially, BHs= background solutions; e.g. spherically symmetric BHs (Schwarschild 1916, Reissner-Nordström 1918) with mass M and electric charge Q

$$ds^2 = -A(r)\mathrm{d}T^2 + B(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right) \tag{1}$$

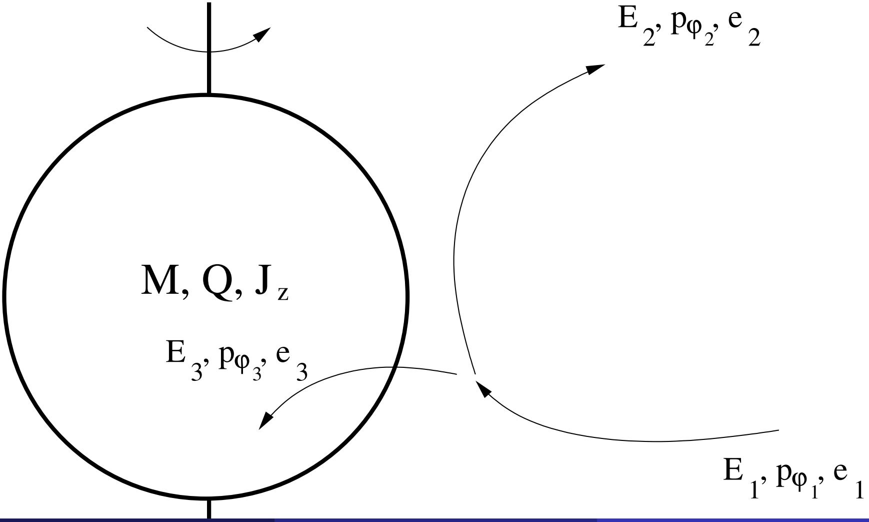
$$\begin{aligned} A(r) &= 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \\ B(r) &= \frac{1}{A(r)}. \end{aligned} \tag{2}$$

Kerr BH 1963: with spin J; then Kerr-Newman with spin J and electric charge Q.

Up to the 1960's BHs were viewed only as passive gravitational wells. For instance, one could think of adiabatically lowering a small mass m at the end of a string until it disappears within the BH, thereby converting its mass-energy mc^2 into work. More realistically, one was thinking of matter orbiting a BH and radiating away its potential energy (up to a maximum, given by the binding energy of the last stable circular orbit around a BH).

Black holes as energy reservoirs

Beyond potential wells: extracting energy from BHs:
(gedanken) "Penrose process" (1969)



Global energetic properties of black holes

A detailed analysis of the efficiency of such gedanken Penrose processes by Christodoulou and Ruffini 70, 71 then led to the understanding of the existence of a fundamental irreversibility in BH dynamics, and to the discovery of the Black Hole mass formula.

Tools:

• Conservation of $E = -p_0$, p_φ and e during the “fall” of the test particle.
• Conservation of $E = -p_0$, p_φ and e during the (quantum) splitting process (near the BH) of the incident test particle 1 into two particles 2 and 3.
• Changes in the total mass-energy M, total angular momentum J and total charge Q of the BH when it absorbs particle 3

$$\begin{array}{rclcl} \delta M & = & E_3 & = & E_1 - E_2, \\ \delta J & = & J_3 & = & J_1 - J_2, \\ \delta Q & = & e_3 & = & e_1 - e_2. \end{array} \tag{3}$$

The Hamilton-Jacobi ("mass-shell") equation reads

$$g^{\mu\nu}\,(p_\mu - eA_\mu)\,(p_\nu - eA_\nu) = -\mu^2, \tag{4}$$

in which $p_\mu = \partial S/\partial x^\mu$, S is the action

$$S = -ET + p_\varphi \varphi + S\,(r,\theta)\,. \tag{5}$$

(4) can then be written explicitly as

$$-\frac{1}{A(r)}\,(p_0 - eA_0)^2 + A(r)p_r^2 + \frac{1}{r^2}\left(p_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2\right) = -\mu^2 \tag{6}$$

which we re-write as

$$(p_0 - eA_0)^2 = A(r)^2 p_r^2 + A(r)\left(\mu^2 + \frac{L^2}{r^2}\right) \tag{7}$$

Using $E = -p_0$ and $-A_0 = +V = +Q/r$ the above expression is quadratic in E (it generalizes the famous flat-spacetime $E^2 = \mu^2 + \mathbf{p}^2$) and one finds two possible solutions for the energy as a function of momenta and charge

$$E = \frac{eQ}{r} \pm \sqrt{A(r)^2 p_r^2 + A(r)\left(\mu^2 + \frac{L^2}{r^2}\right)}. \tag{8}$$

In flat space, $A(r) = 1$, so that, if we ignore charge, we recover the usual Dirac dichotomy on the choice of the $+$ or $-$ sign between particle and antiparticle: $E = \pm\sqrt{\mu^2 + \mathbf{p}^2}$. This shows that one should take the *plus sign* in the equation above.

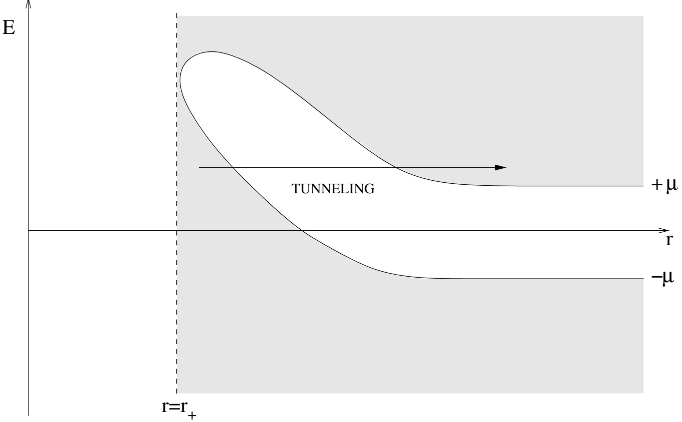


Figure: Classically allowed energy levels (shaded region) as a function of radius, for test particles in the neighborhood of a BH. There exist positive- and negative-energy solutions, corresponding (after second quantization) to particles and anti-particles. Classically (as in the Penrose process) one should consider only the "positive-square-root" energy levels, located in the upper shaded region. The white region is classically forbidden. Note the possibility of tunneling (this corresponds to particle creation via a non-thermal mechanism).

As particle 3 is absorbed by the BH, we can compute its (conserved) energy when it crosses the horizon, i.e., in the limit where $r = r_+(M, Q)$, where $r_+(M, Q) = M + \sqrt{M^2 - Q^2}$ is the outer solution of $A(r) = 0$. This simplifies the expression of E_3 to

$$E_3 = \frac{e_3\, Q}{r_+} + |p^r|, \tag{9}$$

where we have introduced the contravariant component $p^r = g^{rr} p_r = A(r) p_r$, which has a finite limit on the horizon.
Using $E_3 = \delta M$ and $e_3 = \delta Q$, then yields

$$\delta M = \frac{Q \delta Q}{r_+(M, Q)} + |p^r|. \tag{10}$$

Note the presence of the *absolute value* of p^r (coming from the limit of a positive square-root).

Irreversibility in black hole physics

From the positivity of $|p^r|$ we find the following inequality (due to Christodoulou and Ruffini) , expressing the fundamental irreversibility property of BH energetics (here, with $J = 0$ for simplicity):

$$\delta M \geq \frac{Q \delta Q}{r_+(M, Q)}. \tag{11}$$

There exist two types of processes, the *reversible* ones with an '$=$' sign in (11), and the *irreversible* ones with an '$>$' sign. The former ones are reversible because if a BH first absorbs a particle of charge $+e$ with vanishing $|p^r|$ (so that $\delta' M = eQ/r_+(M, Q)$ and $\delta' Q = e$), and then a particle of charge $-e$ with vanishing $|p^r|$ ($\delta'' M = -eQ/r_+(M, Q)$ and $\delta'' Q = -e$), it will be left, at the end, in the same state as the original one (with mass $M + \delta' M + \delta'' M = M$ and charge $Q + \delta' Q + \delta'' Q = Q$). Evidently, such reversible transformations are delicate to perform, and one expects that irreversibility will occur in most BH processes. The situation here is clearly similar to the relation between reversible and irreversible processes in thermodynamics.

BH irreversibility: Kerr-Newman case

The same computation as for the Reissner-Nordström BH can be performed for the Kerr-Newman BH. One obtains in that case, by a slightly more complicated calculation,

$$\delta M - \frac{a\delta J + r_+ Q\delta Q}{r_+^2 + a^2} = \frac{r_+^2 + a^2\cos^2\theta}{r_+^2 + a^2}|p^r|. \tag{12}$$

in which $r_+(M,J,Q) = M + \sqrt{M^2 - Q^2 - a^2}$. We recall that $a = J/M$, and that one has the bound $Q^2 + (J/M)^2 \leq M^2$.
Using again the fact that $|p^r| \geq 0$ leads to the general Christodoulou-Ruffini inequality

$$\delta M \geq \frac{a\delta J + r_+ Q\delta Q}{r_+^2 + a^2} \tag{13}$$

where we recall that $a = J/M$, and that one has the bound $Q^2 + (J/M)^2 \leq M^2$.

Integrating a sequence of reversible transformations

Consider now a sequence of infinitesimal reversible changes (*i.e.*, $p^r \to 0$) of BH states which are reversibly connected to some initial BH state. This leads to a partial differential equation for δM,

$$\delta M = \frac{a \delta J + r_+ Q \delta Q}{r_+^2 + a^2}. \tag{14}$$

Integrating it yields the Christodoulou-Ruffini mass formula (1971)

$$M^2 = \left(M_{\text{irr}} + \frac{Q^2}{4 M_{\text{irr}}} \right)^2 + \frac{J^2}{4 M_{\text{irr}}^2}. \tag{15}$$

Here the *irreducible mass* $M_{\text{irr}} = \frac{1}{2}\sqrt{r_+^2 + a^2}$ appears as an integration constant. The mass squared thus appears as a function of three basic contributions: irreducible mass , Coulomb energy, and rotational energy.

Free Energy of Black Holes

Inserting the mass formula into the inequality Eq. (13), one finds

$$\delta M_{\text{irr}} \geq 0 \tag{16}$$

with $\delta M_{\text{irr}} = 0$ under reversible transformations and $\delta M_{\text{irr}} > 0$ under irreversible transformations. The irreducible mass M_{irr} can only increase or stay constant. This behaviour is reminiscent of the second law of thermodynamics.

The free energy of a BH is therefore $M - M_{\text{irr}}$, i.e., this is the maximum extractable energy. In this view, BHs are no longer passive geometrical backgrounds but contain stored energy that can be extracted. Actually, the stored energy can be enormous because a BH can store up to 29 % of its mass as rotational energy, and up to 50 % as Coulomb energy!

Hawking's generalization

The irreducible mass is related to the area of the horizon of a Kerr-Newman BH, by $A = 16\pi M_{\mathrm{irr}}^2$ so that

$$\delta A \geq 0 \tag{17}$$

with $\delta A = 0$ in a reversible process , while $\delta A > 0$ in an irreversible one. Hawking (1971)showed that this irreversible evolution of the area of the horizon was a general consequence of Einstein's equations, when assuming the weak energy condition. He also showed that in the merging of two BHs of area A_1 and A_2, the total final area satisfied $A_{\mathrm{tot}} \geq A_1 + A_2$.

Such results evidently evoque the second law of thermodynamics. This suggests to consider the analog of the first law of thermodynamics: $\mathrm{d}E(S, \text{extensive parameters}) = \mathrm{d}W + \mathrm{d}Q$, where the work $\mathrm{d}W$ is linked to the variation of extensive parameters (volume, etc.) and where $\mathrm{d}Q = T\mathrm{d}S$ is the heat exchange.

First law of BH thermodynamics

$$\mathrm{d}M\left(Q,J,A\right) = V\mathrm{d}Q + \Omega\mathrm{d}J + \frac{g}{8\pi}\mathrm{d}A. \tag{18}$$

where

$$\begin{aligned} V &= \frac{Qr_+}{r_+^2 + a^2}, \\ \Omega &= \frac{a}{r_+^2 + a^2}, \end{aligned} \tag{19}$$

and

$$g = \frac{1}{2}\frac{r_+ - r_-}{r_+^2 + a^2} = \frac{\sqrt{M^2 - a^2 - Q^2}}{r_+^2 + a^2}. \tag{20}$$

V is interpreted as the electric potential of the BH, and Ω as its angular velocity. Expression (18) resembles the usual form of the first law of thermodynamics in which the area term has to be interpreted as some kind of entropy. The parameter g is called the "surface gravity". [In the Schwarschild case, it reduces to M/r_+^2 (in $G = 1$ units), i.e., the usual formula for the surface gravitational acceleration $g = GM/R^2$.].

Bekenstein's proposal (1972, 1973)

Bekenstein went further in taking seriously (and no longer as a simple analogy) the thermodynamics of BHs. He gave several arguments (using Carnot-cycles, or Heisenberg's uncertainty principle $\delta r \delta p_r \geq \frac{1}{2}\hbar$) leading to attributing to a BH an entropy of the form

$$S_{\mathrm{BH}} = \hat{\alpha} \frac{c^3}{\hbar G} A, \tag{21}$$

with a dimensionless coefficient $\hat{\alpha} = \mathcal{O}(1)$, without being able to fix in a unique, and convincing, manner the value of $\hat{\alpha}$.

This result in turn implies (by applying the law of thermodynamics) that one should attribute to a BH a temperature equal to

$$T_{BH} = \frac{1}{8\pi\hat{\alpha}} \frac{\hbar}{c} g. \tag{22}$$

Hawking's radiation (1974)

This attribution of a finite temperature to a BH looked rather strange in view of the definition of a BH has being "black", i.e., as allowing no radiation to come out of it. In particular, Stephen Hawking resisted this idea, and tried to prove it wrong by studying quantum field theory in a BH background. However, much to his own surprise, he so discovered (in 1974) the phenomenon of quantum radiation from BH horizons which remarkably vindicated the physical correctness of Bekenstein's suggestion. Hawking's calculation also unambiguously fixed the numerical value of $\hat{\alpha}$ to be

$$\hat{\alpha} = \frac{1}{4} \tag{23}$$

Black Holes as Dissipative Membranes

Summarizing so far: The results on BH dynamics and thermodynamics of the early 1970's modified the early view of BHs as passive potential wells by endowing them with *global* dynamical and thermodynamical quantities, such as mass, charge, irreducible mass, entropy, and temperature. Now, we shall review the further changes in viewpoint brought by work in the mid and late 1970's (Hartle-Hawking 72, Hanni-Ruffini 73, Damour 78, 79, 82, Znajek 78) which attributed *local* dynamical and thermodynamical quantities to BHs, and led to considering BH horizons as some kind of *dissipative branes*.

Basic idea: Excise the interior of a BH, and replace the description of the interior BH physics by quantities and phenomena taking place entirely on the "surface of the BH" (i.e., the horizon).

In the following , we shall no longer consider only Kerr-Newman BHs (i.e., stationary BHs in equilibrium, which are not distorted by sources at infinity). We shall consider more general non-stationary BHs distorted by outside forces.

Black hole surface electrodynamics

In order to replace the internal electrodynamics of the BH by surface effects, we replace the real electromagnetic field $F_{\mu\nu}(x)$ by $F_{\mu\nu}(x)\Theta_{\mathrm{H}}$, where Θ_H is a Heaviside-like step function, equal to 1 outside the BH and 0 inside. Then we consider the modified Maxwell equations satisfied by this Θ_H-modified electromagnetic field.

$$\begin{aligned} \nabla_\nu \left(F^{\mu\nu}\Theta\right) &= \left(\nabla_\nu F^{\mu\nu}\right)\Theta + F^{\mu\nu}\nabla_\nu\Theta \\ &= 4\pi\left(J^\mu\Theta + j_H^\mu\right), \end{aligned} \tag{24}$$

where we have introduced a *BH surface current* j_H^μ as

$$j_H^\mu = \frac{1}{4\pi} F^{\mu\nu}\nabla_\nu\Theta. \tag{25}$$

j_H^μ contains a Dirac δ-function δ_H which restricts it to the horizon

$$j_H^\mu = K^\mu \delta_{\mathrm{H}}, \tag{26}$$

where K^μ is the “BH surface current density”.

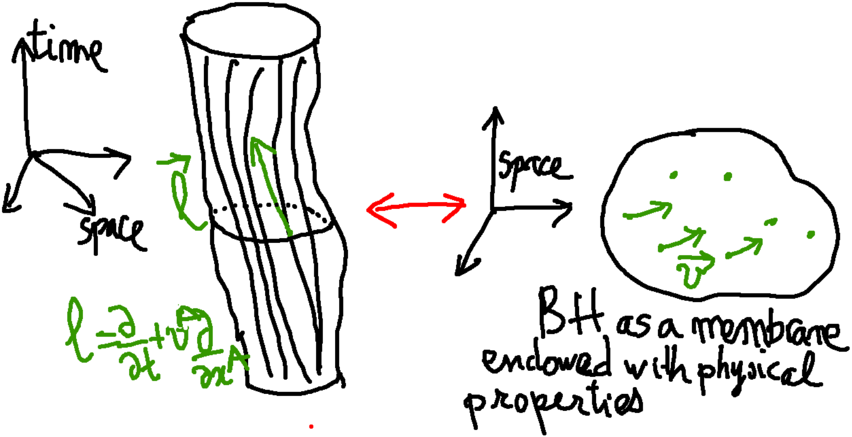
Kinematics of the Horizon as a "fluid"

The horizon is a null hypersurface which by definition is normal to a null covariant vector ℓ_μ satisfying both $\ell_\mu \ell^\mu = 0$ and $\ell_\mu \mathrm{d}x^\mu = 0$ for any infinitesimal displacement $\mathrm{d}x^\mu$ within the hypersurface. In Eddington-Finkelstein-like coordinates $t = x^0, x^1, x^A$ with $A = 2, 3$ (where $x^1 = 0$ on the horizon, and where x^A are some angular-like coordinates on the two-dimensional spatial slice S_t ($x^0 = t, x^1 = 0$) of the horizon) one has

$$\ell^\mu \partial_\mu = \frac{\partial}{\partial t} + v^A \frac{\partial}{\partial x^A}. \tag{27}$$

where v^A can be interpreted as the velocity of some "fluid particles" on the horizon, which are the "constituents" of a null membrane. Similarly to the usual description of the motion of a fluid, one has to keep track of the changes in the distance between two fluid particles as the fluid expands and shears.

Black Hole as a Membrane



Geometry of the Horizon

Distances on the horizon: they are measured by considering the restriction to the horizon of the spacetime metric. As we are considering a null hypersurface, we have

$$\mathrm{d}s^2|_{x^1=0} = \gamma_{AB}\left(t, x^C\right)\left(\mathrm{d}x^A - v^A \mathrm{d}t\right)\left(\mathrm{d}x^B - v^B \mathrm{d}t\right) \tag{28}$$

the area element of the spatial sections S_t

$$\mathrm{d}A = \sqrt{\det \gamma_{AB}}\, \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{29}$$

Deformation tensor of the horizon geometry (Lie derivative along $\vec{\ell}$) :

$$\begin{aligned} D_{AB} &= \frac{1}{2}\left(\partial_t \gamma_{AB} + v^C \partial_C \gamma_{AB} + \partial_A v^C \gamma_{CB} + \partial_B v^C \gamma_{AC}\right) \\ &= \frac{1}{2}\left(\partial_t \gamma_{AB} + v_{A|B} + v_{B|A}\right) \end{aligned} \tag{30}$$

where '$_|$' denotes a covariant derivative w.r.t. the Christoffel symbols of the 2-geometry γ_{AB}.

Charge and Current density on the Horizon

One can decompose the current density K^μ into a time component $\sigma_H = K^0$, and two spatial components K^A tangent to the spatial slices S_t ($t =$ const.) of the horizon,

$$K^\mu \partial_\mu = \sigma_H \partial_t + K^A \partial_A = \sigma_H \ell^\mu + (K^A - \sigma_H v^A)\partial_A \tag{31}$$

The total BH charge can be rewritten as (Hanni-Ruffini 73)

$$Q_H = \oint_H \sigma_H dA, \tag{32}$$

Moreover, an external current injected "normally" to the horizon "closes" onto a combination of currents flowing along the horizon, and/or of an increase in the local horizon charge density:

$$\frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial t} (\sqrt{\gamma}\sigma_H) + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^A} \left(\sqrt{\gamma} K^A\right) = -J^\mu \ell_\mu. \tag{33}$$

Black Hole Surface Resistivity

Defining the electric and magnetic fields on the horizon according to

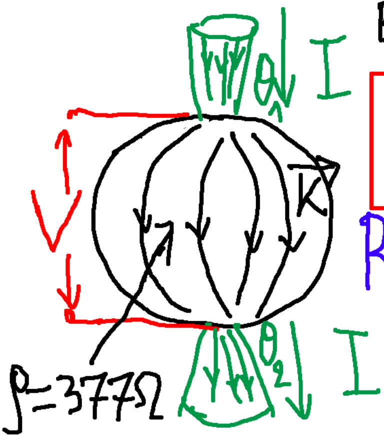
$$\frac{1}{2} F_{\mu\nu} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}|_{\mathrm{H}} = E_A \mathrm{d}x^A \wedge \mathrm{d}t + B_{\perp} \mathrm{d}A. \tag{34}$$

then leads to a BH Ohm's law

$$\vec{E} + \vec{v} \times \vec{B_{\perp}} = 4\pi \left(\vec{K} - \sigma_H \vec{v} \right). \tag{35}$$

From this form of Ohm's law, we can read off that BHs have a *surface electric resistivity equal to $\rho = 4\pi = 377$ Ohm* (Damour 1978, Znajek 1978).

Black Hole Ohm's Law



BH OHM'S LAW

$$\boxed{V = R\,I}$$

$$R = 2 \log \frac{\tan \theta_2/2}{\tan \theta_1/2} \times 30\,\Omega$$

Black Hole Surface Density of Linear Momentum

With

$$\begin{aligned} \nabla_{\vec{\ell}}\, \vec{\ell} &= g\, \vec{\ell}, \\ \nabla_A \vec{\ell} &= \Omega_A \vec{\ell} + D_A^B \vec{e}_B. \end{aligned} \tag{36}$$

one defines the “BH surface density of linear momentum” as

$$\pi_A = -\frac{1}{8\pi}\Omega_A = -\frac{1}{8\pi}\vec{n}\cdot\nabla_A\vec{\ell}. \tag{37}$$

With this definition, one has

$$J_H = \int_S \pi_\varphi \mathrm{d}A, \tag{38}$$

Then, decomposing the deformation tensor into shear and expansion, $D_{AB} = \sigma_{AB} + \frac{1}{2}\theta\gamma_{AB}$, and introducing the following convective derivative $\frac{D\pi_A}{\mathrm{d}t} \equiv (\partial_t + \theta)\,\pi_A + v^B \pi_{A|B} + v^B_{\ |A}\pi_B$,

Black Hole Navier-Stokes Equation

Projecting Einstein's equations along $\ell^\mu e^\gamma_A$, one finds the following (exact) Black Hole Navier-Stokes Equation (Damour 1979)

$$(\partial_t + \theta)\,\pi_A + v^B \pi_{A|B} + v^B_{|A}\pi_B = -\frac{\partial}{\partial x^A}\left(\frac{g}{8\pi}\right) + \frac{1}{8\pi}\sigma^B_{A|B} - \frac{1}{16\pi}\partial_A\theta - \ell^\mu T_{\mu A} \tag{39}$$

The usual Navier-Stokes equation for a viscous fluid reads

$$(\partial_t + \theta)\,\pi_i + v^k \pi_{i,k} = -\frac{\partial}{\partial x^i} p + 2\eta\sigma^k_{i,k} + \zeta\theta_{,i} + f_i, \tag{40}$$

where π_i is the momentum density, p the pressure, η the shear viscosity, $\sigma_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) - \mathrm{Trace}$, the shear tensor, ζ the bulk viscosity, $\theta = v^i_{,i}$ the expansion rate, and f_i the external force density.

Black Hole Viscosity

From the above Black Hole Navier-Stokes Equation, one reads off (in particular) the following value of the Black Hole surface shear viscosity

$$\eta = +\frac{1}{16\pi} \tag{41}$$

When divided by the entropy density found by Hawking ($s \equiv S/A = \frac{1}{4}$), the latter shear viscosity yields the ratio

$$\frac{\eta}{s} = \frac{1}{4\pi}, \tag{42}$$

a result of recent interest in connection with the AdS/CFT correspondence (Kovtun, Son, and Starinets 05,07).

Conclusions

- Up to 1960's, BHs considered as *passive objects*, i.e. potential wells.
- In the early 1970's the study of the *dynamics* of BHs was initiated by Penrose 69, Christodoulou and Ruffini 70,71, Hawking 71, and Bardeen, Carter and Hawking 73. They studied the *global dynamics* of BHs was considered, i.e. their total mass, their total angular momentum, their total irreducible mass, and the variation of these quantities. Key results: (i) irreversibility, (ii) BH mass formula, (iii) BH entropy
- Later works by Hartle and Hawking 72, Hanni and Ruffini 73, Damour 78,79,82, and Znajek 78, considered the *local dynamics of BH horizons*. In this new approach (which was later called the "membrane paradigm" (Thorne 86)) a BH horizon is interpreted as a brane with dissipative properties, such as, for instance, an electrical resistivity ρ, equal to 377 Ohms (Damour 78, Znajek 78) , and a Navier-Stokes-like equation with surface (shear) viscosity, equal to $\eta = \frac{1}{16\pi}$ (Damour 79, 82). The "viscous" properties of horizons has recently raised some renewed interest (Kovtun, Son, and Starinets 05,07, Strominger et al. 11).