The wave functions of the q-deformed Haldane-Shastry model

(Uglov-Lamers model)

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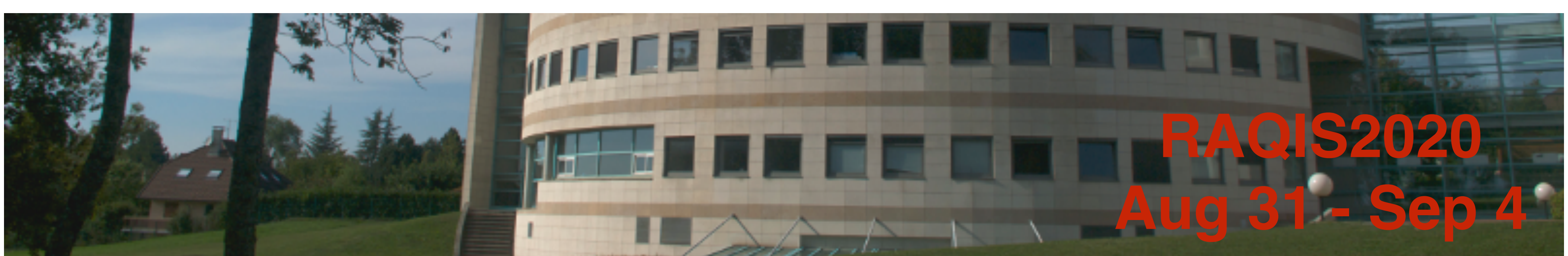
w/ J. Lamers, V. Pasquier, arXiv:2004.13210



Long-range interacting models

- Long range deformations of the Heisenberg model are important for various applications (e.g. AdS/CFT, condensed matter)
- Very few cases are well understood
- Unified framework for nearest-neighbour and long range interaction?
- The role of DAHA (double affine Hecke algebra)
- Relation with the separation of variables?

- Here: solution of the **q-Haldane-Shastry model** via the spin Ruijenaars-Schneider model and Macdonald polynomials

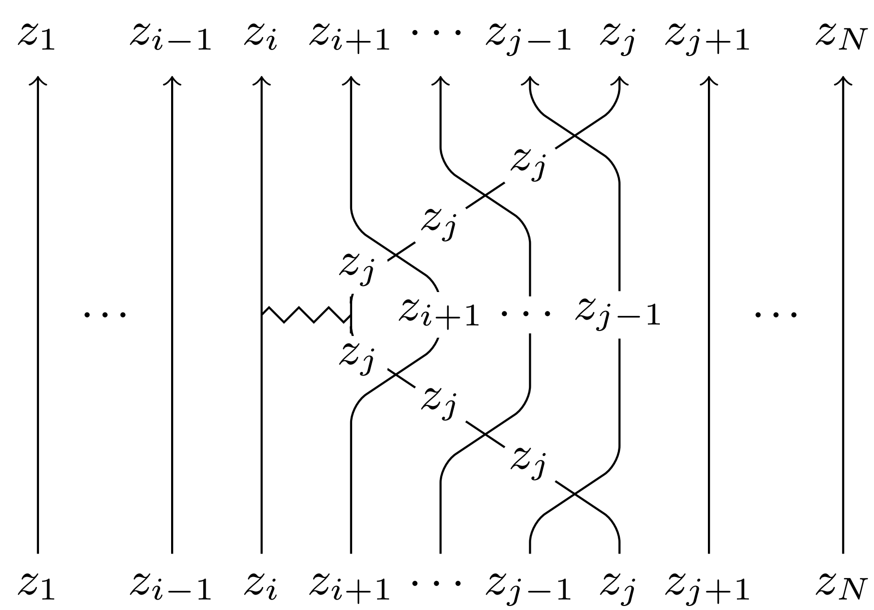
The q-Haldane Shastry Hamiltonian

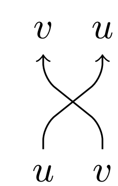
[Bernard, Gaudin, Haldane, Pasquier, 93; Uglov 95; Lamers 18]

$$H = -\frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\, S_{[i,j]}$$

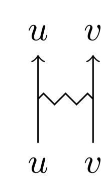
$$[N] := \frac{\mathsf{q}^N - \mathsf{q}^{-N}}{\mathsf{q} - \mathsf{q}^{-1}}$$

$S_{[i,j]} :=$





$:= \check{R}(u/v)$



$$:= -(\mathsf{q} - \mathsf{q}^{-1})\, \check{R}'(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathsf{q}^{-1} & -1 & 0 \\ 0 & -1 & \mathsf{q} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

q-antisymmetrizer

XXZ R-matrix

$$\check{R}(u) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u\, g(u) & f(u) & 0 \\ 0 & f(u) & g(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad f(u) := \frac{u-1}{\mathsf{q}\, u - \mathsf{q}^{-1}}, \quad g(u) := \frac{\mathsf{q} - \mathsf{q}^{-1}}{\mathsf{q}\, u - \mathsf{q}^{-1}}$$

The q-Haldane Shastry Hamiltonian

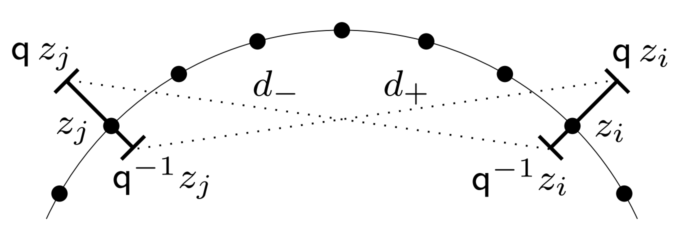
$$H = -\frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\ S_{[i,j]}$$

$$[N] := \frac{\mathsf{q}^N - \mathsf{q}^{-N}}{\mathsf{q} - \mathsf{q}^{-1}}$$

$$V(z_i, z_j) = \frac{z_i\, z_j}{(\mathsf{q}\, z_i - \mathsf{q}^{-1} z_j)(\mathsf{q}^{-1} z_i - \mathsf{q}\, z_j)}$$

$$z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$$

point-splitting interaction (q real):



Recap: the usual Haldane Shastry Hamiltonian

q $\to 1$

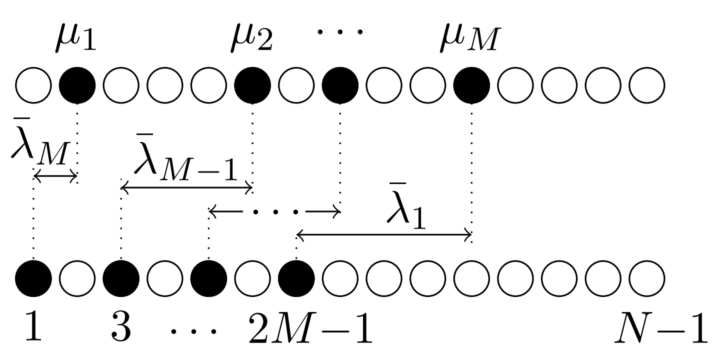
$$H_{\rm HS} = -\sum_{i\neq j} V(z_i, z_j)\; P_{ij}$$

[Haldane, 88; Shastry, 88]

$$V(z_i, z_j) = \frac{z_i z_j}{(z_i - z_j)^2} = \frac{1}{\sin^2 \pi(i-j)/N}$$

$$z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$$

The model is Yangian symmetric (huge degeneracy) and the spectrum is encoded by motifs:



M magnon motif

$$\bar\lambda_m = \mu_{M-m+1} - 2(M-m) - 1 \qquad \mu_{m+1} > \mu_m + 1$$

"vacuum" M magnon motif

$$E(\mu) - E_0 = \sum_{m=1}^{M} \varepsilon(\mu_m) = \sum_{m=1}^{M} \mu_m (N - \mu_m)$$

The q-Haldane Shastry Hamiltonian

$$H = -\frac{[N]}{N} \sum_{i<j} V(z_i, z_j)\; S_{[i,j]}$$

"chiral"

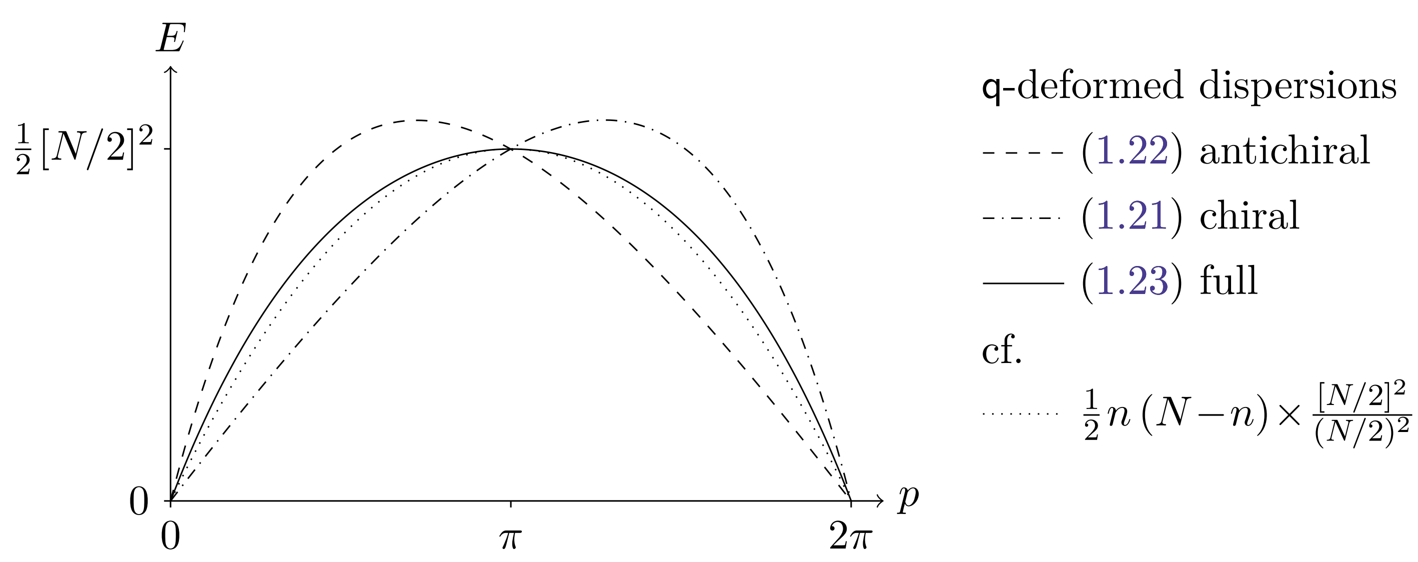
$$\bar{H} = -\frac{[N]}{N} \sum_{i>j} V(z_i, z_j)\; S_{[i,j]}$$

"antichiral"

$$\varepsilon(n) = \frac{1}{\mathsf{q} - \mathsf{q}^{-1}} \left(\frac{\mathsf{q}^{-n}}{\mathsf{q}^{-N}} [n] - \frac{n}{N} [N] \right)$$

$$\bar{\varepsilon}(n) = \frac{1}{\mathsf{q} - \mathsf{q}^{-1}} \left(\frac{\mathsf{q}^{n}}{\mathsf{q}^{N}} [n] - \frac{n}{N} [N] \right) = \varepsilon(n)\big|_{\mathsf{q} \mapsto \mathsf{q}^{-1}} = \varepsilon(N-n)$$

$$[H, \bar{H}] = 0$$



The Yangian symmetry is replaced by the quantum affine symmetry

qHS, spin Macdonald-Ruijsenaars model and DAHA

The q-Haldane-Shastry model is a particular limit of a more general, DAHA based model

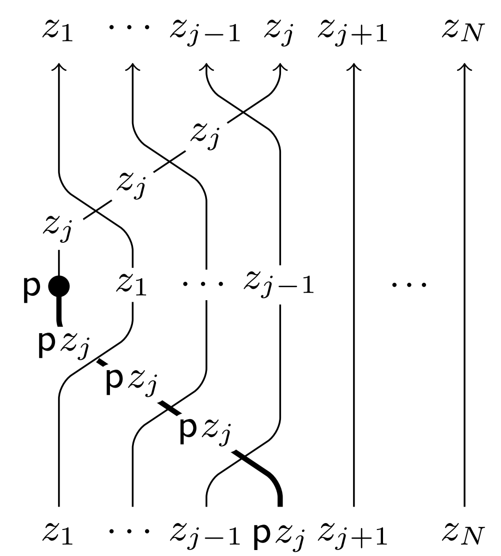
[Bernard, Gaudin, Haldane, Pasquier, 93; Uglov 95]

introduce a new parameter, p, and the multiplication operator $\hat{\mathsf{p}}_j \, z_i = \mathsf{p}^{\delta_{ij}} \, z_i \, \hat{\mathsf{p}}_j$

spin Macdonald-Ruijsenaars operator $\widetilde{D}_1$:

$$A_j := \prod_{k(\neq j)}^{N} \frac{\mathsf{q}\, z_j - \mathsf{q}^{-1} z_k}{z_j - z_k} = \prod_{k(\neq j)}^{N} f_{jk}^{-1}$$

$$\widetilde{D}_1 = \sum_{j=1}^{N} A_j$$



in the non-relativistic limit $\mathsf{p} = \mathsf{q}^{2\hbar/g}$, $\mathsf{q} \to 1$

it becomes the spin Calogero-Sutherland Hamiltonian:

$$\widetilde{H}_{\rm nr} = \sum_{i=1}^{N} (z_i \, \partial_{z_i})^2 - 2 \sum_{i<j}^{N} \frac{z_i \, z_j}{(z_i - z_j)^2} \; g \, (g - P_{ij})$$

$g \to \infty$ (freezing) $\longrightarrow$ Haldane-Shastry

Algebraic setup: Hecke algebra and DAHA

$t^{1/2} = \mathsf{q}$ and $q = \mathsf{p}$

Hecke algebra (generalises the permutations algebra, e.g. for spin variables)

braid relations: $T_i\, T_{i+1}\, T_i = T_{i+1}\, T_i\, T_{i+1}\,, \qquad T_i\, T_j = T_j\, T_i \quad \text{if } |i-j|>1\,,$

Hecke condition: $(T_i - t^{1/2})\,(T_i + t^{-1/2}) = 0\,.$

there exists a realisation of the Hecke algebra on polynomials

$$T_i^{\text{pol}} := -t^{-1/2}\,(t\, z_i - z_{i+1})\,\partial_i + t^{1/2} \qquad \partial_i := (z_i - z_{i+1})^{-1}\,(1 - s_i)$$

coordinate permutation

introduce extra operators Y_i (mutually commuting): affine Hecke algebra (AHA)

$$T_i^{-1}\, Y_i\, T_i^{-1} = Y_{i+1}\,, \qquad T_i\, Y_j = Y_j\, T_i \quad \text{if } j \neq i, i+1$$

together with the multiplication by z_i : double affine Hecke algebra (DAHA)

$$T_i\, Z_i\, T_i = Z_{i+1}\,, \qquad T_i\, Z_j = Z_j\, T_i \quad \text{if } j \neq i, i+1$$

$$Y_i = t^{(N-2i-1)/2}\, R^{\text{pol}}_{i,i+1}\, R^{\text{pol}}_{i,i+2} \cdots R^{\text{pol}}_{iN}\, \hat{q}_i\, R^{\text{pol}}_{i1} \cdots R^{\text{pol}}_{i,i-2}\, R^{\text{pol}}_{i,i-1} \qquad R^{\text{pol}}_{i,i+1} = t^{-1/2}\, T_i^{\text{pol}}\, s_i$$

the centre of the AHA is generated by the symmetric polynomials $\Delta(u) := \prod_{i=1}^{N} (1 + u\, Y_i) = \sum_{r=0}^{N} u^r\, e_r(\boldsymbol{Y})$

Algebraic setup: Hecke algebra and DAHA

Now we want to introduce a spin model and a Hecke algebra realised on spins T^{sp}

XXZ R-matrix: $\check{R}(u) = t^{1/2}\,\dfrac{u\,T^{\mathrm{sp}} - (T^{\mathrm{sp}})^{-1}}{t\,u - 1}$ $\qquad R(u) \coloneqq P\,\check{R}(u)$

obeys Yang-Baxter: $\check{R}_{12}(u/v)\,\check{R}_{23}(u)\,\check{R}_{12}(v) = \check{R}_{23}(v)\,\check{R}_{12}(u)\,\check{R}_{23}(u/v)$

XXZ model: $L^{\mathrm{sp}}_a(u;\boldsymbol{z}) \coloneqq R_{aN}(u/z_N)\cdots R_{a2}(u/z_2)\,R_{a1}(u/z_1)$ ← inhomogeneities

To define the long-range spin Macdonald-Ruijsenaars model introduce a bosonic space:

$$\widetilde{\mathcal{H}} = \bigcap_{i=1}^{N} \ker(T_i^{\mathrm{sp}} - T_i^{\mathrm{pol}})$$

and the action of the Macdonald operators on these spaces:

$$\widetilde{D}_1 \coloneqq Y_1 + \cdots + Y_N\big|_{\widetilde{\mathcal{H}}} = \widetilde{Y}_1 + \cdots + \widetilde{Y}_N$$

Quantum affine symmetry of the model

One defines the L matrix of the model via:

$$\widetilde{L}_a(u) \coloneqq R_{aN}(u\,Y_N)\cdots R_{a2}(u\,Y_2)\,R_{a1}(u\,Y_1)$$ (inhomogeneities became operators)

$\widetilde{L}_a(u)$ preserves the physical space $\widetilde{\mathcal{H}}$

The center of the quantum affine algebra is generated by the quantum determinant

$$\mathrm{qdet}_a\, L_a(u) = A(t\,u)\,D(u) - t^{1/2}\,B(t\,u)\,C(u)$$

$$\mathrm{qdet}_a\, L_a(u;\mathbf{z}) = t^{N/2}\prod_{i=1}^{N}\frac{u-z_i}{tu-z_i}$$

therefore the Hamiltonians $D_r = e_r(\mathbf{Y})$, and their restrictions $\widetilde{D}_r = e_r(\widetilde{\mathbf{Y}})$, commute with the quantum affine algebra, whence the high symmetry of the model

$$[\widetilde{L}_a(u), \Delta(v)] = 0 \qquad\qquad \Delta(u) \coloneqq \prod_{i=1}^{N}(1+u\,Y_i) = \sum_{r=0}^{N} u^r\, e_r(\boldsymbol{Y})$$

Back to the q-Haldane-Shastry model: freezing

To remove the dynamical degrees of freedom one sends $q \to 1$

this freezes the particles in the equilibrium positions $z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$

[Polychronakos, 93; Haldane, Talstra, 95; Uglov 95]

e.g. $$\widetilde{D}_1 = [N] + (q-1)\,\delta\widetilde{D}_1 + \mathcal{O}(q-1)^2$$

$$\delta\widetilde{D}_1 \sim H = -\frac{[N]}{N}\sum_{i<j} V(z_i,z_j)\,S_{[i,j]}$$ **[Lamers 18]**

the higher order Hamiltonians can be similarly obtained by expanding $\delta\widetilde{\Delta}(u)$ in powers ou u

$$\widetilde{\Delta}(u) = \widetilde{\Delta}_0(u) + (q-1)\;\delta\widetilde{\Delta}(u) + \mathcal{O}(q-1)^2$$

the eigenvalues can be obtained from the spectrum of the Macdonald operator, *e.g.*

$$\widetilde{E}_1(\widetilde{\lambda}) = \textstyle\sum_i t^{(N-2\,i+1)/2}\,q^{\widetilde{\lambda}_i}$$ $\widetilde{\lambda}$ a partition

The eigenvectors of the q-Haldane-Shastry model

We have an explicit expression of the (equivalent of) highest weight vectors

one for each multiplet of the quantum affine algebra (for each motif)

$$|\mu\rangle = \sum_{i_1<\cdots<i_M}^{N} \Psi_\mu(i_1,\cdots,i_M)\,|i_1,\cdots,i_M\rangle\!\rangle\,, \qquad |i_1,\cdots,i_M\rangle\!\rangle \coloneqq \sigma^-_{i_1}\cdots\sigma^-_{i_M}\,|{\uparrow}\cdots{\uparrow}\rangle$$

the component where all the reversed spins are at the left is particularly simple

$$\Psi_\mu(1,\cdots,M) = \langle\!\langle 1,\cdots,M|\mu\rangle = \mathrm{ev}_\omega\,\widetilde{\Psi}_{\lambda(\mu)}(z_1,\cdots,z_M)$$

$$\widetilde{\Psi}_\lambda(z_1,\cdots,z_M) \coloneqq \left(\prod_{m<n}^{M} (\mathsf{q}\,z_m - \mathsf{q}^{-1}z_n)\,(\mathsf{q}^{-1}z_m - \mathsf{q}\,z_n)\right) P^\star_\lambda(z_1,\cdots,z_M)$$

$P^\star_\lambda$ is a Macdonald polynomial with parameters $q^* = (t^*)^{1/2} = \mathrm{q}^2$

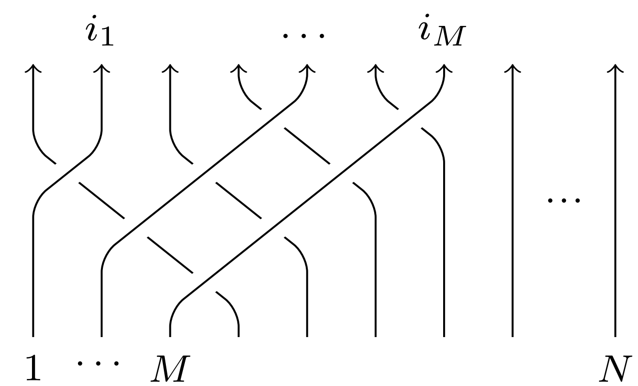
ev_ω sets $z_j \longmapsto \omega^j = \mathrm{e}^{2\pi \mathrm{i} j/N}$

The eigenvectors of the q-Haldane-Shastry model

The other components are slightly more involved and they correspond to transport of the indices using the Hecke generators

$$\Psi_\mu(i_1,\cdots,i_M) = \langle\langle i_1,\cdots,i_M|\mu\rangle = \mathrm{ev}_\omega\Big(T^{\mathrm{pol}}_{\{i_1,\cdots,i_M\}}\widetilde{\Psi}_{\lambda(\mu)}(z_1,\cdots,z_M)\Big)$$

$$T_{\{i_1,\cdots,i_M\}} = T_{(i_1,\cdots,1)}\cdots T_{(i_M,\cdots,M)} =$$



$$T_{(j,j-1,\cdots,i)} = T_{j-1}\cdots T_i$$

This result nicely generalises the construction of the eigenvectors of the ordinary Haldane-Shastry model

The highest weight condition imposes that the length of the partition $\lambda(\mu)$ is strictly equal to M (as for HS)

Open questions

- Construction of the eigenvectors directly from the freezing procedure?
- Crystal limit $q \to \infty$?
- How to build the other vectors than the highest weights in the multiplet? Equivalent of the Gelfand-Tsetlin construction **[Uglov 95; Takemura, Uglov,97]**
- Fermionic version and q-wedges ? **[Kashiwara, Miwa, Stern, 95]**
- Large size limit (CFT) and vertex operator representation?

- Other models? How to interpolate between XXX and Haldane-Shastry (Inozemtsev model **[Klabbers, Lamers, to appear]**)?
- How rigid is the construction of DAHA? One can imagine many deformations of the XXX model **[Beisert et al. 07]** ; how to incorporate them in the algebraic formalism?