Multiple action rules for monodromy matrix entries and highest coefficient

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The super double Yangian

$DY(\mathfrak{gl}(m|n))$ is graded algebra generated by the modes $\mathrm{T}_{i,j}^{(\ell)}$, $\ell \in \mathbb{Z}$, $1 \leq i,j \leq N+1$ of the T-operators

$$\mathrm{T}^{\pm}(u) = \mathbb{I} \otimes \mathbf{1} + \sum_{\substack{\ell \geq 0 \\ \ell < 0}} \sum_{i,j=1}^{N+1} \mathrm{E}_{ij} \otimes \mathrm{T}_{i,j}^{(\ell)} u^{-\ell-1},$$

Monodromy matrix elements $\mathrm{T}_{i,j}^{\pm}(u)$ subject to the relations

$$\mathrm{R}(u,v) \cdot (\mathrm{T}^{\mu}(u) \otimes \mathbb{I}) \cdot (\mathbb{I} \otimes \mathrm{T}^{\nu}(v)) = (\mathbb{I} \otimes \mathrm{T}^{\nu}(v)) \cdot (\mathrm{T}^{\mu}(u) \otimes \mathbb{I}) \cdot \mathrm{R}(u,v)$$

$\mu, \nu = \pm$

Gauss decomposition of monodromy matrix

$$\mathrm{T}^{\pm}_{i,j}(u) = \mathrm{F}^{\pm}_{j,i}(u)k^{\pm}_{i}(u) + \sum_{1\leq\ell<i} \mathrm{F}^{\pm}_{j,\ell}(u)k^{\pm}_{\ell}(u)\mathrm{E}^{\pm}_{\ell,i}(u),$$

$$\mathrm{T}^{\pm}_{i,i}(u) = k^{\pm}_{i}(u) + \sum_{1\leq\ell<i} \mathrm{F}^{\pm}_{i,\ell}(u)k^{\pm}_{\ell}(u)\mathrm{E}^{\pm}_{\ell,i}(u),$$

$$\mathrm{T}^{\pm}_{j,i}(u) = k^{\pm}_{i}(u)\mathrm{E}^{\pm}_{i,j}(u) + \sum_{1\leq\ell<i} \mathrm{F}^{\pm}_{i,\ell}(u)k^{\pm}_{\ell}(u)\mathrm{E}^{\pm}_{\ell,j}(u)\,.$$

Simple currents

$$F_i(u) = \mathrm{F}^{+}_{i+1,i}(u) - \mathrm{F}^{-}_{i+1,i}(u)\,, \quad E_i(u) = \mathrm{E}^{+}_{i,i+1}(u) - \mathrm{E}^{-}_{i,i+1}(u)$$

Commutation relations for the currents

$F_i(t)$, $E_i(t)$ and the Cartan currents $k_i^{\pm}(t)$:

$$k_i^{\pm}(u)F_i(v)k_i^{\pm}(u)^{-1} = f_{[i]}(v,u)\ F_i(v),$$
$$k_{i+1}^{\pm}(u)F_i(v)k_{i+1}^{\pm}(u)^{-1} = f_{[i+1]}(u,v)\ F_i(v),$$

$$((u-v)\epsilon_{i,m} - c_{[i]})\ F_i(u)F_i(v) = ((u-v)\epsilon_{i,m} + c_{[i]})\ F_i(v)F_i(u),$$

$$(u-v)\ F_i(u)F_{i+1}(v) = (u-v-c_{[i+1]})\ F_{i+1}(v)F_i(u),$$

$$[E_i(u), F_j(v)\} = E_i(u)F_j(v) - (-)^{([i]+[i+1])([j]+[j+1])}F_j(v)E_i(u)$$
$$= \delta_{i,j}c_{[i+1]}\delta(u,v)\Big(k_{i+1}^{-}(u)\cdot k_i^{-}(u)^{-1} - k_{i+1}^{+}(v)\cdot k_i^{+}(v)^{-1}\Big)$$

Notation

$$f_{[i]}(u,v) = \frac{u - v + c_{[i]}}{u - v}, \qquad g_{[i]}(u,v) = \frac{c_{[i]}}{u - v}$$

$$\lambda_j(\bar{u}) = \prod_{u_\ell \in \bar{u}} \lambda_j(u_\ell), \qquad f_{[i]}(\bar{u},\bar{v}) = \prod_{u_\ell \in \bar{u},\ v_{\ell'} \in \bar{v}} f_{[i]}(u_\ell, v_{\ell'}).$$

$$\mathrm{T}_{i,j}(\bar{u}) = \prod_{\ell} \mathrm{T}_{i,j}(u_\ell), \qquad \text{for} \qquad [i] + [j] = 0 \mod 2.$$

$$\Delta_x(\bar{u}) = \prod_{1 \le \ell < \ell' \le a} x(u_{\ell'}, u_\ell) \quad \text{and} \quad \Delta'_x(\bar{u}) = \prod_{1 \le \ell < \ell' \le a} x(u_\ell, u_{\ell'})$$

Projections

Counit mapping $\varepsilon : U_F \to \mathbb{C}$ is defined by the rules

$$\varepsilon(F_i^\ell) = 0 \quad \text{and} \quad \varepsilon(k_j^\ell) = 0\,.$$

Subalgebras U_F, U^+ are

$$U_F^- = U_F \cap U^-\,, \qquad U_F^+ = U_F \cap U^+\,.$$

$$P_f^+(\mathcal{F}_- \cdot \mathcal{F}_+) = \varepsilon(\mathcal{F}_-)\,\mathcal{F}_+\,, \quad P_f^-(\mathcal{F}_- \cdot \mathcal{F}_+) = \mathcal{F}_-\,\varepsilon(\mathcal{F}_+)\,, \quad \mathcal{F}_\pm \in U_F^\pm\,,$$

Bethe vectors

Ordered product of currents

$$\mathcal{F}(\bar{t}) = \overrightarrow{\prod_{1\leq a\leq N}} \left(\overrightarrow{\prod_{1\leq \ell\leq r_a}} F_a(t_\ell^a) \right)$$

Eigenvector is a projection onto the positive modes of currents $\mathcal{F}$

$$\mathbb{B}(\bar{t}) = P_f^+ \left(\mathcal{F}(\bar{t})\right) \prod_{s=1}^{N} \lambda_s(\bar{t}^s)|0\rangle$$

Ideals of the Yangian double

Let $I \subset DY(\mathfrak{gl}(m|n))$ be the ideal in the Yangian double generated by the elements of the form $\mathcal{F}_- \cdot \mathcal{F}$ such that $\mathcal{F}_- \in U_F^-$, $\mathcal{F} \in U_F$ and $\varepsilon(\mathcal{F}_-) = 0$ (ε is a counit of $DY(\mathfrak{gl}(m|n))$). The whole ideal I is annihilated by this projection: $P_f^+(I) = 0$.
Let $K \in DY(\mathfrak{gl}(m|n))$ be the ideal generated by the elements which contain any combination of the 'negative' Cartan currents $k_j^-(u)$.
Composed currents

$$F_{j,i}(v) = c_{[i+1]} \cdots c_{[j-1]} F_{j,j-1}(v) F_{j-1,j-2}(v) \cdots F_{i+1,i}(v)$$

Presentations for the projections

Proposition

$$\mathrm{T}^{+}_{i,j}(z) \cdot P^{-}_{f}\left(F_{k,l}(w)\right) \sim_{I,K} -\phi_k \; c_{[l,k]} \; \delta_{jl} \; g(z,w) \; \mathrm{T}^{+}_{i,k}(z)$$

L. Frappat, S. Khoroshkin, S. Pakuliak and E. Ragoucy, *Bethe Ansatz for the Universal Weight Function*, (2009)

Proposition

For $i < j$

$$P^{-}_{f}\left(F_i(t^i)\cdots F_j(t^j)\right) = \sum_{\ell=0}^{j-i} c^{-1}_{[i,i+\ell+1]} \prod_{s=i}^{i+\ell-1} g_{[s+1]}(t^{s+1},t^s)$$
$$\times P^{-}_{f}\left(F_{i+\ell+1,i}(t^{i+\ell})\right) P^{-}_{f}\left(F_{i+\ell+1}(t^{i+\ell+1})\cdots F_j(t^j)\right),$$

Decomposition of positive projection

$$P_f^+\left(\mathcal{F}_1(\bar{t}^1)\cdots\mathcal{F}_N(\bar{t}^N)\right) = P_f^+\left(\mathcal{F}_1(\bar{t}^1)\cdots\mathcal{F}_{N-1}(\bar{t}^{N-1})\right)\cdot\mathcal{F}_N(\bar{t}^N)$$
$$-\sum_{\ell=1}^{N}\mathbb{C}_\ell\,\overline{\mathrm{Sym}}_{\bar{t}^\ell,\ldots,\bar{t}^N}\Big[\mathbb{G}_\ell(\bar{t}^{\ell-1},\ldots,\bar{t}^N)\;c^{-1}_{[\ell,N+1]}P_f^-\left(F_{N+1,\ell}(t_1^N)\right)$$
$$\times\,P_f^+\left(\mathcal{F}_1(\bar{t}^1)\cdots\mathcal{F}_{\ell-1}(\bar{t}^{\ell-1})\mathcal{F}_\ell(\bar{t}_1^\ell)\cdots\mathcal{F}_{N-1}(\bar{t}_1^{N-1})\right)\cdot\mathcal{F}_N(\bar{t}_1^N)\Big]$$
$$+\mathbb{W},$$

$\mathbb{W}$ denotes terms which have the structure $P_f^-\left(F_{j_1,i_1}(w_1)\right)P_f^-\left(F_{j_2,i_2}(w_2)\right)\mathcal{F}$ with $j_1\geq j_2$ and some element $\mathcal{F}\in\overline{U}_F$.

One more additional lemma

$$\mathrm{T}^{+}_{i,j}(z)\cdot P^{+}_{f}\left(\mathcal{F}_1\cdots\mathcal{F}_N\right)\sim_{I,K}\sum_{q=j}^{N+1}\overline{\mathrm{Sym}}_{\bar{t}^{j},\ldots,\bar{t}^{q-1}}\Big[\phi_q\ \mathbb{C}^{q-1}_{j}\ \mathbb{Z}^{q}_{j}(z;\bar{t})$$
$$\times\ \mathrm{T}^{+}_{i,q}(z)\cdot\mathcal{F}_1\cdots\mathcal{F}_{j-1}\mathcal{F}'_{j}\cdots\mathcal{F}'_{q-1}\Big]\cdot\mathcal{F}_q\cdots\mathcal{F}_N$$

Action formulae

After some transformations ...

$$\mathrm{T}^{+}_{i,j}(z)\cdot\mathbb{B}(\bar{t}) = \sum_{p=1}^{i}\sum_{q=j}^{N+1}\varphi_{p,q}\ \mathbb{C}_p^{i-1}\ \mathbb{C}_j^{q-1}$$

$$\times\mathrm{Sym}_{\bar{t}^p,\ldots,\bar{t}^{i-1},\bar{t}^j,\ldots,\bar{t}^{q-1}}\Big[\mathbb{D}(\bar{t})\ \mathbb{Y}(z,\bar{t})\ \Lambda(z;\bar{t})\ \mathbb{B}(\{z,\bar{t}\}')\Big],$$

$$\Lambda(z;\bar{t}) = \frac{\lambda_{p+1}(t^p_{r_p})\cdots\lambda_i(t^{i-1}_{r_{i-1}})}{(\lambda_{p+1}(z)\cdots\lambda_{j-1}(z))^{\theta_{p+1,j-1}}}$$

here $\theta_{i,j}$ is a step function, $\varphi_{p,q}$ gives sign

$$\mathbb{C}_\ell^{\ell'} = \prod_{s=\ell}^{\ell'}\frac{1}{(r_s-1)!}.$$

$\mathfrak{gl}(2|2)$ example

Term with maximal number of summation — "maximal action"
All action rules can be found from only rule for $T_{N,1}$

$$\begin{aligned}
&\mathrm{T}_{4,4}(z)\mathbb{B}_{a,b,d}(\bar{u};\bar{v};\bar{w})\\
&= \varphi_{1,4}\lambda_1(z)\sum \frac{f(u_i,\bar{u}_i)g(v_j,\bar{v}_j)f_{[3]}(w_k,\bar{w}_k)h(\bar{v}_j,z)}{f(v_j,\bar{u})f_{[3]}(w_k,\bar{v})}\\
&\times g(z,u_i)g(v_j,u_i)g_{[3]}(w_k,v_j)\frac{\lambda_2(u_i)\lambda_3(v_j)\lambda_4(w_k)}{\lambda_1(u_i)\lambda_2(v_j)\lambda_3(w_k)}\\
&\times \mathbb{B}_{a,b,d}(\{\bar{u}_i,z\};\{\bar{v}_j,z\};\{\bar{w}_k,z\}).
\end{aligned}$$

Apply

$$\mathrm{T}^{+}_{i,j}(z) \cdot P^{-}_{f}\left(\mathrm{F}^{-}_{k,l}(w)\right) \sim_{I,K} -\phi_k\ \delta_{jl}\ g(z,w)\ \mathrm{T}^{+}_{i,k}(z)$$

to the action rule

$$\begin{aligned} & P^{+}_{f}\left(\mathcal{F}(\bar{u},\bar{v},\bar{w})\right) \\ = \sum(\dots) - \overline{\mathrm{Sym}}_{\bar{u},\bar{v},\bar{w}} & \left[\frac{f(v_1,\bar{u}_1)f_{[3]}(w_1,\bar{v}_1)}{(a-1)!(b-1)!(d-1)!}\right. \\ \times P^{-}_{f}\left(F_1(u_1)F_2(v_1)F_3(w_1)\right) \cdot & \left. P^{+}_{f}\left(\mathcal{F}(\bar{u}_1,\bar{v}_1,\bar{w}_1)\right)\right] + \mathbb{W}. \end{aligned}$$

$$P^{-}_{f}(F_1(u_1)F_2(v_1)F_3(w_1)) = g(u_1,v_1)g_{[3]}(w_1,v_1)P^{-}_{f}(F_{4,1}(w_1)) + \mathbb{W}.$$

The recursion for $T_{N+1,1}$ via $T_{N+1,N+1}$

$$
\begin{aligned}
& T_{4,1}(z)\mathbb{B}_{a,b,d}(\bar{u};\bar{v};\bar{w}) \\
&= \mathrm{Sym}\Big[\sum f(v_j,\bar{u}_i)f_{[3]}(w_k,\bar{v}_j)g(u_i,v_j)g(v_j,w_k)g(w_k,z) \\
&\times \frac{f(\bar{u}_i,u_i)g(\bar{v}_j,v_j)f_{[3]}(\bar{w}_k,w_k)}{f_{[3]}(\bar{w}_k,v_j)f_{[3]}(w_k,\bar{v}_j)f_{[3]}(w_k,v_j)f(\bar{v}_j,u_i)f(v_j,\bar{u}_i)f(v_j,u_i)} \\
&\times \frac{1}{\lambda_1(u_i)\lambda_2(v_j)\lambda_3(w_k)} \\
&\times\ T_{4,4}(z)\mathbb{B}_{a-1,b-1,d-1}(\bar{u}_i;\bar{v}_j;\bar{w}_k)\lambda_1(u_i)\lambda_2(v_j)\lambda_3(w_k)\Big],
\end{aligned}
$$

General formula and arbitrary $T_{i,j}$

$$\bar{u} \longrightarrow \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}, \bar{u}_{\mathrm{III}}\}, \qquad \bar{v} \longrightarrow \{\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}}, \bar{v}_{\mathrm{III}}\}, \qquad \text{etc.}$$

Taking the set $\bar{u}_k$, $\bar{v}_j$ empty we come to the "lower" action rules

$$T_{N+1,N}(\bar{z})\mathbb{B}_{\vec{k}}(\bar{x}) = \lambda_1(\bar{z}) \sum \prod_{i=1}^{N} \alpha_i(\bar{t}_1^i) \prod_{i=1}^{N-1} f(\bar{t}_1^i, \bar{t}_3^i)$$

$$\times \prod_{i=1}^{N-1} \frac{K(\bar{t}_1^{i+1}|\bar{t}_1^i)}{f(\bar{t}_1^{i+1}, \bar{t}_1^i)} \frac{K(\bar{t}_1^1|\bar{z})}{f(\bar{t}_1^1, \bar{z})} \frac{f(\bar{t}_1^N; \bar{t}_2^N; \bar{t}_3^N)}{\prod_{i=1}^{N-1} f(\bar{t}_1^{i+1}, \bar{t}_3^i)} \frac{K(\bar{z}|\bar{t}_2^N)}{f(\bar{z}, \bar{t}_2^N)} \mathbb{B}_{\vec{k}-\vec{n}}(\bar{x}')$$

Reshetikhin's formula

The scalar product in general case has form

$$S_{a,b} = \sum \lambda_1(\bar{u}^C_{\mathrm{I}})\lambda_1(\bar{u}^B_{\mathrm{II}})\lambda_2(\bar{u}^C_{\mathrm{II}})\lambda_2(\bar{u}^B_{\mathrm{I}})\lambda_2(\bar{v}^C_{\mathrm{I}})\lambda_2(\bar{v}^B_{\mathrm{II}})\lambda_3(\bar{v}^C_{\mathrm{II}})\lambda_3(\bar{v}^B_{\mathrm{I}})$$
$$\times f(\bar{u}^C_{\mathrm{II}}, \bar{u}^C_{\mathrm{I}})f(\bar{u}^B_{\mathrm{I}}, \bar{u}^B_{\mathrm{II}})f(\bar{v}^C_{\mathrm{II}}, \bar{v}^C_{\mathrm{I}})f(\bar{v}^B_{\mathrm{I}}, \bar{v}^B_{\mathrm{II}})f(\bar{v}^C_{\mathrm{I}}, \bar{u}^C_{\mathrm{II}})f(\bar{v}^B_{\mathrm{II}}, \bar{u}^B_{\mathrm{I}})$$
$$\times Z_{a_{\mathrm{I}},b_{\mathrm{I}}}(\bar{u}^C_{\mathrm{I}}; \bar{u}^B_{\mathrm{I}}|\bar{v}^C_{\mathrm{I}}; \bar{v}^B_{\mathrm{I}})Z_{a_{\mathrm{II}},b_{\mathrm{II}}}(\bar{u}^B_{\mathrm{II}}; \bar{u}^C_{\mathrm{II}}|\bar{v}^B_{\mathrm{II}}; \bar{v}^C_{\mathrm{II}}).$$

where Z's are so called "highest coefficients"

Highest coefficient and recursion

$$Z(\{\bar{s}\}|\{\bar{x}\}) = \frac{1}{\lambda_1(\bar{t}^1)\dots\lambda_N(\bar{t}^N)}$$
$$\times T_{2,1}(\bar{t}^1)\,T_{3,2}(\bar{t}^2)\dots\mathbb{T}_{m+1,m}(\bar{t}^m)\dots\mathbb{T}_{N+1,N}(\bar{t}^N)\mathbb{B}_{\bar{k}}(\bar{t}),$$

$$Z^N(\{\bar{s}\}|\{\bar{x}\}) = \lambda_1(\bar{z})\mathcal{N}\sum\prod_{i=1}^{N}\alpha_i(\bar{t}_1^i)\prod_{i=1}^{N-1}\frac{K(\bar{t}_1^{i+1}|\bar{t}_1^i)}{f(\bar{t}_1^{i+1},\bar{t}_1^i)}\frac{K(\bar{t}_1^1|\bar{s}^N)}{f(\bar{t}_1^1,\bar{s}^N)}$$
$$\times\prod_{i=1}^{N-1}\frac{f(\bar{t}_1^i,\bar{t}_3^i)}{f(\bar{t}_1^{i+1},\bar{t}_3^i)}\frac{K(\bar{s}^N|\bar{t}_2^N)}{f(\bar{s}^N,\bar{t}_2^N)}\frac{Z^{N-1}(\{\bar{t}_3\}|\{\bar{x}'\})}{\prod_{i=1}^{N-1}f_{[i]}(\bar{t}_3^{i+1},\bar{t}_3^i)}\Bigg|_{\bar{t}_2^N=\bar{s}^N;\bar{t}_1^N=\bar{x}^N}$$

Drinfeld coproduct

$$\Delta^{(D)}(F_i(z)) = F_i(z) \otimes \mathbf{1} + k_{i+1}^{+}(z) \left(k_i^{+}(z)\right)^{-1} \otimes F_i(z)$$
$$\Delta^{(D)}(k_j^{\pm}(z)) = k_j^{\pm}(z) \otimes k_j^{\pm}(z),$$
$$\Delta^{(D)}(E_i(z)) = \mathbf{1} \otimes E_i(z) + E_i(z) \otimes k_{i+1}^{-}(z) \left(k_i^{-}(z)\right)^{-1}$$

$$\mathbb{D}(\bar{t}) = \prod_{s=p}^{i-1} \frac{f_{[s]}(t^s_{r_s}, \bar{t}^s_{r_s})}{[(-1)^{(r_s-1)} h(t^s_{r_s}, \bar{t}^s_{r_s})]^{\delta_{s,m}}} \prod_{s=j}^{q-1} \frac{f_{[s]}(\bar{t}^s_1, t^s_1)}{h(\bar{t}^s_1, t^s_1)^{\delta_{s,m}}}$$

for $p < i$ and $q = j$

$$\mathbb{Y}(z, \bar{t}) = f_{[p]}(z, \bar{t}^{p-1})\ f_{[j]}(\bar{t}^j, z) \prod_{s=p}^{i-1} h(\bar{t}^s_{r_s}, z)^{\delta_{s,m}} \prod_{s=i}^{j-1} h(\bar{t}^s, z)^{\delta_{s,m}}$$
$$\times \frac{g(z, t^p_{r_p}) \prod_{s=p}^{i-2} g_{[s+1]}(t^{s+1}_{r_{s+1}}, t^s_{r_s})}{\prod_{s=p-1}^{i-2} f_{[s+1]}(t^{s+1}_{r_{s+1}}, \bar{t}^s)}$$