Tetrahedron equation and matrix product method

Atsuo Kuniba (Univ. Tokyo)

Joint work with S. Maruyama and M. Okado

Reference
Multispecies totally asymmetric zero range process I, II
Journal of Integrable Systems (2016)

RAQIS’16  26 Aug. 2016, Univ. of Geneva
n-species Totally Asymmetric Zero-Rang Process (n-TAZRP)

n=3 example

Species 1 2 3

Smaller species particles have priority to hop to the left neighbor site

allowed

forbidden
We consider \( n \)-TAZRP on 1D periodic chain, which is a Markov process governed by the master equation.

Local state (site variable):

\[
\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n, \quad \alpha_a = \#(\text{species } a \text{ particles})
\]

\((\gamma, \delta) > (\alpha, \beta) \quad \overset{\text{def}}{\iff} \quad \text{local transition } (\gamma, \delta) \rightarrow (\alpha, \beta) \text{ is allowed (priority rule obeyed)}\)

Local Markov matrix:

\[
h|\gamma, \delta\rangle = \sum_{(\gamma, \delta) > (\alpha, \beta)} (|\alpha, \beta\rangle - |\gamma, \delta\rangle)
\]

Markov matrix:

\[
H = \sum_{i \in \mathbb{Z}_L} h_{i,i+1}
\]

We consider \( n \)-TAZRP on 1D periodic chain, which is a Markov process governed by the master equation:

\[
\frac{d}{dt} \langle P(t) \rangle = H \langle P(t) \rangle
\]
\[ |P(t)\rangle = \sum_{(\sigma_1, \ldots, \sigma_L) \in S(m)} \mathbb{P}(\sigma_1, \ldots, \sigma_L; t) |\sigma_1, \ldots, \sigma_L\rangle \]

Set of configurations in the sector \( m=(m_1, \ldots, m_n) \)

\( m_a = \#(\text{species } a \text{ particles}) \)
Probability of the configuration at time $t$

$$|P(t)\rangle = \sum_{(\sigma_1, \ldots, \sigma_L) \in S(m)} \mathbb{P}(\sigma_1, \ldots, \sigma_L; t) |\sigma_1, \ldots, \sigma_L\rangle$$

Set of configurations in the sector $m=(m_1, \ldots, m_n)$

$m_a = \#$(species $a$ particles)

- Problem in non-equilibrium statistical mechanics
- Stochastic dynamics of $n$-species particles with priority constraint within the same departure site (zero-range interaction)
\[ |P(t)\rangle = \sum_{(\sigma_1, \ldots, \sigma_L) \in S(m)} \mathbb{P}(\sigma_1, \ldots, \sigma_L; t) |\sigma_1, \ldots, \sigma_L\rangle \]

- Problem in non-equilibrium statistical mechanics

- Stochastic dynamics of \(n\)-species particles with priority constraint within the same departure site (zero-range interaction)

- Example of Integrable Probability:

  Matrix product construction of Steady State \( \rightarrow \) tetrahedron equation

  Associated with Stochastic R matrix for \(U_q(A^{(1)}_n)\) (arXiv:1604:08304)

These features become most manifest for multispecies setting \(n>1\)
Steady states: \( H \mid P \rangle = 0 \)

Each sector \( m \) has the unique steady state

\[
\mid \bar{P}_L(m) \rangle = \mid \xi_L(m) \rangle + C \mid \xi_L(m) \rangle + \cdots + C^{L-1} \mid \xi_L(m) \rangle
\]

\( (L = \text{chain length}, \ m = \text{sector}, \ C = \text{cyclic shift}) \)

Example from 3-TAZRP

\[
\mid \xi_2(1, 1, 1) \rangle = 2 \mid 1, 23 \rangle + \mid 2, 13 \rangle + 3 \mid 3, 12 \rangle + 6 \mid 0, 123 \rangle, \\
\mid \xi_3(1, 1, 1) \rangle = 5 \mid 1, 2, 3 \rangle + \mid 1, 3, 2 \rangle + 9 \mid 0, 1, 23 \rangle + 3 \mid 0, 2, 13 \rangle + 6 \mid 0, 3, 12 \rangle + 12 \mid 0, 12, 3 \rangle \\
\quad + 3 \mid 0, 13, 2 \rangle + 3 \mid 0, 23, 1 \rangle + 18 \mid 0, 0, 123 \rangle, \\
\mid \xi_4(1, 1, 1) \rangle = 17 \mid 0, 1, 2, 3 \rangle + 3 \mid 0, 1, 3, 2 \rangle + 12 \mid 0, 1, 0, 23 \rangle + 3 \mid 0, 2, 1, 3 \rangle + 7 \mid 0, 2, 3, 1 \rangle + 8 \mid 0, 2, 0, 13 \rangle \\
\quad + 9 \mid 0, 3, 1, 2 \rangle + \mid 0, 3, 2, 1 \rangle + 20 \mid 0, 3, 0, 12 \rangle + 24 \mid 0, 0, 1, 23 \rangle + 6 \mid 0, 0, 2, 13 \rangle + 10 \mid 0, 0, 3, 12 \rangle \\
\quad + 30 \mid 0, 0, 12, 3 \rangle + 6 \mid 0, 0, 13, 2 \rangle + 4 \mid 0, 0, 23, 1 \rangle + 40 \mid 0, 0, 0, 123 \rangle, \\
\mid \xi_2(2, 1, 1) \rangle = 2 \mid 1, 123 \rangle + \mid 2, 113 \rangle + 3 \mid 3, 112 \rangle + 2 \mid 11, 23 \rangle + \mid 12, 13 \rangle + 6 \mid 0, 1123 \rangle, \\
\mid \xi_3(2, 1, 1) \rangle = 3 \mid 1, 1, 23 \rangle + 2 \mid 1, 2, 13 \rangle + \mid 1, 3, 12 \rangle + 5 \mid 1, 12, 3 \rangle + \mid 1, 13, 2 \rangle + 5 \mid 2, 3, 11 \rangle + \mid 2, 11, 3 \rangle \\
\quad + 9 \mid 0, 1, 123 \rangle + 3 \mid 0, 2, 113 \rangle + 6 \mid 0, 3, 112 \rangle + 9 \mid 0, 11, 23 \rangle + 3 \mid 0, 12, 13 \rangle + 3 \mid 0, 13, 12 \rangle \\
\quad + 3 \mid 0, 23, 11 \rangle + 12 \mid 0, 112, 3 \rangle + 3 \mid 0, 113, 2 \rangle + 3 \mid 0, 123, 1 \rangle + 18 \mid 0, 0, 1123 \rangle, \\
\]

n=1-TAZRP has trivial (uniform) steady states
Result: Matrix product formula

Steady state probability

\[ P(\sigma_1, \ldots, \sigma_L) = \text{Tr}_{F \otimes n(n-1)/2} (X_{\sigma_1} \cdots X_{\sigma_L}) \]

\( F = \text{Fock space of } q\text{-boson at } q=0 \)

**TAZRP operators**

Piece of *layer transfer matrices* of 3D lattice model satisfying Tetrahedron equation

\( q\text{-boson and Fock space} \)

\[ A_q = \langle a^+, a^-, k \rangle \text{ acts on } F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle \text{ by } \]

\[ a^+ |m\rangle = |m + 1\rangle, \quad a^- |m\rangle = (1 - q^{2m}) |m - 1\rangle, \quad k |m\rangle = q^m |m\rangle \]
TAZRP operators

\[ X_\alpha = X_\alpha(z = 1) \]

\[ X_\alpha(z) = \sum z^{a_1 + \cdots + a_n} \]

\[ \alpha = (\alpha_1, \ldots, \alpha_n) \quad \alpha_1 + \cdots + \alpha_n \]

\[ \in A_0 ^{\otimes n(n-1)/2} \]

Edges summed over \( \mathbb{Z}_{\geq 0} \)

\[ \begin{array}{c}
  b \\
  i \quad \downarrow \\
  j \quad \rightarrow \quad a \\
  \delta^{a+b}_{i+j} \theta(a \geq j)(a^+)^j k^{a-j}(a^-)^b \in A_{q=0}
\end{array} \]

\[ \theta(\text{true}) = 1, \quad \theta(\text{false}) = 0, \quad \delta^{a+b}_{i+j} = \theta(a + b = i + j) \]

n=2 Example

\[ X_{\alpha_1, \alpha_2}(z) = \sum_{j \geq 0} z^{\alpha_1 + \alpha_2 + j} \alpha_1 + \alpha_2 \]

\[ \alpha_2 \rightarrow \quad j + \alpha_1 \]

\[ = z^{\alpha_1 + \alpha_2} \sum_{j \geq 0} z^j (a^+)^j k^{\alpha_1}(a^-)^{\alpha_2} \]
Derivation

To show the steady state condition for TAZRP operators

$$x^{|\beta|} \sum_{(\gamma, \delta) \geq (\alpha, \beta)} X_\gamma(x) X_\delta(y) = (x \leftrightarrow y) \quad (|\beta| = \beta_1 + \cdots + \beta_n)$$

... Highly non-local relation
Derivation

To show the steady state condition for TAZRP operators

\[
x^{|\beta|} \sum_{(\gamma, \delta) \geq (\alpha, \beta)} X_\gamma(x) X_\delta(y) = (x \leftrightarrow y) \quad (|\beta| = \beta_1 + \cdots + \beta_n)
\]

\ldots\text{Highly non-local relation}

**Strategy of the proof**

- Tetrahedron equation (Single local relation)
- Bilinear relations of layer transfer matrices
- \( q \to 0 \)
Layer transfer matrix with boundary condition $b, i$

$n = 3$ Example

$$T(z)_i^b = \chi'_b \chi_i \sum_{a,j} \chi'_a \chi_j$$

Each 3D vertex is a $q$-boson acting on the Fock space on the blue lines.
3D R-operators

\[ \hat{R}_{ij}^{ab}(z) = \delta_{i+j} \sum_{\lambda+\mu=b} (-1)^{\lambda} q^{\lambda+\mu^2} (i^{\lambda} \lambda) (j^{\mu} \mu) (a^{-\mu}(a^{+})^{-\lambda}k^{i+\lambda-\mu} \right. \]
Define 3D R-operators $\hat{R}_{ij}^{ab}(z)$ and $\hat{S}_{ij}^{ab}(z)$ by

$$\hat{R}_{ij}^{ab}(z) = \delta_{i,j} z^{a-b} \sum_{\lambda+\mu=b} (-1)^{\lambda q^2} \binom{i}{\mu} \binom{j}{\lambda} (a^{-})^{\mu} (a^{+})^{j-\lambda} k^{i+\lambda-\mu}$$

$$\hat{S}_{ij}^{ab}(z) = \delta_{i,j} z^{-1} \sum_{\lambda+\mu=b} (-1)^{\lambda} q^2 \binom{i}{\mu} \binom{j}{\lambda} (a^{-})^{\mu} (a^{+})^{j-\lambda} k^{i+\lambda-\mu}$$

Define 3D R-operators $\mathcal{R}(z), \mathcal{S}(z) \colon F \otimes F \otimes F \to F \otimes F \otimes F$ by

$$\mathcal{R}(z) (|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes \hat{R}_{ij}^{ab}(z) |k\rangle$$

$$\mathcal{S}(z) (|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes \hat{S}_{ij}^{ab}(z) |k\rangle$$
Fact (reducible to Kapranov-Voevodsky 1994)

As operators on $F^\otimes 6$ the tetrahedron eq. holds: $(z_{ij} = z_i / z_j)$

$$S(z_{12})_{126}S(z_{34})_{346}R(z_{13})_{135}R(z_{24})_{245} = R(z_{24})_{245}R(z_{13})_{135}S(z_{34})_{346}S(z_{12})_{126}$$
Fact (reducible to Kapranov-Voevodsky 1994)

As operators on $F^\otimes 6$ the tetrahedron eq. holds: \( z_{ij} = z_i/z_j \)

\[ S(z_{12})_{126} S(z_{34})_{346} R(z_{13})_{135} R(z_{24})_{245} = R(z_{24})_{245} R(z_{13})_{135} S(z_{34})_{346} S(z_{12})_{126} \]

Background and relevant topics:

- Intertwiner of Soibelman’s representations of quantized coordinate ring of SL4
- Transition coefficients of the PBW bases of $U_q^+(sl_3)$
- Quantum geometry interpretation (Bazhanov-Mangazeev-Sergeev 2008)
Theorem (Bilinear relation of layer transfer matrices)

\[ \sum_{b,b',i,i' \mid b+b'=s, i+i'=r} \left| x^{b+i} y^{b'}^{i'} \right| T(x)^b_i T(y)^{b'}_{i'} = (x \leftrightarrow y) \]
\[ \forall s, r \in (\mathbb{Z}_{\geq 0})^n \]

Generalizes the commutativity corresponding to \( s = r = (0,\ldots,0) \)
Theorem (Bilinear relation of layer transfer matrices)

\[
\sum_{b,b',i',i''} x |b| + i |y| b' + i' | T(x)_{i}^{b} T(y)_{i'}^{b'} = (x \leftrightarrow y)
\]

\forall s, r \in (\mathbb{Z}_{\geq 0})^{n}

Generalizes the commutativity corresponding to \( s = r = (0,\ldots,0) \)

Follows from repeated applications of the tetrahedron eq.
At $q=0$, Layer transfer matrix is frozen to TAZRP operators.

$$T(z)^{0,\ldots,0,r}_{0,\ldots,0,r} = z^{-r} \sum_{a,j} z^{\|j\|}$$

(n=3 example)

$$= z^{-r} \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} X_\alpha(z) \otimes \text{simply described operators}$$

TAZRP operator
At $q=0$, Layer transfer matrix is frozen to TAZRP operators

\[
T(z)_{0; \ldots; 0, r} = z^{-r} \sum_{a, j} z^{j_1} \sum_{j_2} \sum_{j_3} T_{a, j, j_1, j_2, j_3} (n=3 \text{ example})
\]

Stationary condition for TAZRP operators

Bilinear relation of layer transfer matrices

Tetrahedron eq.
Concluding remarks

A parallel story holds for
n-species *Totally Asymmetric Simple Exclusion Process (n-TASEP)*

TAZRP and TASEP correspond to the two situations in which
type A quantum R matrices are factorized into solutions of
the tetrahedron equation as follows:

<table>
<thead>
<tr>
<th>Tetrahedron:</th>
<th>3D R operator</th>
<th>3D L operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yang-Baxter:</td>
<td>R matrix for</td>
<td>R matrix for</td>
</tr>
<tr>
<td></td>
<td>symm. tensor rep.</td>
<td>anti-symm. tensor rep.</td>
</tr>
<tr>
<td>Markov process:</td>
<td>n-TAZRP</td>
<td>n-TASEP</td>
</tr>
<tr>
<td></td>
<td>(today’s talk)</td>
<td>(arXiv:1506.04490, 1509.09018)</td>
</tr>
</tbody>
</table>
Integrable origin of n-TAZRP

\[ q = 0 \quad \mu = 0 \]

U\(_q(\mathbb{A}^{(1)}_n)\)-Zero Range Process associated with Stochastic R matrix


Quantum R matrix in a special gauge satisfying the axioms of Markov matrix

Matrix elements generalize Povolotsky’s transition rate for \( q\)-Hahn process (n=1)

\[ q \sum_{1 \leq i < j \leq n} (\beta_i - \gamma_i) \gamma_j \left( \frac{\mu}{\lambda} \right)^{\gamma_1 + \cdots + \gamma_n} \frac{(\lambda; q)_{\gamma_1 + \cdots + \gamma_n} (\mu; q)_{\beta_1 + \cdots + \beta_n - \gamma_1 - \cdots - \gamma_n}}{\prod_{i=1}^{n} (q; q)_{\gamma_i} (q; q)_{\beta_i - \gamma_i}} \]
Matrix product formula for $U_q(A^{(1)}_2)$-ZRP \hspace{1em} (K-Okado, arXiv:1608.02779)

(Discrete time Markov process with inhomogeneity $\mu_1, \ldots, \mu_L$)

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \text{Tr}(X_{\sigma_1}(\mu_1) \cdots X_{\sigma_L}(\mu_L)),$$

$$X_{\alpha}(\mu) = \mu^{-\alpha_1-\alpha_2} \frac{(\mu; q)_{\alpha_1+\alpha_2}}{(q; q)_{\alpha_1}(q; q)_{\alpha_2}} \frac{(a^+; q)_\infty}{(\mu^{-1}a^+; q)_\infty} \frac{k^{\alpha_2}}{(-qk; q)_{\alpha_1}}(a^-)^{\alpha_1}$$
Matrix product formula for $U_q(A^{(1)}_2)$-ZRP  
(K-Okado, arXiv:1608.02779)

(Discrete time Markov process with inhomogeneity $\mu_1, \ldots, \mu_L$)

\[ P(\sigma_1, \ldots, \sigma_L) = \text{Tr}(X_{\sigma_1}(\mu_1) \cdots X_{\sigma_L}(\mu_L)), \]

\[ X_\alpha(\mu) = \mu^{-\alpha_1-\alpha_2} \frac{(\mu; q)_{\alpha_1+\alpha_2}}{(q; q)_{\alpha_1} (q; q)_{\alpha_2}} \frac{(a^+; q)_{\infty}}{(\mu^{-1}a^+; q)_{\infty}} \frac{k^{\alpha_2}}{(-qk; q)_{\alpha_1}} (a^-)^{\alpha_1} \]

A $q$-boson representation of Zamolodchikov-Faddeev algebra

\[ X(\mu) \otimes X(\lambda) = \tilde{S}(\lambda, \mu) \left[ X(\lambda) \otimes X(\mu) \right] \]

satisfying an auxiliary condition

$q=0, \mu_i =0$ case agrees with the tetrahedron result for 2-TAZRP