ASEP with open boundaries and Koornwinder polynomials

Luigi Cantini

RAQIS’16 - Recent Advances in Quantum Integrable Systems
University of Geneva
Particles propagating under the effect of an external field and interacting with two reservoirs at different chemical potential $\rho_L, \rho_R$
Introduction

Particles propagating under the effect of an external field and interacting with two reservoirs at different chemical potential $\rho_L, \rho_R$

No detailed balance: Macroscopic particle current
The Asymmetric Simple Exclusion Process (ASEP)

- One dimensional lattice
- **Exclusion**: at most one particle per site
- **Asymmetric**: jump rate to the right $t^{1/2}$, to the left $t^{-1/2}$
- Particles enter with rate $\alpha$ from left, with rate $\delta$ from right
- Particles leave with rate $\gamma$ from left, with rate $\beta$ from right
Applications

Luigi Cantini

ASEP and Koornwinder Polynomials
Impurities

Here we want to take into account the presence of impurities that are NOT EXCHANGED with the reservoirs.
Two species ASEP

The impurity (second-class particle) has the same dynamics as a normal (first-class particle), while the first-class particles treat it as a hole.

Blockade effect

It is convenient to think at first class, second class particles and empty sites as three kinds of particles
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*Blockade effect*

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Master equation and Markov generator

Probabilities of the configurations \( \{w\} \) evolve under a Master equation

\[
\frac{d}{dt} P_w(t) = \sum_{w' \neq w} \mathcal{M}(w \rightarrow w') P_{w'}(t) - \sum_{w' \neq w} \mathcal{M}(w' \rightarrow w) P_w(t)
\]

Using a vector representation for the probabilities

\[
P_{N,m}(t) = \sum_{w \in Q(N,m)} P_w(t) w
\]

The Master equation reads

\[
\frac{d}{dt} P(t) = \mathcal{M} P(t)
\]

Where the Markov generator is given by the sum of local terms

\[
\mathcal{M} = \sum_{i=1}^{N-1} e_i + f_1 + f_N.
\]
Master equation and Markov generator

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Where the Markov generator is given by the sum of local terms

\[
\mathcal{M} = \sum_{i=1}^{N-1} e_i + f_1 + f_N.
\]

Length of the chain

Number of impurities
Define operators $T_0, T_1, \ldots, T_N$ as

$$T_0 = \alpha^{-\frac{1}{2}} \delta^{-\frac{1}{2}} f_1 + \alpha^\frac{1}{2} \delta^{-\frac{1}{2}} \mathbf{1}$$

$$T_N = \beta^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} f_N + \beta^\frac{1}{2} \gamma^{-\frac{1}{2}} \mathbf{1}$$

and for $1 \leq i \leq N - 1$

$$T_i = e_i + t^{-\frac{1}{2}} \mathbf{1}$$

They satisfy the commutation relations of the generators of the affine Hecke algebra $\hat{C}_N$

$$T_i - T_i^{-1} = t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}}$$

$$T_i T_j = T_j T_i \text{ if } |i - j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ if } i \neq 0, N - 1$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$$

$$T_N T_{N-1} T_N T_{N-1} = T_{N-1} T_N T_{N-1} T_N$$

with $t_0^{\frac{1}{2}} = \alpha^{\frac{1}{2}} \delta^{-\frac{1}{2}}, t_N^{\frac{1}{2}} = \beta^{\frac{1}{2}} \gamma^{-\frac{1}{2}}$ and $t_i = t$ for $1 \leq i \leq N - 1$. 
Integrability

\[ \tilde{R}_i(z) = 1 + \frac{z - 1}{t^{1/2}z - t^{-1/2}}e_i \]

\[ K_1(z|a, b) = 1 + \frac{(z^2 - 1)}{(z - a)(z - b)}\delta^{-1}f_1 \]

\[ K_N(z|c, d) = 1 + \frac{(1 - z^2)}{(cz - 1)(dz - 1)}\gamma^{-1}f_N \]

Where we assume \( t \neq 1 \) and parametrize the boundary rates as

\[ \alpha = \frac{(t^{1/2} - t^{-1/2})ab}{(a - 1)(b - 1)}, \quad \delta = \frac{t^{-1/2} - t^{1/2}}{(a - 1)(b - 1)} \]

\[ \beta = \frac{(t^{1/2} - t^{-1/2})cd}{(c - 1)(d - 1)}, \quad \gamma = \frac{t^{-1/2} - t^{1/2}}{(c - 1)(d - 1)} \]
Integrability

- Yang-Baxter equation (YBE)

\[ \mathcal{R}_i(yz^{-1})\mathcal{R}_{i+1}(xz^{-1})\mathcal{R}_i(xy^{-1}) = \mathcal{R}_{i+1}(xy^{-1})\mathcal{R}_i(xz^{-1})\mathcal{R}_{i+1}(yz^{-1}) \]

- Boundary Yang-Baxter equations [Sklyanin, Cherednik]

\[ \mathcal{R}_1(xy^{-1})K_1(y)\mathcal{R}_1(x^{-1}y^{-1}))K_1(x) = K_1(x)\mathcal{R}_1(x^{-1}y^{-1})K_1(y)\mathcal{R}_1(xy^{-1}), \]

\[ \mathcal{R}_{N-1}(xy^{-1})K_N(x)\mathcal{R}_{N-1}(xy)K_N(y) = K_N(y)\mathcal{R}_{N-1}(xy)K_N(x)\mathcal{R}_{N-1}(xy^{-1}), \]

- Unitarity

\[ \mathcal{R}_i(z)\mathcal{R}_i(z^{-1}) = 1, \quad K_1(x)K_1(x^{-1}) = 1, \quad K_N(x)K_N(x^{-1}) = 1. \]
In the $m = 0$ sector $\mathcal{M}$ is equivalent to a spin 1/2 chain with non-diagonal boundaries, many, many works [Cao et al., Pasquier Lazarescu, Nepomechie et al., Maillet et al.,...]

For $m \neq 0$ in the bulk it is a $U_q(SU(3))$ spin chain.

In this talk we focus on the Stationary measure

$$\mathcal{MP}_{N,m} = 0$$

Usually dealt with by the Matrix Product Ansatz [Derrida, Evans, Hakim, Pasquier]

Rich combinatorics [Corteel, Williams, Mandelshtam, Viennot,...]

Boundary induced phase transitions [Krug,...]

Here I’ll describe an approach based on

Exchange/reflection equations.
Spin chains spectrum et al.

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In this talk we focus on the *Stationary measure* $\mathcal{M} \mathcal{P}_{N,m} = 0$

- Usually dealt with by the Matrix Product Ansatz [Derrida, Evans, Hakim, Pasquier]
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*Exchange/reflection equations.*
Stationary measure: exchange/reflection equations

We introduce a vector

$$
\Psi_{N,m}(z) = \sum_{w \in \mathcal{Q}(N,m)} \psi_w(z)w
$$

solution of the following exchange/reflection equations

$$
\check{R}_i(z_i z_{i+1}^{-1}) \Psi_{N,m}(z) = s_i \Psi_{N,m}(z)
$$
$$
K_1(z_1) \Psi_{N,m}(z) = s_0 \Psi_{N,m}(z)
$$
$$
K_N(z_N) \Psi_{N,m}(z) = s_N \Psi_{N,m}(z).
$$

where

$$
\begin{align*}
    s_i f(\ldots, z_i, z_{i+1}, \ldots) &= f(\ldots, z_{i+1}, z_i, \ldots) \\
    s_0 f(z_1, \ldots) &= f(z_1^{-1}, \ldots) \\
    s_N f(\ldots, z_N) &= f(\ldots, z_N^{-1})
\end{align*}
$$
The exchange equations have unique solution (up to multiplication by a function invariant under the action of $s_0, s_i, s_N$)

Under specialization $z_i = 1$, the vector $\Psi_{N,m}(1)$ becomes proportional to the stationary measure

$$\Psi_{N,m}(1) \propto P_{N,m}$$
Stationary measure: exchange/reflection equations

Proposition

- The exchange equations have unique solution (up to multiplication by a function invariant under the action of $s_0, s_i, s_N$)
- Under specialization $z_i = 1$, the vector $\Psi_{N,m}(1)$ becomes proportional to the stationary measure

$$\Psi_{N,m}(1) \propto \mathcal{P}_{N,m}$$
Exchange/reflection equations in components

\[ \psi_{\ldots, \bullet, \bullet, \ldots}(z) = s_i \psi_{\ldots, \bullet, \bullet, \ldots}(z) \]

\[ \psi_{\ast, \ldots}(z) = s_0 \psi_{\ast, \ldots}(z) \]

\[ \psi_{\ldots, \ast}(z) = s_N \psi_{\ldots, \ast}(z) \]

\[ \psi_{\ldots, \bullet, \bullet, \ldots}(z) = t_1^{\frac{1}{2}} \hat{T}_i \psi_{\ldots, \bullet, \bullet, \ldots}(z) \]

\[ \psi_{\ast, \ldots}(z) = t_0^{\frac{1}{2}} \hat{T}_0 \psi_{\ast, \ldots}(z) \]

\[ \psi_{\ldots, \bullet}(z) = t_N^{\frac{1}{2}} \hat{T}_N \psi_{\ldots, \bullet}(z) \]
Affine Hecke Again: Noumi representation

Noumi introduced a representation of $\hat{C}_N$ depending on 6 parameters $a, b, c, d, t, q$, acting on $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$

$$\hat{T}_i = t^{\frac{1}{2}} - (t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} z_{i+1}) \partial_i$$

$$\hat{T}_0 = t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}} \frac{(z_1 - a)(z_1 - b)}{z_1} \partial_0$$

$$\hat{T}_N = t_N^{\frac{1}{2}} - t_N^{-\frac{1}{2}} \frac{(c z_N - 1)(d z_N - 1)}{z_N} \partial_N,$$

where $t_0 = -q^{-1}ab$ $t_N = -cd$ and

$$\partial_i = \frac{1 - s_i}{z_i - z_{i+1}}, \quad \partial_0 = \frac{1 - s_0}{z_1 - q z_1^{-1}}, \quad \partial_N = \frac{1 - s_N}{z_N - z_N^{-1}}.$$ 

Where $s_i, s_N$ are as before but

$$s_0 f(z_1, \ldots) = f(q z_1^{-1}, \ldots)$$

For ASEP $q=1$
Affine Hecke Again: Noumi representation

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$$\widehat{T}_i = t^{\frac{1}{2}} - \left( t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} z_{i+1} \right) \partial_i$$  \hspace{1cm} (1)

$$\widehat{T}_0 = t_0^{\frac{1}{2}} - t_0^{-\frac{1}{2}} \frac{(z_1 - a)(z_1 - b)}{z_1} \partial_0$$  \hspace{1cm} (2)

$$\widehat{T}_N = t_N^{\frac{1}{2}} - t_N^{-\frac{1}{2}} \frac{(cz_N - 1)(dz_N - 1)}{z_N} \partial_N,$$  \hspace{1cm} (3)

where $t_0 = -q^{-1}ab$ $t_N = -cd$ and

$$\partial_i = \frac{1 - s_i}{z_i - z_{i+1}}, \quad \partial_0 = \frac{1 - s_0}{z_1 - qz_1^{-1}}, \quad \partial_N = \frac{1 - s_N}{z_N - z_N^{-1}}.$$

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For ASEP $q=1$

$$s_0 f(z_1, \ldots) = f(qz_1^{-1}, \ldots)$$
Non-symmetric Koornwinder [Noumi, Sahi, Stokman, . . .]

The commutative subalgebra \( \mathcal{Y}_N \) generated by elements \( Y_1^{\pm 1}, \ldots, Y_N^{\pm 1} \) [Lusztig]

\[
Y_i = (T_i \ldots T_{N-1})(T_N \ldots T_0)(T_1^{-1} \ldots T_i^{-1}).
\]

Its common eigenfunctions are the \textit{nonsymmetric Koornwinder polynomials} \( E_\alpha(z) \) (\( \alpha \in \mathbb{Z}^N \))

\[
E_\alpha(z) = z^\alpha + \sum_{z^\beta \prec z^\alpha} c_\beta z^\beta,
\]

\[
\hat{Y}_i E_\alpha(z) = \omega_i(\alpha) E_\alpha(z).
\]

By using the exchange/reflection relations it is easy to show that

\[
\psi_{m \ldots 0 \ldots m}(z) = E_{-1 \ldots -1 0 \ldots 0}(z)
\]

(Actually \( E_{-1 \ldots -1 0 \ldots 0}(z) \) doesn’t depend on \( q \)).
Non-symmetric Koornwinder [Noumi, Sahi, Stokman, ...]

The commutative subalgebra $\mathcal{Y}_N$ generated by elements $Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}$ [Lusztig]

$$Y_i = (T_i \ldots T_{N-1})(T_N \ldots T_0)(T_1^{-1} \ldots T_{i-1}^{-1}).$$

Its common eigenfunctions are the nonsymmetric Koornwinder polynomials $E_\alpha(z)$ ($\alpha \in \mathbb{Z}^N$)

$$E_\alpha(z) = z^\alpha + \sum_{z^\beta \prec z^\alpha} c_\beta z^\beta,$$

$$\hat{Y}_i E_\alpha(z) = \omega_i(\alpha) E_\alpha(z).$$

By using the exchange/reflection relations it is easy to show that

$$\psi_{\underbrace{\circ \ldots \circ}_{N-m}} \ast \underbrace{\ast \ldots \ast}_{m} (z) = E_{\underbrace{-1 \ldots -1}_{N-m}} \underbrace{0 \ldots 0}_{m}(z)$$

(Actually $E_{\underbrace{-1 \ldots -1}_{N-m}} \underbrace{0 \ldots 0}_{m}(z)$ doesn’t depend on $q$).
Symmetric Macdonald-Koornwinder polynomials

[Koornwinder]

Koornwinder q-difference operator

\[ D_{q,t} = \sum_{i=1}^{N} \Phi_i(z_i)(T_{q,z_i} - 1) + \Phi_i(z_i^{-1})(T_{q,z_i}^{-1} - 1) \]

where \( T_{q,z_i} \) is the \( i \)-th q-shift operator

\[ T_{q,z_i} f(z_1, \ldots, z_i, \ldots, z_N) = f(z_1, \ldots, qz_i, \ldots, z_N) \]

and

\[ \Phi_i(z) = \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)} \prod_{j=1}^{N} \frac{(1 - tzz_j)(1 - tzz_j^{-1})}{(1 - zz_j)(1 - zz_j^{-1})} \]
Symmetric Macdonald-Koornwinder polynomials

The symmetric Macdonald-Koornwinder polynomials $P_\lambda(z)$

- Laurent polynomials in $N$ variables
- Labeled by a partition $\lambda$, coefficient of $z^\lambda$ in $P_\lambda(z)$ is 1
- Eigenfunctions of $\mathcal{D}_{q,t}$

\[
\mathcal{D}_{q,t} P_\lambda(z) = d_\lambda P_\lambda(z)
\]

\[
d_\lambda = \sum_{i=1}^{N} \left[ q^{-1} abcd t^{2n-i-1} (q^\lambda_i - 1) + t^{i-1} (q^{-\lambda_i} - 1) \right]
\]

- They are multivariate generalization of the Askey-Wilson polynomials

\[
P_{\{m\}}(z) \propto p_m(x; a, b, c, d|q).
\]
Symmetric Macdonald-Koornwinder polynomials

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$$D_{q,t}P_{\lambda}(z) = d_{\lambda}P_{\lambda}(z)$$

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The symmetric Macdonald-Koornwinder polynomials $P_{\lambda}(z)$

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- They are multivariate generalization of the Askey-Wilson polynomials

$$P_{\{m\}}(z) \propto p_m(x; a, b, c, d|q).$$
Weighted partition function

Let \( \bullet(w) \) be the number of first class particles in configuration \( w \). We define the **weighted partition function** as

\[
Z_N(\xi; z; a, b, c, d) := \sum_{w \in Q(N,m)} \xi^{2\bullet(w)} \psi_w(z).
\]

**Theorem**

The **weighted partition function** is given by

\[
Z_{N,m}(\xi; z; a, b, c, d) = \xi^{N-m} P_{1N-m0m}(\xi z | \xi z; a_\xi, b_\xi, c_\xi, d_\xi)
\]

with \( a_\xi = \xi a, b_\xi = \xi b, c_\xi = \xi^{-1} c, d_\xi = \xi^{-1} d \).
Current and density

- Using certain boundary recursion relations, we obtain the current

\[ \langle J_{N,m} \rangle = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) \frac{Z_{N-1,m}(1)}{Z_{N,m}(1)}. \]

- For the density of first class particles we have

\[ \langle \rho_{N,m}^\bullet \rangle = \left( \frac{1}{2N} \frac{\partial}{\partial \xi} \log Z_{N,m}(\xi; 1) \right) \bigg|_{\xi=1}, \]

- In order to determine their asymptotic behavior \( N \to \infty, \rho^\bullet = m/N \), we use an integral representation of the Macdonald-Koornwinder polynomials.
Current and density

- Using certain boundary recursion relations, we obtain the current
  \[
  \langle J_{N,m} \rangle = \left( t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) \frac{Z_{N-1,m}(1)}{Z_{N,m}(1)}.
  \]

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Current and density

- Using certain boundary recursion relations, we obtain the current

\[ \langle J_{N,m} \rangle = (t^{1/2} - t^{-1/2}) \frac{Z_{N-1,m}(1)}{Z_{N,m}(1)}. \]

- For the density of first class particles we have

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- In order to determine their asymptotic behavior \( N \to \infty, \rho_* = m/N \), we use an integral representation of the Macdonald-Koornwinder polynomials.
Integral representation

Using the Cauchy identity for Macdonald-Koornwinder polynomials [Mimachi] and assuming $t < 1$, we can write

$$Z_{N,m}(\xi; \mathbf{z}) = r_m^{-1}(a_\xi, b_\xi, c_\xi, d_\xi | t) \times$$

$$\xi^{N-m} \oint_C \frac{dx}{4\pi ix} \Pi(\xi, x) w(x; a_\xi, b_\xi, c_\xi, d_\xi | t) p_m(x; a_\xi, b_\xi, c_\xi, d_\xi | t)$$

$p_m(x; a, b, c, d | t)$ is the $m$-th Askey-Wilson polynomial of base $t$ in the variable $\frac{x+x^{-1}}{2}$,

$$w(x; a, b, c, d | t) = \frac{(x^2, x^{-2}; t)_\infty}{(ax, ax^{-1}, bx, bx^{-1}, cx, cx^{-1}, dx, dx^{-1}; t)_\infty},$$

$$\Pi(\mathbf{z}, x) = \prod_{1 \leq i \leq N} (z_i + z_i^{-1} - x - x^{-1})$$

$$r_m(a, b, c, d | t) = \frac{(abcdt^{2m}; t)_\infty}{(t^{m+1}, abt^m, act^m, adt^m, bct^m, bdt^m, cdt^m; t)_\infty}.$$
Using the Cauchy identity for Macdonald-Koornwinder polynomials [Mimachi] and assuming $t < 1$, we can write

$$Z_{N,m}(\xi; z) = r_m^{-1}(a\xi, b\xi, c\xi, d\xi | t) \times$$

$$\xi^{N-m} \oint_C \frac{dx}{4\pi i x} \prod(\xi z, x) w(x; a\xi, b\xi, c\xi, d\xi | t) \rho_m(x; a\xi, b\xi, c\xi, d\xi | t)$$

**Remarks**

- This formula at $z = 1$ generalizes the result of Uchiyama, Sasamoto and Wadati obtained for $m = 0$.

- At $z = 1$ improves a much more complicated formula obtained by Uchiyama.

- Comparison with results of Corteel, Mandelshtam and Williams: combinatorial representations of $P_\lambda$ in terms of Rhombic Staircase Tableaux?
Phase diagram

In terms of the parameter $x_0 = -\frac{1+\rho_*}{1-\rho_*}$, we find three regions

\begin{align*}
x_0 < a, c : & \quad J = \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{4}(1 - \rho_*^2), \quad \rho^* = \frac{1-\rho_*}{2} \\
a < x_0, c : & \quad J = \frac{a(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}{(1-a)^2}, \quad \rho^* = \frac{a}{a-1} - \rho_* > \frac{1-\rho_*}{2} \\
c < x_0, a : & \quad J = \frac{c(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}{(1-c)^2}, \quad \rho^* = \frac{1}{1-c} < \frac{1-\rho_*}{2}
\end{align*}
Conclusion

- The approach to the study of the stationary measure through exchange relations can be applied to any Yang-Baxter integrable stochastic process.
- Relations with multivariate orthogonal polynomials.
- Generic Macdonald-Koornwinder polynomials at $q = 1$ appear in the multispecies generalization of the open ASEP \([\text{LC, Garbali, de Gier, Wheeler}]\)
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