Representations of quantum toroidal algebras —Recent topics—

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Joint with Feigin, Miwa, Mukhin

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Brief history

- Let $\mathfrak{g}$ be a complex simple Lie algebra. The affinization of $\mathfrak{g}$ is a Kac-Moody Lie algebra, also realized as a one-dimensional central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t^\pm]$. The toroidal Lie algebra is a further affinization, realized as a two-dimensional central extension of the double loop algebra $\mathfrak{g} \otimes \mathbb{C}[s^\pm, t^\pm]$.

- **Quantum toroidal algebras** are a quantization of toroidal Lie algebras. They were first introduced by geometric methods in

  V. Ginzburg, M. Kapranov and E. Vasserot,

The case $\mathfrak{g} = \mathfrak{gl}_n$ is special, in that one can introduce two deformation parameters in the algebra. From now on we consider only the $\mathfrak{gl}_n$ case.
• Some early studies include the Schur-Weyl duality


and the construction of Fock modules


• Basic theory on the structure and representations has been established by a series of works by Miki (algebraic method à la Chari-Pressley and Beck)

Among others, the $\mathfrak{gl}_1$ toroidal algebra is particularly interesting for its simplicity and extra symmetry. Originally introduced by Miki, also rediscovered by several authors, in connection with:
Macdonald functions and Virasoro/W-algebras,


double affine Hecke algebra (DAHA),

O. Schiffmann, arXiv:1004.2575,

and others.
Very little has been known about concrete representations of quantum toroidal algebras (the only example: Fock representation). Our goal here is to explain an elementary/explicit construction of a large family of modules.

This talk is based on papers by B.Feigin, T.Miwa, E.Mukhin and MJ:

- Representations of quantum toroidal \( \mathfrak{gl}_N \), arXiv:1204.5378, and earlier ones by B.Feigin, E.Feigin, T.Miwa, E.Mukhin and MJ:
Plan of this talk

We shall mainly focus on the $\mathfrak{gl}_1$ case.

1. Basic definitions and known facts
2. Macmahon modules
3. Construction via Fock modules
4. Macmahon modules with boundary conditions
5. Resonances
6. $\mathfrak{gl}_n$ case
Quantum $\mathfrak{gl}_1$ toroidal algebra $\mathcal{E}$

The algebra $\mathcal{E}$ is defined from the following data.

parameters: $q_1, q_2, q_3 \in \mathbb{C}^\times$, $q_1 q_2 q_3 = 1$

generators: $E_k, F_k, H_r, K_0^\pm, C^{1/2}$ ($k \in \mathbb{Z}, r \in \mathbb{Z}\{0\}$)

We set

$$E(z) = \sum_{k \in \mathbb{Z}} E_k z^{-k}, \quad F(z) = \sum_{k \in \mathbb{Z}} F_k z^{-k},$$

$$K^\pm(z) = K_0^\pm \exp(\pm(q_2^{1/2} - q_2^{-1/2}) \sum_{\pm r > 0} H_r z^{-r}),$$

$$g(z, w) = (z - q_1 w)(z - q_2 w)(z - q_3 w),$$

$$\bar{g}(z, w) = (q_1 z - w)(q_2 z - w)(q_3 z - w).$$
Relations:

\[ K_0^\pm, C^{1/2} \text{: central invertible,} \]

\[ K^\pm(z)K^\pm(w) = K^\pm(w)K^\pm(z), \]

\[ g(Cz, w)\bar{g}(z, Cw)K^+(z)K^-(w) = g(z, Cw)\bar{g}(Cz, w)K^-(w)K^+(z), \]

\[ g(z, w)K^\pm(C^{\mp1/2}z)E(w) = \bar{g}(z, w)E(w)K^\pm(C^{\mp1/2}z), \]

\[ \bar{g}(z, w)K^\pm(C^{\pm1/2}z)F(w) = g(z, w)F(w)K^\pm(C^{\pm1/2}z), \]

\[ [E(z), F(w)] = \frac{1}{g(1, 1)}(\delta(Cw/z)K^+(C^{1/2}w) - \delta(Cz/w)K^-(C^{1/2}z)), \]

\[ g(z, w)E(z)E(w) = \bar{g}(z, w)E(w)E(z), \]

\[ \bar{g}(z, w)F(z)F(w) = g(z, w)F(w)F(z), \]

\[ [E_0, [E_1, E_{-1}]] = [F_0, [F_1, F_{-1}]] = 0. \]
Remark

1. Various names:
   - $(q, \gamma)$-analog of $\mathcal{W}_{1+\infty}$ (Miki),
   - Ding-Iohara algebra (Feigin, Shiraishi, · · · ),
   - elliptic Hall algebra (Schiffmann)

2. $\mathcal{E}$ is symmetric in $q_1, q_2, q_3$ (not true for $\mathcal{E}_n, n \geq 2$)

3. $\mathcal{E}$ is isomorphic to the spherical DAHA (Schiffmann 2010)
There exists an order 4 automorphism $\psi$ of $\mathcal{E}$ such that

\begin{align*}
H_1 &\mapsto c_1 E_0, \quad H_{-1} \mapsto c_2 F_0, \\
E_0 &\mapsto -c_1^{-1} H_{-1}, \quad F_0 \mapsto -c_2^{-1} H_1, \\
C &\mapsto K_0, \quad K_0 \mapsto C^{-1}
\end{align*}
• Hereafter we assume that $q_i$’s are generic, i.e.,

$$q_1^{j_1} q_2^{j_2} q_3^{j_3} = 1 \text{ only if } j_1 = j_2 = j_3.$$

• An $E$-module $V$ is of level $(x, y)$ if $C$ acts as scalar $x$ and $(K_0^+)^{-1} K_0^-$ as $y$.

• In the rest of this talk we consider modules of level $(1, y)$, i.e. $C = 1$. Then the operators $K^\pm(z)$ are mutually commutative.
Lowest weight modules

An $\mathcal{E}$-module $V$ is called **lowest weight** if $V = \mathcal{E} v$ with

$$F(z)v = 0, \quad K^\pm(z)v = \phi^\pm(z)v, \quad Cv = v,$$

for some formal power series $\phi^\pm(z)$ in $z^{\mp 1}$. $(\phi^+(z), \phi^-(z))$ is called the lowest weight of $V$.

Assign degrees by

$$\deg E_k = 1, \quad \deg F_k = -1, \quad \deg H_r = 0.$$

A graded $\mathcal{E}$-module $V = \bigoplus_{k \in \mathbb{Z}} V_k$ is called **quasi-finite** if

$$\dim V_k < \infty \quad (\forall k).$$
Classification [Miki 2007]

1) An irreducible lowest weight module $M$ is in one-to-one correspondence with its lowest weight $(\phi^+(z), \phi^-(z))$.

2) $M$ is quasi-finite if and only if $\phi^\pm(z)$ are expansions of a common rational function $\phi(z)$ in $z^{\mp1}$.
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So, given a rational function $\phi(z)$ which is regular and non-zero at $z = 0, \infty$, there corresponds a unique irreducible lowest weight $\mathcal{E}$-module $M$. From the definition, however, it is hard to tell how it looks like. For instance, what is the character

$$\chi(M) = \sum_k q^k \dim M_k.$$
Macmahon modules

A Macmahon module $M(u, K)$ is the irreducible lowest weight module whose lowest weight is

$$\phi(z) = \frac{1 - Ku/z}{1 - u/z} \quad (u, K \in \mathbb{C}).$$

We shall construct $M(u, K)$ explicitly.
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Main features:

1. $M(u, K)$ has a basis parametrized by all plane partitions,
2. $K^\pm(z)$ act diagonally on them with simple joint spectrum,
3. $E(z)$ adds a box and $F(z)$ removes a box, with explicitly given matrix coefficients.
A plane partition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \cdots)$ is a sequence of ordinary partitions (Young diagrams) satisfying

$$\lambda^{(1)} \supset \lambda^{(2)} \supset \lambda^{(3)} \supset \cdots, \quad \lambda^{(N)} = \emptyset \ (N \gg 0).$$

Denote the set of all plane partitions by $\pi$. We visualize $\lambda \in \pi$ as a 3-dimensional Young diagram $Y_\lambda$, where the $k$-th Young diagram $\lambda^{(k)}$ is “sitting on the $k$-th floor”.
3D Young diagram $Y_\lambda$

$\lambda = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$
Let $M(u, K)$ be a vector space with basis $|\lambda\rangle$ ($\lambda \in \pi$). There is an action of $\mathcal{E}$ on $M(u, K)$ of the form

$$K^{\pm}(z)|\lambda\rangle = \psi_{\lambda}(u/z)|\lambda\rangle,$$

$$(1 - q_1)E(z)|\lambda\rangle = \sum_{(i, j, k) : \text{concave}} \psi_{\lambda, i, j, k} \psi_{\lambda(k), i} \delta(q_3^i q_1^j q_2^k)|\lambda + 1^{(k)}\rangle,$$

$$(q_1^{-1} - 1)F(z)|\lambda\rangle = \sum_{(i, j, k) : \text{convex}} \psi'_{\lambda, i, j, k} \psi'_{\lambda(k), i} \delta(q_3^i q_1^j q_2^k)|\lambda - 1^{(k)}\rangle,$$

All coefficients have factorized form, e.g.,

$$\psi_{\lambda}(u/z) = \frac{1 - Ku/z}{1 - u/z} \prod_{(i, j, k) \in Y_{\lambda}} h(q_3^i q_1^j q_2^k u/z),$$

$$h(z) = \frac{(1 - q_1^{-1}z)(1 - q_2^{-1}z)(1 - q_3^{-1}z)}{(1 - q_1 z)(1 - q_2 z)(1 - q_3 z)}.$$
For generic $K$, $M(u, K)$ is an irreducible, tame, lowest weight $\mathcal{E}$-module of lowest weight $\frac{1 - Ku/z}{1 - u/z}$ and level $(1, K)$. 

By the Macmahon formula for plane partitions, we have $
\chi(M(u, K)) = \prod_{i=1}^{\infty} (1 - q^i)^i.
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Idea of construction

Fock module for $\mathfrak{gl}_\infty$ is constructed as a semi-infinite wedge

$$\mathcal{F}_{\mathfrak{gl}_\infty} = V \wedge V \wedge V \wedge \cdots , \quad V = \mathbb{C}^\infty$$
Idea of construction

Fock module for \( \mathfrak{gl}_\infty \) is constructed as a semi-infinite wedge

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\]

We construct the Macmahon module analogously, following two-step semi-infinite constructions

\[
V(u) \Longrightarrow \mathcal{F}(u) \Longrightarrow M(u, K)
\]

The main points are the Drinfeld coproduct and the structures of zeroes of the matrix coefficients.
Vector representation

Let \( V(u) = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}[u]_i \).

The following formula defines an \( \mathcal{E} \)-module structure on \( V(u) \):

\[
(1 - q_1) E(z)[u]_i = \delta(q_1^i u/z)[u]_{i+1},
\]

\[
(q_1^{-1} - 1) F(z)[u]_i = \delta(q_1^{-1} u/z)[u]_{i-1},
\]

\[
K^{\pm}(z)[u]_i = \psi(q_1^i u/z)[u]_i,
\]

where

\[
\psi(z) = \frac{(1 - q_2 z)(1 - q_3 z)}{(1 - z)(1 - q_2 q_3 z)}.
\]
Drinfeld coproduct

Algebra $\mathcal{E}$ has the following formal coproduct ($C = 1$, for simplicity)

\[
\Delta E(z) = E(z) \otimes 1 + K^-(z) \otimes E(z),
\]
\[
\Delta F(z) = F(z) \otimes K^+(z) + 1 \otimes F(z),
\]
\[
\Delta K^\pm(z) = K^\pm(z) \otimes K^\pm(z),
\]

The right hand side is an infinite series, so it is not defined in the usual sense.
Example. In $V(u) \otimes V(v)$,

$$(1 - q_1)E(z)[u]_i \otimes [v]_j = \delta(q_1^i u/z)[u]_{i+1} \otimes [v]_j$$

$$+ \frac{1 - q_2 q_1^{i-j} u/v}{1 - q_1^{i-j} u/v} \frac{1 - q_3 q_1^{i-j} u/v}{1 - q_2 q_3 q_1^{i-j} u/v} \times \delta(q_1^j v/z)[u]_i \otimes [v]_{j+1}$$

For generic $u, v$, $V(u) \otimes V(v)$ is an well-defined irreducible $\mathcal{E}$-module. (For special values of $u/v$, poles can appear.)
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Take $v = q_2^{-1} u$. Then

- Poles do not occur,
- The subspace

\[ \text{span}\{[u]_i \otimes [uq_2^{-1}]_j \mid i > j \} \subset V(u) \otimes V(q_2^{-1} u) \]

is invariant under $E(z)$ because of the zero $1 - q_3 q_1^{i-j} u/v = 0$. 
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$$1 - q_3 q_1^{i-j} u/v = 0 .$$

Easy to check that this subspace is a well-defined $E$-module. (It can be thought of as an analog of $\wedge^2 V(u)$.)
Consider the subspace of the infinite tensor product
\[ \mathcal{F}(u) \subset V(u) \otimes V(uq_2^{-1}) \otimes V(uq_2^{-2}) \otimes \cdots \]
\[ \mathcal{F}(u) = \text{span}\{ |\lambda\rangle | \lambda_1 \geq \lambda_2 \geq \cdots, \lambda_i \in \mathbb{Z}, \lambda_i = 0 (i \gg 0) \} \]
where
\[ |\lambda\rangle = [u]_{\lambda_1} \otimes [uq_2^{-1}]_{\lambda_2 - 1} \otimes [uq_2^{-2}]_{\lambda_3 - 2} \otimes \cdots. \]
The action of $E(z)$ is well defined on $\mathcal{F}(u)$. 
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The action of $E(z)$ is well defined on $\mathcal{F}(u)$. The actions of $F(z), K^\pm(z)$ need interpretation as they involve a formal infinite product. We define a modified action as follows.
Given a partition $\lambda$, take $N$ large enough and consider a vector
\[ |\lambda\rangle^{(N)} = [u]_{\lambda_1} \otimes [uq_2^{-1}]_{\lambda_2-1} \otimes \cdots \otimes [uq_2^{-N+2}]_{-N+2} \otimes [uq_2^{-N+1}]_{-N+1} \]
in the finite tensor product
\[ \mathcal{F}^{(N)}(u) = V(u) \otimes V(uq_2^{-1}) \otimes V(uq_2^{-2}) \otimes \cdots \otimes V(uq_2^{-N+1}). \]

Then the modified action
\[ F^{new}(z)|\lambda\rangle^{(N)} = \beta_N(z) \cdot F(z)|\lambda\rangle^{(N)}, \]
\[ K^{\pm,new}(z)|\lambda\rangle^{(N)} = \beta_N(z) \cdot K^{\pm}(z)|\lambda\rangle^{(N)}, \]
\[ \beta_N(z) = \frac{1 - q_3^N u/z}{1 - q_2^{-1} q_3^N u/z} \]
is compatible with the embedding $\mathcal{F}^{(N)}(u) \hookrightarrow \mathcal{F}^{(N+1)}(u)$.
With the above definition, $\mathcal{F}(u)$ is a well-defined $E$-module. $\mathcal{F}(u)$ is an irreducible, tame, lowest weight module of lowest weight $\frac{1 - q_2 u/z}{1 - u/z}$ and level $(1, q_2)$.

All matrix coefficients are factorized. e.g.

$$\langle \lambda | K^\pm(z) | \lambda \rangle = \prod_{(i, j): \text{concave}} \frac{1 - q_3 q_1 q_2^2 u/z}{1 - q_3 q_1 q_2 u/z} \prod_{(i, j): \text{convex}} \frac{1 - q_3 q_1 q_2 u/z}{1 - q_3 q_1 q_2^2 u/z},$$

In this form, $\mathcal{F}(u)$ was found by Feigin-Tsymbaliuk with a geometric method.
One can repeat the same construction, using $\mathcal{F}(u)$ in place of $V(u)$. We use

Lemma: Let $a, b \in \mathbb{Z}_{\geq 0}$, and let

$$v/u = q_1^{-a}q_2q_3^{-b}.$$ 

Then the subspace

$$\text{span}\{|\lambda\rangle \otimes |\mu\rangle \mid \lambda_i + a \geq \mu_{i+b} \ (\forall i)\} \subset \mathcal{F}(u) \otimes \mathcal{F}(v)$$

is an $\mathcal{E}$-submodule.
Choosing $a = b = 0$ in the Lemma, consider the subspace of an infinite tensor product of Fock modules

$$M(u, K) \subset \mathcal{F}(u) \otimes \mathcal{F}(u q_2) \otimes \mathcal{F}(u q_2^2) \otimes \cdots$$

$$M(u, K) = \text{span}\{ |\lambda\rangle \mid \lambda \in \pi \}$$

where for $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots)$ we set

$$|\lambda\rangle = |\lambda^{(1)}\rangle \otimes |\lambda^{(2)}\rangle \otimes \cdots \otimes |\emptyset\rangle \otimes \cdots .$$

As before, $E(z)$ has a well-defined action on $M(u, K)$. For each $\lambda \in \pi$, the modification

$$F^{new}(z)|\lambda\rangle^{(N)} = \frac{1 - Ku/z}{1 - u/z} \cdot F(z)|\lambda\rangle^{(N)} ,$$

$$K^{\pm, new}(z)|\lambda\rangle^{(N)} = \frac{1 - Ku/z}{1 - u/z} \cdot K^{\pm}(z)|\lambda\rangle^{(N)} ,$$

leads to the action of $F(z), K(z)$ on $M(u, K)$. 
Macmahon modules with boundary conditions

More generally, let $\alpha, \beta, \gamma$ be given partitions. Let $M_{\alpha, \beta, \gamma}(u, K)$ be the vector space spanned by infinite plane partitions $\mu$ such that

$$\mu^{(k)}_i = \alpha_k \quad (i \gg 0),$$
$$\mu^{(k)}_i = \gamma_i \quad (k \gg 0),$$
$$\mu^{(k)}_i = \infty \quad \text{if } 1 \leq i \leq \beta_k.$$

Let $\omega$ be the minimal plane partition with this boundary condition.

There is an action of $\mathcal{E}$ on $M_{\alpha, \beta, \gamma}(u, K)$. It is an irreducible, tame, lowest weight module of level $(1, K)$ and lowest vector $|\omega\rangle$. 
Macmahon module with boundary \((\alpha, \beta, \gamma)\).
The minimal diagram \(Y_\omega\) is shown.
Lowest weights of $M_{\alpha, \beta, \gamma}(u, K)$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>Lowest weight</th>
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<tbody>
<tr>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\frac{1 - Ku/z}{1 - u/z}$</td>
</tr>
<tr>
<td>${1}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\frac{(1 - Ku/z)(1 - q_1 q_2 u/z)}{(1 - q_1 u/z)(1 - q_2 u/z)}$</td>
</tr>
<tr>
<td>${2}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
<td>$\frac{(1 - q_1^2 u/z)(1 - q_2 u/z)}{(1 - Ku/z)(1 - q_1 q_2 q_3 u/z)}$</td>
</tr>
<tr>
<td>${1}$</td>
<td>${1}$</td>
<td>$\emptyset$</td>
<td>$\frac{(1 - q_2 u/z)(1 - q_1 q_3 u/z)}{(1 - Ku/z)(1 - q_1 q_2 q_3 u/z)^2}$</td>
</tr>
<tr>
<td>${1}$</td>
<td>${1}$</td>
<td>${1}$</td>
<td>$\frac{(1 - q_1 q_2 u/z)(1 - q_1 q_3 u/z)(1 - q_2 q_3 u/z)}{(1 - q_1 q_2 u/z)(1 - q_1 q_3 u/z)(1 - q_2 q_3 u/z)}$</td>
</tr>
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Resonances

When the parameters $K$, $u$ or $q_1, q_2, q_3$ take special values, $M_{\alpha,\beta,\gamma}(u, K)$ become reducible. We call this a resonant case.

We consider only the case where $q_1, q_2, q_3$ are generic but $K$ is special:

$$K = q_2^m q_3^n \quad (m, n \in \mathbb{Z}_{\geq 0}),$$

In this case, the action of $F(z)$ cannot remove the boxes at $(a + t, b + t, c + t) \ (y \in \mathbb{Z}_{\geq 0})$, where

$$(a, b, c) \in Y_\omega, \quad (a - 1, b - 1, c - 1) \notin Y_\omega.$$

Correspondingly there is a series of submodules

$$M_{\alpha,\beta,\gamma}(u, K) = M_{\alpha,\beta,\gamma}^{m,n,0}(u) \supset M_{\alpha,\beta,\gamma}^{m,n,1}(u) \supset M_{\alpha,\beta,\gamma}^{m,n,2}(u) \supset \cdots$$
Let

\[ N_{\alpha,\beta,\gamma}^{m,n}(u) := M_{\alpha,\beta,\gamma}^{m,n,0}(u) / M_{\alpha,\beta,\gamma}^{m,n,1}(u) \]

be the first irreducible quotient.

For example, if \( K = q_2 \) then \( M_{\emptyset,\emptyset,\emptyset}^{1,0,1}(u) \) contain all diagrams containing the box \((0,0,1)\). Hence

\[ N_{\emptyset,\emptyset,\emptyset}^{1,0}(u) = \mathcal{F}(u). \]
Characters of $\mathcal{N}^{n,n}_{\alpha,\emptyset,\emptyset}(u)$

In general, it is a challenge to determine the character of $\mathcal{N}^{m,n}_{\alpha,\beta,\gamma}(u)$. The following result is a special case.

\[ \chi_k = \bar{\chi}_k, \quad \chi_{-k} = q^k \bar{\chi}_k \quad (k \in \mathbb{Z}_{\geq 0}), \]
\[ \bar{\chi}_k = \frac{1}{(q)_2} \sum_{j=0}^{\infty} (-1)^j q^{j(j+1)/2+jk}. \]

\[ \chi(\mathcal{N}^{n,n}_{\alpha,\emptyset,\emptyset}) = \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} \prod_{i=1}^{n} \chi(\sigma(\alpha+\rho)-\rho)_i. \]
Almost all constructions for the $\mathfrak{gl}_1$-quantum toroidal algebra carry over to the case of $\mathfrak{gl}_n$. Minor differences are:

- Algebra is symmetric in $q_3, q_1$ but $q_2$ plays a different role.
- Boxes in (plane) partitions are colored $((i, j, k)$ has color $i - j \mod n)$,
- Macmahon modules with boundary conditions can be defined only when partition $\gamma$ is colorless.
Summary

We have obtained a new family of modules of quantum toroidal algebras. The actions of generators are explicit and the joint spectrum of the $K(z)$'s is simple. This allows a combinatorial study of these modules.

Open questions:
1. Characters
2. Resonances in $K$ (e.g. $K = q_1 q_2 q_3$)
3. Resonances in $q_1, q_2, q_3$
4. Are there integrable models which have $E$ as its symmetry?
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4. *Are there integrable models which have $\mathcal{E}$ as its symmetry?*
THANK YOU VERY MUCH FOR YOUR ATTENTION