Correlation functions of the integrable spin-s XXZ spin chains and some related topics

Tetsuo Deguchi

Ochanomizu University, Tokyo, Japan

June 15, 2010

[1,2] T.D. and Chihiro Matsui, NPB 814[FS](2009)405 ;831[FS](2010)359

Partially in collaboration with Jun Sato

(For SCP model and the $sl_2$ loop algebra symmetry)

\[\text{RAQIS10, LAPTH, Annecy, France. The talk is given on June 15, 2010.}\]
Introduction

Contents

- 0: Integrable spin-$s$ XXZ spin chains through fusion method

- Part I:
  (1): Quantum Inverse Scattering Problem for the spin-$s$ XXZ spin chain (Several tricks such as Hermitian elementary matrices)
  (2): Multiple-integral representation of arbitrary correlation functions of the spin-$s$ XXZ spin chain

- Part II:
  Quantum Inverse Scattering Problem (QISP) for the super-integrable chiral Potts model (SCP model)
**Key idea:**

We construct the spin-\(s\) representation \(V^{(2s)}\) of \(U_q(sl_2)\) in the 2\(s\)th tensor product of spin-1/2 representations \(V^{(1)}\):

\[
V^{(2s)} \subset V_1^{(1)} \otimes \cdots \otimes V_{2s}^{(1)}
\]

\[
|2s, n\rangle = \left(\Delta^{(2s-1)}(X^-)\right)^n |0\rangle_1 \otimes \cdots \otimes |0\rangle_{2s}
\]

We define **dual vectors** by anti-involution \(\ast\) of \(U_q(sl_2)\):

\[
\langle 2s, n| = (|2s, n\rangle)^* \]

If \(q\) is complex, they are different from Hermitian conjugate vectors:

\[
\langle 2s, n| = (|2s, n\rangle)^\dagger \]

We thus introduce as **Hermitian operators** \(\hat{E}^{m,n(2s)} = |2s, m\rangle \langle 2s, n|\)
PART 0: Review: Exact methods for deriving the multiple-integral representation of XXZ CFs

- The $q$-vertex operator approach (infinite system, no external field, zero temperature)
  M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki,

- An algebraic approach solving the $q$ KZ equation
  M. Jimbo and T. Miwa,

- Algebraic Bethe-ansatz approach (finite system, external fields)
  N. Kitanine, J.M. Maillet and V. Terras,
We define the \( R \)-matrix and the monodromy matrix \( T_{0,12\ldots L}(\lambda; \{w_j\}) \) by

\[
R_{12}(\lambda_1, \lambda_2) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b(u) & c(u) & 0 \\
0 & c(u) & b(u) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
T_{0,12\ldots L}(\lambda; \{w_j\}) = R_{0L}(\lambda, w_L)R_{0L-1}(\lambda, w_{L-1}) \cdots R_{02}(\lambda, w_2)R_{01}(\lambda, w_1)
\]

The operator-valued matrix element of the monodromy matrix give the creation and annihilation operators

\[
T_{0,12\ldots L}(u; \{w_j\}) = \begin{pmatrix}
A(u) & B(u) \\
C(u) & D(u)
\end{pmatrix}
\]

The transfer matrix, \( t(u) \), is given by the trace of the monodromy matrix with respect to the 0th space:

\[
t(u; w_1, \ldots, w_L) = \text{tr}_0 (T_{0,12\ldots L}(u; \{w_j\}))
\]

\[
= A(u; \{w_j\}) + D(u; \{w_j\})
\]
The Yang-Baxter equations: Essence of the integrability

Figure: The Yang-Baxter equations:
\[ R_{2,3}(v)R_{1,3}(u+v)R_{1,2}(u) = R_{1,2}(u)R_{1,3}(u+v)R_{2,3}(v) \]

Spectral parameter \( u \) is expressed by the angle between lines 1 and 2, where the intersection corresponds to \( R_{1,2}(u) \).
Review: algebraic BA derivation of the multiple-integral representation of the spin-1/2 XXZ correlation functions

- Quantum Inverse Scattering Problem (QISP)
  Local spin-1/2 operators expressed by $A, B, C, D$ (spin-1/2)

- Scalar product of the BA:

  If $\{\mu_j\}$ or $\{\lambda_j\}$ are Bethe roots, we have

  $$\langle 0 | C(\mu_1) \cdots C(\mu_M) B(\lambda_1) \cdots B(\lambda_M) | 0 \rangle = \det \Psi' \quad \text{(Slavnov's formula)}$$

  $$\langle 0 | C(\mu_1) \cdots C(\mu_M) B(\lambda_1) \cdots B(\lambda_M) | 0 \rangle /
  \langle 0 | C(\lambda_1) \cdots C(\lambda_M) B(\lambda_1) \cdots B(\lambda_M) | 0 \rangle = \det (\Psi'/\Phi')$$

  $\Phi'$: the Gaudin matrix

- Integral equations for the matrix elements of $\Psi'/\Phi'$

  To evaluate the matrix elements of $\left( \Psi'/\Phi' \right)$ by solving the integral equations (cf. Izergin)
PART I: Higher-spin XXZ spin chains

Hamiltonians of the spin-1/2 and spin-\(s\) XXZ spin chains

The spin-1/2 XXZ chain the Hamiltonian under the P. B. C.

\[
\mathcal{H}_{\text{XXZ}} = \frac{1}{2} \sum_{j=1}^{L} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right). \tag{2}
\]

Here \(\sigma_j^a\) (\(a = X, Y, Z\)) are Pauli matrices on the \(j\)th site. We define \(q\) by

\[
\Delta = (q + q^{-1})/2 \quad (q = \exp \eta)
\]

For \(-1 < \Delta \leq 1\), \(\mathcal{H}_{\text{XXZ}}\) is gapless. (\(\Delta = \cos \zeta\) by \(q = e^{i\zeta}\), \(0 \leq \zeta < \pi\).) For \(\Delta > 1\) or \(\Delta < -1\), it is gapful. (\(\Delta = \pm \cosh \zeta\) by \(q = e^{-\zeta}\), \(0 < \zeta\).)

Integrable spin-\(s\) XXZ Hamiltonian \(\mathcal{H}_{\text{XXZ}}^{(2s)}\) is given by the logarithmic derivative of the spin-\(s\) XXZ transfer matrix. (We express it by qCGC.)

\[
s = 1 \text{ XXX case} \quad \mathcal{H}_{\text{XXX}}^{(2)} = J \sum_{j=1}^{N_s} \left( \vec{S}_j \cdot \vec{S}_{j+1} - (\vec{S}_j \cdot \vec{S}_{j+1})^2 \right). \tag{3}
\]

Hereafter we shall often set \(\ell = 2s\).
References on the spin-\(s\) XXX or XXZ correlation functions

Relevant algebraic Bethe-ansatz studies on the spin-\(s\) XXX CFs


Relevant studies on the higher-spin XXZ and XYZ chains


Fusion construction

First trick: **Applying the $R$-matrix $R^+$ in the homogeneous grading to the fusion construction**

Through a similarity transformation we transform $R$ to $R^+$

$$R_{12}^+(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c^-(u) & 0 \\ 0 & c^+(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4)$$

$$c^\pm(u) = e^{\pm \sinh \eta/\sinh(u + \eta)} \quad b(u) = \sinh u/\sinh(u + \eta)$$

The $R^+$ gives the intertwiner of the affine quantum group $U_q(\widehat{sl_2})$ in the homogeneous grading:

$$R_{12}^+(u) (\Delta(a))_{12} = (\tau \circ \Delta(a))_{12} R_{12}^+(u) \quad a \in U_q(sl_2)$$

where $\tau$ denotes the permutation operator: $\tau(a \otimes b) = b \otimes a$ for $a, b \in U_q(sl_2)$, and $(\Delta(a))_{12} = \pi_1 \otimes \pi_2 (\Delta(a))$. 
Projection operators and the fusion construction

We define permutation operator $\Pi_{12}$ by

$$\Pi_{12}v_1 \otimes v_2 = v_2 \otimes v_1,$$  \hspace{1cm} (5)

and then define $\tilde{R}$ by

$$\tilde{R}_{12}^+(u) = \Pi_{12}R_{12}^+$$  \hspace{1cm} (6)

We define spin-1 projection operator by

$$P_{12}^{(2)} = \tilde{R}_{12}^+(u = \eta)$$  \hspace{1cm} (7)

We define spin $- \ell/2$ projection operator recursively as follows.

$$P_{12\ldots\ell}^{(\ell)} = P_{12\ldots\ell-1}^{(\ell-1)} \tilde{R}_{\ell-1,\ell}^+ ((\ell - 1)\eta) P_{12\ldots\ell-1}^{(\ell-1)}$$  \hspace{1cm} (8)
Figure: Projector $P^{(2)}$ is given by the special value of $R$-matrix $\tilde{R}(\eta)$, and hence it commutes with other $R$-matrices. (Cf. Kulish, Sklyanin, and Reshetikhin (1981)).
We define monodromy matrix $T_{0}^{(1,2s)}(\lambda_0; \xi_1, \ldots, \xi_{N_s})$ acting on the tensor product $V^{(1)}(\lambda_0) \otimes (V^{(2s)}(\xi_1) \otimes \cdots \otimes V^{(2s)}(\xi_{N_s}))$ as follows.

$$T_{0}^{(1,2s)}(\lambda_0; \xi_1, \ldots, \xi_{N_s}) = P_{12\ldots L}^{(2s)} \cdot R_{0,12\ldots L}^{+}(\lambda_0; w_{1}^{(2s)}, \ldots, w_{L}^{(2s)}) \cdot P_{12\ldots L}^{(2s)}.$$

Here inhomogenous parameters $w_{j}$ are given by complete $2s$-strings

$$w_{2s(p-1)+k}^{(2s)} = \xi_p - (k - 1)\eta \quad (p = 1, 2, \ldots, N_s; k = 1, \ldots, 2s).$$

More precisely, we shall put them as almost complete $2s$-strings

$$w_{2s(p-1)+k}^{(2s; \epsilon)} = \xi_p - (k - 1)\eta + \epsilon r_k \quad (p = 1, 2, \ldots, N_s; k = 1, \ldots, 2s).$$

We express the matrix elements of the monodromy matrix as follows.

$$T_{0,12\ldots N_s}^{(1,2s)}(\lambda; \{\xi_k\}_{N_s}) = \left( \begin{array}{cc}
A^{(2s)}(\lambda; \{\xi_k\}_{N_s}) & B^{(2s)}(\lambda; \{\xi_k\}_{N_s}) \\
C^{(2s)}(\lambda; \{\xi_k\}_{N_s}) & D^{(2s)}(\lambda; \{\xi_k\}_{N_s})
\end{array} \right).$$

$$A^{(2s)}(\lambda; \{\xi_k\}_{N_s}) = P_{12\ldots L}^{(2s)} \cdot A^{(1)}(\lambda; \{w_{j}^{(2s)}\}_{L}) \cdot P_{12\ldots L}^{(2s)}.$$
Here quantum spaces $V_j^{(2s)}(\xi_j)$ are $(2s + 1)$-dimensional (vertical lines). Variables $a_j$ and $b_j$ take values $0, 1, \ldots, 2s$ while the auxiliary space $V_0^{(\ell)}(\lambda_0)$ is $(\ell + 1)$-dimensional (horizontal line). Variables $c_j$ take values $0, 1, \ldots, \ell$.

$$L = 2sN_s$$

Spin-1/2 chain of $L$-sites with inhomogeneous parameters $w_1, \ldots, w_L$, while the spin-$s$ chain of $N_s$ sites with $\xi_1, \ldots, \xi_{N_s}$. 

Figure: Matrix element of the monodromy matrix $(T_{\alpha,\beta}^{(\ell,2s)})_{a_1,\ldots,a_{Ns}}^{b_1,\ldots,b_{Ns}}$. 

$$c_1 = \alpha \begin{array}{c|c|c|c|c|c} a_1 & a_2 & \cdots & a_{Ns} \\ c_2 & c_{Ns} + 1 = \beta \\ b_1 & b_2 & \cdots & b_{Ns} \end{array}$$
We now define $T^{(\ell, 2s)}_0 (\lambda_0; \xi_1, \ldots, \xi_{N_s})$ acting on the tensor product $V^{(\ell)}_0 (\lambda_0) \otimes (V^{(2s)}(\xi_1) \otimes \cdots \otimes V^{(2s)}(\xi_{N_s}))$ as follows.

$$T^{(\ell, 2s)}_{0, 12\cdots N_s} = P^{(\ell)}_{a_1 a_2 \cdots a_\ell} T^{(1, 2s)}_{a_1, 12\cdots N_s} (\lambda_{a_1}) T^{(1, 2s)}_{a_2, 12\cdots N_s} (\lambda_{a_1} - \eta) \cdots T^{(1, 2s)}_{a_\ell, 12\cdots N_s} (\lambda_{a_1} - (\ell - 1)\eta) P^{(\ell)}_{a_1 a_2 \cdots a_\ell}.$$
AB Derivation of the multiple-integral representation for the integrable spin-$s$ XXZ correlation functions

- □ Quantum Inverse Scattering Problem (QISP)
  Spin-$s$ local operators expressed in terms of $A, B, C, D$ (spin-1/2)

- □ Scalar product:
  If $\{\lambda_j\}$ are Bethe roots, we have the determinant expression:

  \[
  \lim_{\epsilon \to 0} \langle 0 | C^{(2s)}(\lambda_1) \cdots C^{(2s)}(\lambda_M) B^{(2s)}(\lambda_1) \cdots B^{(2s)}(\lambda_M) | 0 \rangle = \det \Psi^{(2s)}
  \]

  \[
  \frac{\langle 0 | C^{(2s)}(\mu_1) \cdots C^{(2s)}(\mu_M) B^{(2s)}(\lambda_1) \cdots B^{(2s)}(\lambda_M) | 0 \rangle}{\langle 0 | C^{(2s)}(\lambda_1) \cdots C^{(2s)}(\lambda_M) B^{(2s)}(\lambda_1) \cdots B^{(2s)}(\lambda_M) | 0 \rangle} = \det \left( \Psi^{(2s)} / \Phi^{(2s)} \right)
  \]

- □ Integral equations for the spin-$s$
  To evaluate the matrix elements of $\left( \Psi^{(2s)} / \Phi^{(2s)} \right)$ by solving the integral equations
Scheme of exact derivation of spin-$s$ XXZ correlation functions

(1) Algebraic part:
Quantum Inverse Scattering Problem (QISP) Formula, Slavnov’s scalar product formula, Algebraic Bethe ansatz, Fusion method, projectors, Hermitian elementary matrices

(2) Physical part:
Fundamental conjecture: In the region $0 \leq \zeta < \pi/2s$, the spin-$s$ XXZ ground state is given by a Bethe-ansatz eigenvector of $2s$-strings with small deviations


The low-lying excitation spectrum of the spin-$s$ XXZ chain should be characterized by the level $k$ SU(2) WZWN model with $k = 2s$.

Spin-$s$ Gaudin’s matrix, spin-$s$ integral equations
The quantum algebra $U_q(sl_2)$ is an associative algebra over $\mathbb{C}$ generated by $X^\pm, K^\pm$ with the following relations:

$$
KK^{-1} = KK^{-1} = 1, \quad KX^\pm K^{-1} = q^{\pm 2}X^\pm, \quad [X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}. \quad (11)
$$

The algebra $U_q(sl_2)$ is also a Hopf algebra over $\mathbb{C}$ with comultiplication

$$
\Delta(X^+) = X^+ \otimes 1 + K \otimes X^+, \quad \Delta(X^-) = X^- \otimes K^{-1} + 1 \otimes X^-, \quad \Delta(K) = K \otimes K, \quad (12)
$$

and antipode: $S(K) = K^{-1}, S(X^+) = -K^{-1}X^+, S(X^-) = -X^-K,$

and coproduct: $\epsilon(X^\pm) = 0$ and $\epsilon(K) = 1.$
\[ [n]_q = (q^n - q^{-n})/(q - q^{-1}) \]: the \( q \)-integer of an integer \( n \).

\[ [n]_q! \]: the \( q \)-factorial for an integer \( n \).

\[ [n]_q! = [n]_q [n - 1]_q \cdots [1]_q . \]  \hspace{1cm} (13)

For integers \( m \geq n \geq 0 \), the \( q \)-binomial coefficient is defined by

\[ \left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[m - n]_q! [n]_q!} \] \hspace{1cm} (14)

We define \( ||\ell,0\rangle \) for \( n = 0,1,\ldots,\ell \) by

\[ ||\ell,0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_\ell . \] \hspace{1cm} (15)

Here \( |\alpha\rangle_j \) for \( \alpha = 0,1 \) denote the basis vectors of the spin-1/2 rep. We define \( ||\ell,n\rangle \) for \( n \geq 1 \) and evaluate them as follows.

\[ ||\ell,n\rangle = \left( \Delta^{(\ell-1)}(X^-) \right)^n ||\ell,0\rangle \frac{1}{[n]_q!} \]

\[ = \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma_{i_1}^- \cdots \sigma_{i_n}^- |0\rangle q^{i_1 + i_2 + \cdots + i_n - n\ell + n(n-1)/2} . \] \hspace{1cm} (16)
We have conjugate vectors \( \langle \ell, n|| \) explicitly as follows.

\[
\langle \ell, n|| = \left[ \begin{array}{c} \ell \\ n \end{array} \right]^{-1}_q q^{n(\ell-n)} \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \langle 0| \sigma_{i_1}^+ \cdots \sigma_{i_n}^+ q^{i_1+\cdots+i_n-n\ell+n(n-1)/2} \rangle.
\]

(17)

Here the normalization conditions: \( \langle \ell, n|| || \ell, n \rangle = 1 \).

Conjugate vectors are given by \( * \) anti-involution \( \langle \ell, n|| = (|| \ell, n \rangle)^* \) where

\[
(X^-)^* = q^{-1} X^+ K^{-1}, \quad (X^+)^* = q^{-1} X^- K, \quad K^* = K
\]

with \( (\Delta(a))^* = \Delta(a^*), \quad (ab)^* = b^* a^*, \quad (a)^{**} = a, \) for \( a \in U_q(sl(2)) \).

In the massive regime where \( q = \exp \eta \) with real \( \eta \), the conjugate vectors \( \langle \ell, n|| \) are Hermitian conjugate to vectors \( || \ell, n \rangle \).

However, in the massless regime \( |q| = 1 \) and \( q \neq \pm 1 \), they are not.
For an integer $\ell \geq 0$ we define $\langle \ell, n\|\|$ for $n = 0, 1, \ldots, n$, by
\[
\langle \ell, n\| = \left( \begin{array}{c} \ell \\ n \end{array} \right)^{-1} \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \langle 0|\sigma^+_{i_1}\cdots\sigma^+_{i_n} q^{-(i_1+\cdots+i_n)+n\ell-n(n-1)/2}. \tag{18}
\]
They are conjugate to $\|\ell, n\rangle$: $\langle \ell, m\|\|\ell, n\rangle = \delta_{m,n}$ . Here we have denoted the binomial coefficients as follows.
\[
\left( \begin{array}{c} \ell \\ n \end{array} \right) = \frac{\ell!}{(\ell-n)!n!}. \tag{19}
\]
Setting $\langle \ell, n\|\|\ell, n\rangle = 1$, vectors $\|\ell, n\rangle$ are given by
\[
\|\ell, n\rangle = \sum_{1 \leq i_1 < \cdots < i_n \leq \ell} \sigma^-_{i_1}\cdots\sigma^-_{i_n} |0\rangle q^{-(i_1+\cdots+i_n)+n\ell-n(n-1)/2}
\times \left[ \begin{array}{c} \ell \\ n \end{array} \right]_q q^{-n(\ell-n)} \left( \begin{array}{c} \ell \\ n \end{array} \right)^{-1}. \tag{20}
\]
Hermitian elementary matrices

In the massless regime we define new elementary matrices $\widetilde{E}^{m,n}_{\ell}$ by

$$ \widetilde{E}^{m,n}_{\ell} = \langle 2s, m | \langle 2s, n | $$

for $m, n = 0, 1, \ldots, 2s$.  \hfill (21)

We define projection operators by

$$ P_{12\ldots\ell}^{(\ell)} = \sum_{n=0}^{\ell} \langle \ell, n | \langle \ell, n | . $$ \hfill (22)

Let us now introduce another set of projection operators $\widetilde{P}_{1\ldots\ell}^{(\ell)}$ as follows.

$$ \widetilde{P}_{1\ldots\ell}^{(\ell)} = \sum_{n=0}^{\ell} \langle \ell, n | \langle \ell, n | . $$ \hfill (23)
Projector $\tilde{P}^{(\ell)}_{1 \ldots \ell}$ is idempotent: $(\tilde{P}^{(\ell)}_{1 \ldots \ell})^2 = \tilde{P}^{(\ell)}_{1 \ldots \ell}$. In the massless regime where $|q| = 1$, it is Hermitian: $\left(\tilde{P}^{(\ell)}_{1 \ldots \ell}\right)^\dagger = \tilde{P}^{(\ell)}_{1 \ldots \ell}$. From the definition, we show the following properties:

\begin{align}
       P^{(\ell)}_{12 \ldots \ell} \tilde{P}^{(\ell)}_{1 \ldots \ell} &= P^{(\ell)}_{12 \ldots \ell}, \quad (24) \\
\tilde{P}^{(\ell)}_{1 \ldots \ell} P^{(\ell)}_{12 \ldots \ell} &= \tilde{P}^{(\ell)}_{1 \ldots \ell}. \quad (25)
\end{align}

In the tensor product of quantum spaces, $V^{(2s)}_1 \otimes \cdots \otimes V^{(2s)}_{N_s}$, we define $\tilde{P}^{(2s)}_{12 \ldots L}$ by

\begin{equation}
\tilde{P}^{(2s)}_{12 \ldots L} = \prod_{i=1}^{N_s} \tilde{P}^{(2s)}_{2s(i-1)+1}. \quad (26)
\end{equation}

Here we recall $L = 2sN_s$. 
Spin-$s$ XXZ Hamiltonian expressed by the $q$-Clebsch-Gordan coefficients

$$
\mathcal{H}^{(2s)}_{\text{XXZ}} = \frac{d}{d\lambda} \log \tilde{t}_{12...N_s}^{(2s,2s+)}(\lambda) \Big|_{\lambda=0, \xi_j=0} = \sum_{i=1}^{N_s} \frac{d}{du} \tilde{R}_{i,i+1}^{(2s,2s)}(u) \Big|_{u=0}
$$

where $\tilde{t}(u) = P_{12...L}^{(2s)} t(u)$. Here, the elements of the $R$-matrix for $V(l_1) \otimes V(l_2)$ are given by (cf. [T.D. and K. Motegi])

$$
\tilde{R} |l_1, a_1\rangle \otimes |l_2, a_2\rangle = \sum_{b_1, b_2} \tilde{R}_{a_1, a_2}^{b_1, b_2} |l_1, b_1\rangle \otimes |l_2, b_2\rangle,
$$

$$
\tilde{R}_{a_1, a_2}^{b_1, b_2} = \delta_{a_1+a_2, b_1+b_2} N(l_1, a_1) N(l_2, a_2) \sum_{j=0}^{\min(l_1,l_2)} N(l_1 + l_2 - 2j, a_1 + a_2)^{-1}
$$

\times \rho_{l_1+l_2-2j} \begin{bmatrix} l_2 & l_1 & l_1+l_2-2j \\ b_1 & b_2 & a_1+a_2 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_1+l_2-2j \\ a_1 & a_2 & a_1+a_2 \end{bmatrix}
$$
For $m \geq n$ we have

$$
\tilde{E}_{i}^{m, n}(\ell+) = \left( \begin{array}{c} \ell \\
\end{array} \right) \left[ \begin{array}{c} \ell \\
m \\
\end{array} \right]_{q} \left[ \begin{array}{c} \ell \\
n \\
\end{array} \right]_{q}^{-1} \tilde{P}_{1 \ldots L}^{(i-1)\ell} \prod_{\alpha=1}^{(i-1)\ell} (A + D)(w_{\alpha})
$$

$$
\times \prod_{k=1}^{n} D(w_{(i-1)\ell+k}) \prod_{k=n+1}^{m} B(w_{(i-1)2s+k}) \prod_{k=m+1}^{L} A(w_{(i-1)\ell+k})
$$

$$
\times \prod_{\alpha=i\ell+1}^{\ell N_{s}} (A + D)(w_{\alpha}) \tilde{P}_{1 \ldots L}^{(\ell)}. \quad (27)
$$

For $m \leq n$ we have a similar formula.
We express a factor of \((2s + 1) \times (2s + 1)\) elementary matrices in terms of a \(2s\)th product of \(2 \times 2\) elementary matrices with entries \(\{\epsilon_j, \epsilon'_j\}\) as follows.

\[
\tilde{E}^{i_b, j_b (2s +)} = C(\{i_k, j_k\}) \tilde{P}_{12...L}^{(2s)} \cdot \prod_{k=1}^{2s} e_{k}^{\epsilon'_k, \epsilon_k} \cdot \tilde{P}_{12...L}^{(2s)}.
\] (28)

Here, \(C(\{i_k, j_k\})\) is given by

\[
C(\{i_b, j_b\}) = \begin{pmatrix} 2s \\ j_b \end{pmatrix} q \begin{pmatrix} 2s \\ i_b \end{pmatrix}^{-1} q \begin{pmatrix} 2s \\ j_b \end{pmatrix}^{-1} \cdot
\] (29)

Here \(\epsilon_\beta\) and \(\epsilon'_\beta\) (\(\beta = 1, \ldots, 2s\)) are given by

\[
\epsilon_{2s(b-1)+\beta} = \begin{cases} 1 & (1 \leq \beta \leq j_b) \\ 0 & (j_b < \beta \leq 2s) \end{cases}; \quad \epsilon'_{2s(b-1)+\beta} = \begin{cases} 1 & (1 \leq \beta \leq i_b) \\ 0 & (i_b < \beta \leq 2s) \end{cases}.
\] (30)
The fundamental conjecture of the spin-$s$ ground state
The spin-$s$ ground state $|\psi_g^{(2s)}\rangle$ is given by $N_s/2$ sets of $2s$-strings for the region: $0 \leq \zeta < \pi/2s$

$$\lambda^{(\alpha)}_a = \mu_a - (\alpha - 1/2)\eta + \epsilon^{(\alpha)}_a, \quad \text{for } a = 1, 2, \ldots, N_s/2 \text{ and } \alpha = 1, 2, \ldots, 2s.$$  

Deviations are given by $\epsilon^{(\alpha)}_a = \sqrt{-1}\delta^{(\alpha)}_a$ where $\delta^{(\alpha)}_a$ are real and decreasing w.r.t. $\alpha$, and $|\delta^{(\alpha)}_a| > |\delta^{(\alpha+1)}_a|$ for $\alpha < s$, $|\delta^{(\alpha)}_a| < |\delta^{(\alpha+1)}_a|$ for $\alpha > s$.

It is shown analytically through the finite-size corrections of the spin-1 XXZ chain (A. Klümper, M. Batchelor and P.A. Pearce, J. Phys. A: 24 (1991)).
The density of string centers, $\rho_{\text{tot}}(\mu)$, is given by

$$\rho_{\text{tot}}(\mu) = \frac{1}{N_s} \sum_{p=1}^{N_s} \frac{1}{2\zeta \cosh(\pi(\mu - \xi_p)/\zeta)}$$  \hspace{1cm} (31)

For the homogeneous chain where $\xi_p = 0$ for $p = 1, 2, \ldots, N_s$, we denote the density of string centers by $\rho(\lambda)$.

$$\rho(\lambda) = \frac{1}{2\zeta \cosh(\pi \lambda/\zeta)}$$  \hspace{1cm} (32)
We define a spin-$s$ correlation function by

$$F^{(2s+)}(\{\epsilon_j, \epsilon'_j\}) = \frac{\langle \psi_{g}^{(2s+)} | \prod_{i=1}^{m} \tilde{E}_{i}^{m_i, n_i}^{(2s+)} | \psi_{g}^{(2s+)} \rangle}{\langle \psi_{g}^{(2s+)} | \psi_{g}^{(2s+)} \rangle} \quad (33)$$

Applying the formulas, we reduce it to the following:

$$F^{(2s+)}(\{\epsilon_j, \epsilon'_j\}) = \frac{\langle \psi_{g}^{(2s+)} | \prod_{j=1}^{2sm} e_{j}^{\epsilon_j, \epsilon'_j} | \psi_{g}^{(2s+)} \rangle}{\langle \psi_{g}^{(2s+)} | \psi_{g}^{(2s+)} \rangle} \quad (34)$$

Here $\epsilon_j, \epsilon'_j = 0, 1$. We send inhomogeneous parameters $w_j$ to a set of complete strings.

$$w_{2s(b-1)+\beta} = w_{2s(b-1)+\beta}^{(2s;\epsilon)} \rightarrow w_{2s(b-1)+\beta}^{(2s)} = \xi_b - (\beta - 1)\eta. \quad (35)$$

Let us define $\alpha^{-}$ and $\alpha^{+}$ by

$$\alpha^{-} = \{j; \epsilon_j = 0\}, \quad \alpha^{+} = \{j; \epsilon'_j = 1\}. \quad (36)$$
For sets $\alpha^-$ and $\alpha^+$ we define $\tilde{\lambda}_j$ for $j \in \alpha^-$ and $\tilde{\lambda}'_j$ for $j \in \alpha^+$, respectively, by the following relation:

$$(\tilde{\lambda}'_{j_{\text{max}}}, \ldots, \tilde{\lambda}'_{j_{\text{min}}}, \tilde{\lambda}_{j_{\text{min}}}, \ldots, \tilde{\lambda}_{j_{\text{max}}}) = (\lambda_1, \ldots, \lambda_{2sm}).$$

(37)

We have

$$F^{(2s+)}(\{\epsilon_j, \epsilon'_j\}) =$$

$$= \left( \int_{\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty+i(2s-1)\xi+i\epsilon}^{\infty+i(2s-1)\xi+i\epsilon} \right) d\lambda_1$$

$$\cdots \left( \int_{\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty+i(2s-1)\xi+i\epsilon}^{\infty+i(2s-1)\xi+i\epsilon} \right) d\lambda_{s'}$$

$$\left( \int_{-\infty-i\epsilon}^{-\infty-i(2s-1)\xi-i\epsilon} + \cdots + \int_{-\infty-i\epsilon}^{-\infty-i(2s-1)\xi-i\epsilon} \right) d\lambda_{s'+1}$$

$$\cdots \left( \int_{-\infty-i\epsilon}^{-\infty-i(2s-1)\xi-i\epsilon} + \cdots + \int_{-\infty-i\epsilon}^{-\infty-i(2s-1)\xi-i\epsilon} \right) d\lambda_m$$

$$\times Q(\{\epsilon_j, \epsilon'_j\}; \lambda_1, \ldots, \lambda_{2sm}) \det S(\lambda_1, \ldots, \lambda_{2sm})$$

(38)
Here factor \( Q \) is given by
\[
Q(\{\epsilon_j, \epsilon'_j\}) = (-1)^{\alpha +} \prod_{j \in \alpha^-} \left( \prod_{k=1}^{j-1} \varphi(\tilde{\lambda}_j - w_k^{(2s)}) + \eta \right) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}_j - w_k^{(2s)}) \prod_{1 \leq k < \ell \leq 2sm} \varphi(\lambda_\ell - \lambda_k + \eta + \epsilon_{\ell,k})
\prod_{j \in \alpha^+} \left( \prod_{k=1}^{j-1} \varphi(\tilde{\lambda}'_j - w_k^{(2s)}) - \eta \right) \prod_{k=j+1}^{2sm} \varphi(\tilde{\lambda}'_j - w_k^{(2s)}) \prod_{1 \leq k < \ell \leq 2sm} \varphi(w_k^{(2s)} - w_\ell^{(2s)})
\]  
(39)

The matrix elements of \( S \) are given by
\[
S_{j,k} = \rho(\lambda_j - w_k^{(2s)} + \eta/2) \delta(\alpha(\lambda_j), \beta(k)), \quad \text{for} \quad j, k = 1, 2, \ldots, 2sm.
\]  
(40)

Here \( \delta(\alpha, \beta) \) denotes the Kronecker delta and \( \alpha(\lambda_j) \) are given by \( a \) if \( \lambda_j = \mu_j - (a - 1/2)\eta \) \( (1 \leq a \leq 2s) \), where \( \mu_j \) correspond to centers of complete \( 2s \)-strings. 

In the denominator, we have set \( \epsilon_{k,l} \) associated with \( \lambda_k \) and \( \lambda_l \) as follows.
\[
\epsilon_{k,l} = \begin{cases} 
  i\epsilon & \text{for } \Im(\lambda_k - \lambda_l) > 0 \\
  -i\epsilon & \text{for } \Im(\lambda_k - \lambda_l) < 0.
\end{cases}
\]  
(41)
Examples of multiple integrals

For \( s = 1 \) and \( m = 1 \) (\( w_1^{(2)} = \xi_1, w_2^{(2)} = \xi_1 - \eta \)), we have

\[
\langle \widetilde{E}_1^{11}(2+) \rangle = \langle \psi_g^{(2+)} | \widetilde{E}_1^{11}(2+) | \psi_g^{(2+)} \rangle / \langle \psi_g^{(2+)} | \psi_g^{(2+)} \rangle \\
= 2 \langle \psi_g^{(2+)} | A(w_1^{(2)}) D(w_2^{(2)}) | \psi_g^{(2+)} \rangle / \langle \psi_g^{(2+)} | \psi_g^{(2+)} \rangle \\
= 2 \left( \int_{-\infty+\im \epsilon}^{\infty+\im \epsilon} + \int_{-\infty-\im \zeta+\im \epsilon}^{\infty-\im \zeta+\im \epsilon} \right) d\lambda_1 \left( \int_{-\infty-\im \epsilon}^{\infty-\im \epsilon} + \int_{-\infty-\im \zeta-\im \epsilon}^{\infty-\im \zeta-\im \epsilon} \right) d\lambda_2 \\
\times Q(\lambda_1, \lambda_2) \det S(\lambda_1, \lambda_2)
\] (42)

where \( Q(\lambda_1, \lambda_2) \) is given by

\[
Q(\lambda_1, \lambda_2) = (-1) \frac{\varphi(\lambda_2 - w_2^{(2)}) \varphi(\lambda_1 - w_1^{(2)} - \eta)}{\varphi(\lambda_2 - \lambda_1 + \eta + \epsilon_{2,1}) \varphi(\eta)}
\] (43)

and matrix \( S(\lambda_1, \lambda_2) \) is given by

\[
\begin{pmatrix}
\rho(\lambda_1 - w_1^{(2)} + \eta/2)\delta(\alpha(\lambda_1), 1) & \rho(\lambda_1 - w_2^{(2)} + \eta/2)\delta(\alpha(\lambda_1), 2) \\
\rho(\lambda_2 - w_1^{(2)} + \eta/2)\delta(\alpha(\lambda_2), 1) & \rho(\lambda_2 - w_2^{(2)} + \eta/2)\delta(\alpha(\lambda_2), 2)
\end{pmatrix}
\] (44)
Here we have applied the following formula

$$\tilde{E}_{1}^{1,1}(2+) = 2 \tilde{P}_{1\ldots L}^{(2)} D^{(1+)}(w_{1}) A^{(1+)}(w_{2}) \prod_{\alpha=3}^{2N_{s}} (A^{(1+)} + D^{(1+)})(w_{\alpha}) \tilde{P}_{1\ldots L}^{(2)}.$$ 

The correlation function is expressed in terms of a single product of the multiple-integral representation.

By evaluating the double integral, the integral over $\lambda_{1}$ is decomposed as follows.

$$\left( \int_{-\infty + i\epsilon}^{\infty + i\epsilon} + \int_{-\infty - i\zeta + i\epsilon}^{\infty - i\zeta + i\epsilon} \right) d\lambda_{1} Q(\lambda_{1}, \lambda_{2}) \det S(\lambda_{1}, \lambda_{2})$$

$$= \left( \int_{-\infty - i\zeta/2}^{\infty - i\zeta/2} + \int_{-\infty - i3\zeta/2}^{\infty - i3\zeta/2} \right) d\lambda_{1} Q(\lambda_{1}, \lambda_{2}) \det S(\lambda_{1}, \lambda_{2})$$

$$+ \oint_{\Gamma_{1}} d\lambda_{1} Q(\lambda_{1}, \lambda_{2}) \det S(\lambda_{1}, \lambda_{2}) + \oint_{\Gamma_{2}} d\lambda_{1} Q(\lambda_{1}, \lambda_{2}) \det S(\lambda_{1}, \lambda_{2}).$$
Thus, the integral is calculated as

\[
\langle \tilde{\psi}_g^{(2+)} | \tilde{E}_1^{11} (2+) | \tilde{\psi}_g^{(2+)} \rangle / \left( 2 \langle \tilde{\psi}_g^{(2+)} | \tilde{\psi}_g^{(2+)} \rangle \right) = -2\pi i \int_{-\infty}^{\infty} \frac{\sinh(x - \eta/2) \sinh(x - 3\eta/2)}{\sinh \eta} \rho^2(x) \, dx \\
+ 2 \cosh \eta \int_{-\infty}^{\infty} \frac{\sinh(x - \eta/2)}{\sinh(x + \eta/2)} \rho(x) \, dx \\
- \int_{-\infty}^{\infty} \rho(x) \, dx + (-1)^2 \cosh \eta \int_{-\infty}^{\infty} \frac{\sinh(x - \eta/2)}{\sinh(x + \eta/2)} \rho(x) \, dx
\]

\[
= \frac{\cos \zeta (\sin \zeta - \zeta \cos \zeta)}{2\zeta \sin^2 \zeta}. \quad (45)
\]

We thus obtain

\[
\langle \tilde{E}_1^{1,1} \rangle = \frac{\cos \zeta (\sin \zeta - \zeta \cos \zeta)}{\zeta \sin^2 \zeta}. \quad (46)
\]
Evaluating the integral we obtain the spin-1 EFP with $m = 1$ as follows.

$$\tau^{(2)}(1) = \langle \tilde{E}_1^{2,2} \rangle = \frac{\zeta - \sin \zeta \cos \zeta}{2\zeta \sin^2 \zeta}.$$  \hspace{1cm} (47)

In the XXX limit, we have

$$\lim_{\zeta \to 0} \frac{\zeta - \sin \zeta \cos \zeta}{2\zeta \sin^2 \zeta} = \frac{1}{3}.$$  \hspace{1cm} (48)

The limiting value $1/3$ coincides with the spin-1 XXX result [Kitanine (2001)]. As pointed out in Kitanine (2001), $\langle E_1^{22} \rangle = \langle E_1^{11} \rangle = \langle E_1^{00} \rangle = 1/3$ for the XXX case since it has the rotational symmetry.

Furthermore, due to the uniaxial symmetry we have $\langle \tilde{E}_1^{0,0} \rangle = \langle \tilde{E}_1^{2,2} \rangle$, and hence, directly evaluating the integrals, we confirm

$$\langle \tilde{E}_1^{0,0} \rangle + \langle \tilde{E}_1^{1,1} \rangle + \langle \tilde{E}_1^{2,2} \rangle = 1.$$  \hspace{1cm} (49)
Symmetric expression of the multiple integrals of $F^{(2s+)}(\{\epsilon_j, \epsilon'_j\})$

$$\frac{1}{\prod_{1 \leq \alpha < \beta \leq 2s} \sinh^m(\beta - \alpha)\eta} \prod_{1 \leq k < l \leq m} \frac{\sinh^{2s}(\pi(\xi_k - \xi_l)/\zeta)}{\prod_{j=1}^{2s} \prod_{r=1}^{2s} \sinh(\xi_k - \xi_l + (r - j)\eta)}$$

$$\sum_{\sigma \in S_{2sm}/(S_m)^{2s}} \prod_{j=1}^{\alpha_+} \left( \int_{-\infty+i\epsilon}^{\infty+i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta+i\epsilon}^{\infty-i(2s-1)\zeta+i\epsilon} \right) d\mu_{\sigma j}$$

$$\frac{2sm}{\prod_{j=\alpha_++1}^{2sm}} \left( \int_{-\infty-i\epsilon}^{\infty-i\epsilon} + \cdots + \int_{-\infty-i(2s-1)\zeta-i\epsilon}^{\infty-i(2s-1)\zeta-i\epsilon} \right) d\mu_{\sigma j}$$

$$\times (\text{sgn } \sigma) \left( \prod_{j=1}^{2sm} \prod_{b=1}^{m} \frac{\sinh(\lambda_j - \xi_b + \beta\eta)}{\prod_{b=1}^{m} \cosh(\pi(\mu_j - \xi_b)/\zeta)} \right)$$

$$\times \frac{i^{2sm^2}}{(2i\zeta)^{2sm}} \prod_{\gamma=1}^{2s} \prod_{1 \leq b < a \leq m} \sinh(\pi(\mu_{2s}(a-1)+\gamma - \mu_{2s}(b-1)+\gamma)/\zeta)$$

$$Q(\{\epsilon_j, \epsilon'_j\}; \lambda_{\sigma 1}, \ldots, \lambda_{\sigma (2sm)}) \cdot$$

It is straightforward to take the homogeneous limit: $\xi_k \to 0$. 
Part II: $sl(2)$ loop algebra symmetry and the super-integrable chiral Potts model

\[ \mathcal{H}_{XXZ} = \frac{1}{2} \sum_{j=1}^{L} \left( \sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z \right). \]

When $q$ is a root of unity: $q^{2N} = 1$ where $\Delta = (q + q^{-1})/2$, the $\mathcal{H}_{XXZ}$ (and $\tau_{6V}(z)$) commutes with the $sl_2$ loop algebra, $U(L(sl_2))$ in sector A: $S^Z \equiv 0 \, (\text{mod } N)$ (and in sector B: $S^Z \equiv N/2 \, (\text{mod } N)$):

\[ S^\pm(N) = (S^\pm)^N / [N]!, \quad T^\pm(N) = (T^\pm)^N / [N]! \]

Here $S^\pm$ and $T^\pm$ are generators of $U_q(\hat{sl}_2)$

\[ S^\pm = \sum_{j=1}^{L} q^{\sigma_j^Z/2} \otimes \cdots \otimes q^{\sigma_j^Z/2} \otimes \sigma_j^\pm \otimes q^{-\sigma_j^Z/2} \otimes \cdots \otimes q^{-\sigma_j^Z/2} \]

\[ T^\pm = \sum_{j=1}^{L} q^{-\sigma_j^Z/2} \otimes \cdots \otimes q^{-\sigma_j^Z/2} \otimes \sigma_j^\pm \otimes q^{\sigma_j^Z/2} \otimes \cdots \otimes q^{\sigma_j^Z/2} \]
$S^{\pm(N)}$ and $T^{\pm(N)}$ generate the $sl_2$ loop algebra $U(L(sl_2))$

Generators $h_k$ and $x_k^{\pm}$ ($k \in \mathbb{Z}$) satisfy the defining relations of the $sl_2$ loop algebra ($j, k \in \mathbb{Z}$)

$$[h_j, x_k^{\pm}] = \pm 2x_{j+k}^{\pm}, \quad [x_j^{+}, x_k^{-}] = h_{j+k}$$

$$[h_j, h_k] = 0, \quad [x_j^{\pm}, x_k^{\pm}] = 0$$

We have

$$x_0^{+} = S^{+(N)}, \quad x_0^{-} = S^{-(N)}, \quad (50)$$

$$x_{-1}^{+} = T^{+(N)}, \quad x_{-1}^{-} = T^{-(N)}, \quad (51)$$

$$h_0 = \frac{2}{N} S^Z. \quad (52)$$
Main Reference of PART II.


Many interesting papers relevant to this talk


Definition of D-highest weight vectors (Drinfeld-highest weight)

We call a representation of $U(L(sl_2))$ D-highest weight if it is generated by a vector $\Omega$ satisfying

(i) $\Omega$ is annihilated by generators $x^+_k$:

$$x^+_k \Omega = 0 \quad (k \in \mathbb{Z})$$

(ii) $\Omega$ is a simultaneous eigenvector of generators, $h_k$'s:

$$h_k \Omega = d_k \Omega, \quad \text{for} \quad k \in \mathbb{Z}.$$  

Here $d_k$ denotes the eigenvalues of $h_k$'s.

We call $\Omega$ a highest weight vector.
Definition (D-highest weight polynomial)

Let $\lambda_k$ be eigenvalues as follows:

$$(x_0^+)^k(x_1^-)^k/(k!)^2 \quad \Omega = \lambda_k \Omega \quad (k = 1, 2, \ldots, r)$$

For the sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, we define polynomial $P^\lambda(u)$ by

$$P^\lambda(u) = \sum_{k=0}^{r} \lambda_k (-u)^k,$$

We call $P^\lambda(u)$ the D-highest weight polynomial for highest weight $d_k$.

When $U(L(sl_2))\Omega$ is irreducible, we call it the Drinfeld polynomial of the representation.
Conjecture on highest weight vectors

Conjecture (Fabricius and McCoy)

Bethe eigenvectors are highest weight vectors of the $sl_2$ loop algebra, and they have the Drinfeld polynomials [FM].


It was proved in sector A: $S^Z \equiv 0 \pmod{N}$ when $q^{2N} = 1$ (and in sector B: $S^Z \equiv N/2 \pmod{N}$ when $q^N = 1$ with $N$ being odd):

Taking the limit of infinite rapidities, we have

\[ \hat{A}(\pm \infty) = q^{\pm S^Z}, \quad \hat{B}(\infty) = T^-, \quad \hat{B}(-\infty) = S^-, \]
\[ \hat{D}(\pm \infty) = S^Z, \quad \hat{C}(\infty) = S^+, \quad \hat{C}(-\infty) = T^+, \]  \hspace{1cm} (54)

Let \( N \) be a positive integer.

**Definition (Complete \( N \)-string)**

We call a set of rapidities \( z_j \) a complete \( N \)-string, if they satisfy

\[ z_j = \Lambda + \eta(N + 1 - 2j) \quad (j = 1, 2, \ldots, N). \]  \hspace{1cm} (55)

We call \( \Lambda \) the center of the \( N \)-string.
Sufficient conditions of a highest weight vector

Lemma

Suppose that \( x_0^\pm, x_-^1, x_1^- \) and \( h_0 \) satisfy the defining relations of \( U(L(sl_2)) \), and \( x_k^\pm \) and \( h_k \) (\( k \in \mathbb{Z} \)) are generated from them. If a vector \(|\Phi\rangle\) satisfies the following:

\[
x_0^+ |\Phi\rangle = x_-^1 |\Phi\rangle = 0, \tag{56}
\]
\[
h_0 |\Phi\rangle = r |\Phi\rangle, \tag{57}
\]
\[
(x_0^+)^{(n)}(x_1^-)^{(n)} |\Phi\rangle = \lambda_n |\Phi\rangle \quad \text{for} \ n = 1, 2, \ldots, r, \tag{58}
\]

where \( r \) is a nonnegative integer and \( \lambda_n \) are complex numbers. Then \(|\Phi\rangle\) is highest weight, i.e. we have

\[
x_k^+ |\Phi\rangle = 0 \quad (k \in \mathbb{Z}), \tag{59}
\]
\[
h_k |\Phi\rangle = d_k |\Phi\rangle \quad (k \in \mathbb{Z}), \tag{60}
\]

where \( d_k \) are complex numbers.
Diagonal property

In addition to regular Bethe roots at generic $q, t_1, t_2, \ldots, t_R$, we introduce $kN$ rapidities, $z_1, z_2, \ldots, z_{kN}$, forming a complete $kN$-string:

$z_j = \Lambda + (kN + 1 - 2j)\eta$ for $j = 1, 2, \ldots, kN$.

We calculate the action of $(S_{\xi}^{+(N)})^k(T_{\xi}^{-(N)})^k$ on the Bethe state at $q_0$,

$$|R\rangle = B(\tilde{t}_1) \cdots B(\tilde{t}_R)|0\rangle,$$

and we have explicitly shown

$$\left(S_{\xi}^{+(N)}\right)^k \left(T_{\xi}^{-(N)}\right)^k |R\rangle = \lim_{q \rightarrow q_0} \lim_{\Lambda \rightarrow \infty} \frac{1}{([N]_q!)^k} \left(\hat{C}(\infty)\right)^{kN} \times \frac{1}{([N]_q!)^k} \hat{B}(z_1, \eta) \cdots \hat{B}(z_{kN}, \eta_n) B(t_1, \eta) \cdots B(t_R, \eta)|0\rangle$$

$$= \lambda_k |R\rangle$$
Connection to SCP model

The 1D transverse Ising model

\[ \mathcal{H} = A_0 + h A_1 \]

where \( A_0 \) and \( A_1 \) are given by

\[ A_0 = \sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z, \quad A_1 = \sum_{j=1}^{L} \sigma_j^x \]

The \( A_0 \) and \( A_1 \) satisfy the Dolan-Grady conditions:

\[ [A_i, [A_i, [A_i, A_{1-i}]]] = 16[A_i, A_{1-i}], (i = 0, 1) \]

and generate the Onsager algebra (OA). [L. Onsager (1944)]

\[
\begin{align*}
[A_\ell, A_m] &= 4G_{\ell-m} \\
[G_\ell, A_m] &= 2A_{m+\ell} - 2A_{m-\ell} \\
[G_\ell, G_m] &= 0
\end{align*}
\]

The $Z_N$-symmetric Hamiltonian by von Gehlen and Rittenberg:

$$H_{Z_N} = A_0 + k'A_1$$

$$= \frac{4}{N} \sum_{i=1}^{L} \sum_{m=1}^{N-1} \frac{1}{1 - \omega^{-m}} Z_{i}^{2m} + k' \frac{4}{N} \sum_{i=1}^{L} \sum_{m=1}^{N-1} \frac{1}{1 - \omega^{-m}} X_{i}^{-m} X_{i+1}^{m}$$

Here, $q^N = 1$ and $\omega = q^2$, and the operators $Z_i, X_i \in \text{End}(\mathbb{C}^N \otimes L)$ are defined by

$$Z_i(v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i} \otimes \cdots \otimes v_{\sigma_L}) = q^{\sigma_i} v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i} \otimes \cdots \otimes v_{\sigma_L},$$

$$X_i(v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i} \otimes \cdots \otimes v_{\sigma_L}) = v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_i+1} \otimes \cdots \otimes v_{\sigma_L},$$

for the standard basis $\{v_{\sigma}| \sigma = 0, 1, \cdots, N - 1\}$ of $\mathbb{C}^N$ and under the P.B.C.s: $Z_{L+1} = Z_1$ and $X_{L+1} = X_1$.

The Hamiltonian $H_{Z_N}$ is derived from the expansion of the SCP transfer matrix with respect to the spectral parameter.
The superintegrable $\tau_2$ model

The transfer matrix of the superintegrable $\tau_2$ model commutes with that of the superintegrable chiral Potts (SCP) model.

The $L$-operator of the superintegrable $\tau_2$ model is given by that of the integrable spin-$(N - 1)/2$ XXZ spin chain with twisted boundary conditions.

We define **SCP polynomial** (superintegrable chiral Potts polynomial) by

$$P_{\text{SCP}}(z^N) = \omega^{-p_b} \sum_{j=0}^{N-1} \frac{(1 - z^N) L(z\omega^j)^{-p_a-p_b}}{(1 - z\omega^j) L F_{\text{CP}}(z\omega^j) F_{\text{CP}}(z\omega^j+1)}, \quad (61)$$

Here $F_{\text{CP}}(z) = \prod_{i=1}^{R}(1 + zu_i \omega)$ and $\{u_i\}$ satisfy the Bethe ansatz equations.
Proposition

If \( q^N = 1 \) and \( L \) is a multiple of \( N \), the transfer matrix \( \tau_{\tau_2}(z) \) has the \( sl(2) \) loop algebra symmetry in the sector with \( S^Z \equiv 0 \) \((\text{mod} \ N)\).

Proposition

The superintegrable chiral Potts (SCP) polynomial \( P_{\text{SCP}}(\zeta) \) is equivalent to the \( D \)-highest weight polynomial \( P_D(\zeta) \) in the sector \( S^Z \equiv 0 \) \((\text{mod} \ N)\).
Observations: (1) SCP and the D-highest weight polynomials have the same ‘Bethe roots’. (2) The dimensions are the same for the OA representation and the highest weight representation of the loop algebra of the $\tau_2$ model.

Conjecture

The OA representation space $V_{OA}$ should correspond to the highest weight representation of the $sl(2)$ loop algebra generated by the regular Bethe states $|R, \{z_i\}\rangle$ of $\tau_2$ model.

Corollary

The eigenvectors of the $Z_N$ symmetric Hamiltonian, $|V\rangle$ are given by

$$|V\rangle = (\text{some element of } L(sl_2)) \times B(z_1) \cdots B(z_R)|0\rangle$$

Here $B(z_1) \cdots B(z_R)|0\rangle$ denotes a regular Bethe state of the $\tau_2$ model and generates the highest weight rep. of the $sl(2)$ loop algebra.
$N = 2$ case

The Hamiltonians of the SCP model and the $\tau_2$-model are given in the forms

$$H_{SCP} = \sum_{i=1}^{L} \sigma_{i}^{z} + \lambda \sum_{i=1}^{L} \sigma_{i}^{x} \sigma_{i+1}^{x}, \quad H_{\tau_2} = \sum_{i=1}^{L} (\sigma_{i}^{x} \sigma_{i+1}^{y} - \sigma_{i}^{y} \sigma_{i+1}^{x}),$$

where $\sigma^{x}$, $\sigma^{y}$ and $\sigma^{z}$ are Pauli's matrices. In terms of fermion operators:

$$c_i = \sigma_{i}^{+} \prod_{j=1}^{i-1} \sigma_{j}^{z}, \quad \tilde{c}_{k} = \frac{1}{L} \sum_{i=1}^{L} e^{-1(ki+\frac{\pi}{4})} c_i,$$

the Hamiltonian $H_{\tau_2}$ in the sector with $S_{z} \equiv \frac{1}{2} \sum_{i=1}^{L} \sigma_{i}^{z} \equiv 0 \mod 2$ is written as

$$H_{\tau_2} = \sum_{k \in K} \sin(k) \tilde{c}_{k}^{\dagger} \tilde{c}_{k},$$

where $K = \{ \frac{\pi}{L}, \frac{3\pi}{L}, \ldots, \frac{(L-1)\pi}{L} \}$. 
For even $L$, the generators of the $\mathfrak{sl}_2$ loop algebra symmetry of the $\tau_2$-model is given by

\begin{align*}
 h_n &= \sum_{k \in K} \cot^{2n} \left( \frac{k}{2} \right) (H)_k, \\
 x^+_n &= \sum_{k \in K} \cot^{2n+1} \left( \frac{k}{2} \right) (E)_k, \\
 x^-_n &= \sum_{k \in K} \cot^{2n-1} \left( \frac{k}{2} \right) (F)_k,
\end{align*}

where $(H)_k, (E)_k, (F)_k$ are given by

\[
(H)_k = 1 - \tilde{c}^+_k \tilde{c}_k - \tilde{c}^+_k \tilde{c}^-_k, \quad (E)_k = \tilde{c}^-_k \tilde{c}_k, \quad (F)_k = \tilde{c}^+_k \tilde{c}^+_k.
\]

The reference state $|0\rangle$, generates a $2^{L/2}$-dimensional irreducible representation, a degenerate eigenspace of the Hamiltonian $H_{\tau_2}$.

\[
H_{SCP} = 2 \sum_{k \in K} (H)_k + 2 \lambda \sum_{k \in K} \left( \cos(k)(H)_k + \sin(k)((E)_k + (F)_k) \right).
\]
The $2^{L/2}$ eigenvalues of $H_{SCP}$ and the eigenstates are given by

$$E(K_+; K_-) = 2 \sum_{k \in K_+} \sqrt{1 - 2\lambda \cos(k) + \lambda^2}$$
$$- 2 \sum_{k \in K_-} \sqrt{1 - 2\lambda \cos(k) + \lambda^2},$$

$$|K_+; K_-\rangle = \prod_{k \in K_+} (\cos \theta_k + \sin \theta_k (F)_k) \prod_{k \in K_-} (\sin \theta_k - \cos \theta_k (F)_k) |0\rangle,$$

where $K_+$ and $K_-$ are such disjoint subsets of $K$ that $K = K_+ \cup K_-$ and

$$\tan(2\theta_k) = \frac{2\lambda \sin(k)}{2(\lambda \cos(k) - 1)}.$$
QISP for the SCP model

We consider the spin-1/2 chain of \((N - 1)L\) lattice sites. We set \(\ell = N - 1\).

For \(m = n\) \((0 \leq m \leq N - 1)\), upto gauge transformations, we have

\[
\tilde{E}_i^{n, n (N - 1 +)} = \begin{pmatrix} \ell \\ n \end{pmatrix} \tilde{P}_{1 \cdots (N-1) L}^{(\ell)} \prod_{\alpha = 1}^{(i-1)\ell} (A^{(\ell +)} + e^\varphi D^{(\ell +)})(w^{(\ell +)}_{\alpha}) \prod_{k=1}^{n} D^{(\ell +)}(w^{(\ell +)}_{(i-1)\ell + k}) \times \prod_{k=n+1}^{\ell} A^{(\ell +)}(w_{(i-1)\ell + k}) \prod_{\alpha=i\ell +1}^{\ell L} (A^{(\ell +)} + e^\varphi D^{(\ell +)})(w^{(\ell +)}_{\alpha}) \tilde{P}^{(\ell)}_{1 \cdots L(N-1)}.
\]

Here we set \(e^\varphi = q\).

We next consider the case of \(N = 3\), \(q^3 = 1\) and \(\omega = q^2\).
\[ Z_i = E_{i}^{00} + \omega E_{i}^{11} + \omega^2 E_{i}^{22} \]

\[ = \widetilde{P}^{(2)}_{1 \ldots 2L} \prod_{\alpha=1}^{2(i-1)} (A^{(2+)} + e^{\varphi} D^{(2+)})(w^{(2+)}_{\alpha}) \]

\[ \times \prod_{k=1}^{2} A^{(2+)}(w^{(2+)}_{2(i-1)+k}) \prod_{\alpha=2i+1}^{2L} (A^{(2+)} + e^{\varphi} D^{(2+)})(w^{(2+)}_{\alpha}) \widetilde{P}^{(2)}_{1 \ldots 2L} \]

\[ + \omega \widetilde{P}^{(2)}_{1 \ldots 2L} \prod_{\alpha=1}^{2(i-1)} (A^{(2+)} + e^{\varphi} D^{(2+)})(w^{(2+)}_{\alpha}) D^{(2+)}(w^{(2+)}_{2(i-1)+1}) \]

\[ \times A^{(2+)}(w^{(2+)}_{2(i-1)+2}) \prod_{\alpha=2i+1}^{2L} (A^{(2+)} + e^{\varphi} D^{(2+)})(w^{(2+)}_{\alpha}) \widetilde{P}^{(2)}_{1 \ldots 2L} \]

\[ + \omega^2 \widetilde{P}^{(2)}_{1 \ldots 2L} \prod_{\alpha=1}^{2(i-1)} (A^{(2+)} + e^{\varphi} D^{(2+)})(w^{(2+)}_{\alpha}) \prod_{k=1}^{2} D^{(2+)}(w^{(2+)}_{2(i-1)+k}) \]

\[ \times \prod_{\alpha=2i+1}^{2L} (A^{(2+)} + e^{\varphi} D^{(2+)})(w^{(2+)}_{\alpha}) \widetilde{P}^{(2)}_{1 \ldots 2L} \]
Conclusion

- **I-1: QISP for the spin-\(s\) XXZ spin chains** both in the massless and massive regimes

  New tricks such as Hermitian elementary matrices

- **I-2: Single-product form of the multiple-integral representation of an arbitrary spin-\(s\) XXZ correlation function**

  We have evaluated one-point functions for spin-1 and show

  \[
  \langle \tilde{E}^{00} \rangle + \langle \tilde{E}^{11} \rangle + \langle \tilde{E}^{22} \rangle = 1. \quad (0 \leq \zeta < \pi/2s)
  \]

- **II: QISP for SCP model**

Possible future work:

- Spin-\(s\) correlation functions in the massive regime

- Factorization property: Are all the multiple integrals reduced into single integrals? (A question given by B.M. McCoy)

- Confirmation of conjectures by Au-Yang and Perk for deriving correlation functions of \(Z_N\)-symmetric Hamiltonian (SCP model)
Thank you for your attention!

Reference
[1,2] T.D. and Chihiro Matsui, NPB 814[FS](2009)405 ;831[FS](2010)359

Partially in collaboration with Jun Sato

(For SCP model and the $sl_2$ loop algebra symmetry)