The Kerr-Newman solution

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2. Introduction

The story of this metric began when I was a graduate student at Cambridge University, 1955–58. I started as a student of Professor Philip Hall, the algebraist, moved to theoretical physics, found that I could not remember the zoo of particles that were being discovered at that time and finally settled on general relativity. The reason for this last move was that I met John Moffat, a fellow graduate student, who was proposing a Unified Field theory where the gravitational and electromagnetic fields were replaced by a complex g_{ij} satisfying the now complex equations $G_{ij} = 0$. John and I used the EIH method to calculate the forces between the singularities. To my surprise, if not John's, we got the usual gravitational and EM forces in the lowest order. However, I then realized that we had imposed as "coordinate conditions" the usual $(\sqrt{g}g^{\alpha\beta})_{,\beta} = 0$, i.e. the radiation gauge. Since the metric was complex we were imposing eight real conditions, not fou r. In effect, the theory needed four more field equations.

Although this was a failure, I did become interested in the theory behind the methods then used to calculate the motion of slow moving bodies. The original paper by Einstein, Infeld and Hoffman (1) was an outstanding one but there were problems with it that had still not been resolved. It was realized that although the method appears to give four "equations of motion" $\underset{n}{P^\mu} = 0$ in each approximation order, n, these terms should be added together to give $\Sigma_n \underset{n}{P^\mu} = 0$. How to prove this last step - that was the problem. It was thought that the field equations could be satisfied exactly in each approximation order but this was neither true nor necessary. This original paper was followed by a succession of attempts by Infeld and coworkers to explain why the terms should be added together, culminating with the "New approximation method" (2) where fictitious dipole moments were introduced in each order. John and I wrote a rebuttal of this approach, showing that their proof that it was consistent was false.

A more serious problem with the method was that there had to be more than just the momentum equations as any physicist should have realized. Any proof that the method was consistent should also have explained conservation of angular momentum. It was shown in my thesis that one needs to work with the total field up to a given order,

$$\underset{(n)}{g}^{\mu\nu} = \sum_{s=0}^{n} \underset{s}{g}^{\mu\nu}$$

rather than the individual terms. The approximation can be advanced one step provided that the seven equations of motion for each body, calculated from the lower order fields, are satisfied to the appropriate order (3). The equations of motion for spinning particles with arbitrary multipole moments were calculated but not published. After this the process was extended to relativistic particles (4) and the lowest order forces were calculated for the Einstein (5) and Einstein-Maxwell fields (6). The usual electromagnetic radiation reaction terms were found but the corresponding terms were not calculated for the gravitational field since any such terms are smaller than forces to be calculated in the next iteration.

The reader may wonder what this has to do with the Kerr metric and why it has been discussed here. There are claims by many in the literature that I did not know what I was looking for and did not know what I had found. This is hogwash. Angular momentum was at the forefront of my mind from the time when I realized that it had been overlooked by the EIH and related methods.

Alexei Zinovievich Petrov had published a paper (7) in 1954 where the simultaneous invariants and canonical forms were calculated for the metric and conformal tensors at a general point in an Einstein space.[1] In 1958, my last year at Cambridge, I was invited to attend the relativity seminars at Kings College in London, including one by Felix Pirani where he discussed his 1957 paper on radiation theory (8). He analyzed gravitational shock waves, calculated the possible jumps in the Riemann tensor across the wave fronts, and related these to the Petrov types. At the time I thought that he was stretching when he proposed that radiation was type N, and I said so, a rather stupid thing for a graduate student with no real supervisor to do.[2] It seemed obvious that a superposition of type N solutions would not itself be type N, and that gravitational waves near a macroscopic body would be of general type, not Type N.

Perhaps I did Felix an injustice. His conclusions may have been oversimplified but his paper had some very positive consequences. Andrzej Trautman computed the asymptotic properties of the Weyl tensor for outgoing radiation by generalizing Sommerfeld's work on electromagnetic radiation, confirming that the far field is Type N. Bondi, van der Burg and Metzner (9) then introduced appropriate null coordinates to study gravitational radiation in the far zone, relating this to the results of Petrov and Pirani.

After Cambridge and a brief period at Kings College I went to Syracuse University for 18 months as a research associate of Peter Bergmann. I was then invited to join Joshua Goldberg at the Aeronautical Research Laboratory

[1] This was particularly interesting for empty Einstein spaces where the Riemann and conformal tensors are identical. This paper took a while to be appreciated in the West, probably because the Kazan State University journal was not readily available, but it has been very influential.

[2] My nominal supervisor was a particle physicist and had no interest in general relativity.

Figure 2.1.: Ivor Robinson and Andrzej Trautman constructed all Einstein spaces possessing a hypersurface orthogonal shearfree congruence. Whereas Bondi and his colleagues were looking at spaces with these properties asymptotically, far from any sources, Robinson and Trautman went a step further, constructing exact solutions. (Images courtesy of Andrzej Trautman and the photographer, Marek Holzman.)

in Dayton Ohio.[3] Before Josh went on sabbatical we became interested in the new methods that were entering relativity at that time. Since we did not have a copy of Petrov's paper we rederived his results using projective geometry. He had shown that in an empty Einstein space,

The conformal tensor $\mathfrak{C}$ determines four null "eigenvectors" at each point. This vector is called a principal null vector (PNV), the field of these is called a principal null "congruence" and the metric is called algebraically special (AS) if two of these eigenvectors coincide.

Josh and I used a tetrad formulation to study vacuum Einstein spaces with degenerate holonomy groups (10; 11). The tetrad used consisted of two null vectors and two *real* orthogonal space-like vectors,

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + 2\omega^3\omega^4.$$

We proved that the holonomy group must be an even dimensional subgroup of the Lorentz group at each point, and that if its dimension is less than six, its maximum, coordinates can be chosen so that the metric has the following form:

$$ds^2 = dx^2 + dy^2 + 2du(dv + \rho dx + \tfrac{1}{2}(\omega - \rho_{,x}v)du),$$

where both ρ and ω are independent of v, an affine parameter along the rays,[4] and

$$\begin{aligned}\rho_{,xx} + \rho_{,yy} &= 0\\ \omega_{,xx} + \omega_{,yy} &= 2\rho_{,ux} - 2\rho\rho_{,xx} - (\rho_{,x})^2 + (\rho_{,y})^2\end{aligned}$$

This coordinate system was not quite uniquely defined. If ρ is bilinear in x and y then it can be transformed to zero, giving the well-known plane-fronted wave solutions. These are type N, and have a two-dimensional holonomy groups. The more general metrics are type III with four-dimensional holonomy groups.

When I went to Dayton I knew that Josh was going on sabbatical leave to Kings College. He left in September 1961 to join Hermann Bondi, Andrzej Trautman, Ray Sachs and others there. By this time it was well known that all AS spaces possess a null congruence whose vectors are both geodesic and shearfree. These are the degenerate "eigenvectors" of the conformal tensor at each point, the PNVs. Andrzej suggested to Josh and Ray how they might

[3] There is a claim spread on internet that we were employed to develop an antigravity engine to power spaceships. This is rubbish! The main reason why the US Air Force had created a General Relativity section was probably to show the Navy that they could also do pure research. The only real use that the USAF made of us was when some crackpot sent them a proposal for antigravity or for converting rotary motion inside a spaceship to a translational driving system. These proposals typically used Newton's equations to prove non-conservation of momentum for some classical system.

[4] The simple way that the coordinate v appears is typical of all algebraically special metrics.

prove the converse. This led to the celebrated Goldberg-Sachs theorem (12):

Theorem 1 *A vacuum metric is algebraically special if and only if it contains a geodesic and shearfree null congruence.*

Either the properties of the congruence, geodesic and shear-free, or the property of the Conformal tensor, algebraically degenerate, could be considered fundamental with the other property following from the Goldberg-Sachs theorem. It is likely that most thought that the algebra was fundamental, but I believe that Ivor Robinson and Andrzej Trautman (13) were correct when they emphasized the properties of the congruence instead. They showed that for any Einstein space with a shear-free null congruence *which is also hypersurface orthogonal* there are coordinates for which

$$ds^2 = 2r^2P^{-2}d\zeta d\bar{\zeta} - 2dudr - (\Delta lnP - 2r(lnP)_{,u} - 2m(u)/r)du^2,$$

where ζ is a complex coordinate,

$$\zeta = (x + iy)/\sqrt{2} \quad \Rightarrow \quad 2d\zeta d\bar{\zeta} = dx^2 + dy^2.$$

The one remaining field equation is,

$$\Delta\Delta(lnP) + 12m(lnP)_{,u} - 4m_{,u} = 0, \quad \Delta = 2P^2\partial_\zeta\partial_{\bar{\zeta}}. \tag{2.0.1}$$

The PNV[5] is $\boldsymbol{k} = k^\mu\partial_\mu = \partial_r$, where r is an affine parameter along the rays. The corresponding differential form is $k = k_\mu dx^\mu = du$, so that $\boldsymbol{k}$ is the normal to the surfaces of constant u. The coordinate u is a retarded time, the surfaces of constant r, u are distorted spheres with metric $ds^2 = 2r^2P^{-2}d\zeta d\bar{\zeta}$ and the parameter $m(u)$ is loosely connected with the system's mass. This gives the complete solution for AS spaces with hypersurface-orthogonal rays, subject to the single Robinson-Trautman equation above.

In the summer of 1962 Josh Goldberg and myself attended a pair of meetings at Santa Barbara and Jablonna. I think that the first of these was a month-long meeting in Santa Barbara, designed to get mathematicians and relativists talking to each other. The physicists learnt quite a lot about modern mathematical techniques in differential geometry, but I doubt that the geometers learnt much from the relativists. All that aside, I met Alfred Schild at this conference. He had just persuaded the Texas state legislators to finance a Center for Relativity at the University of Texas, and had arranged for an outstanding group of relativists to join. These included Roger Penrose and Ray Sachs, but neither could come immediately and so I was invited to visit for the 62-63 academic year.

After Santa Barbara, we attended a conference Jablonna near Warsaw. This was the third precursor to the triennial meetings of the GRG society and

[5]The letters $\boldsymbol{k}$ and k will be used throughout this article to denote the PNV.

could be called GR3. Robinson and Trautman (14) presented a paper on *"Exact Degenerate Solutions"* at this conference. They spoke about their well-known solution and also showed that when the rays are not hypersurface orthogonal coordinates can be chosen so that

$$ds^2 = -P^2[(d\xi - ak)^2 + (d\eta - bk)^2] + 2d\rho k + ck^2,$$

where, as usual, k is the PNV. They knew that the components k_α were independent of ρ, but a, b, c and P could still have been functions of all four coordinates.

3. The NUT roadblock and its removal.

I was studying the structure of the Einstein equations during the latter half of 1962, using the new (to physicists) methods of tetrads and differential forms. I had written out the equations for the curvature using a complex null tetrad and associated self-dual bivectors, and had examined their integrability conditions. In particular, I was interested in the same problem that Robinson and Trautman were investigating but there was a major problem holding this work back. Alan Thompson had also come to Austin that year and was also interested in these methods. Although there seemed to be no reason why there should not be many algebraically special spaces, Alan kept quoting a result from a preprint of a paper by Newman, Tambourino and Unti (15) in which they had "proved" that the only possible space with a diverging and rotating PNV is NUT space, a one-parameter generalization of the Schwarzschild metric that is not asymptotically flat. This result was obtained using the new Newman-Penrose spinor formalism (N-P). Their equations are essentially the same as those obtained by people such as myself using self-dual bivectors: only the names are different. I did not understand how the equations that I was studying could possibly lead to their claimed result, but presumed it was so since I did not have a copy of their paper.

Finally, Alan lent me a preprint of this paper in the spring of 1963. I read through it quickly, trying to see where their hunt for solutions had died. The N-P formalism assigns a different Greek letter to each component of the connection, so I did not try to read it carefully, just rushed ahead until I found what appeared to be the key equation,

$$\tfrac{1}{3}(n_1 + n_2 + n_3)a^2 = 0, \tag{3.0.1}$$

where the n_i were all small integers. Their sum was not zero so this gave $a = 0$. I did not know what a represented, but its vanishing seemed to be disastrous and so I looked more carefully to see where this equation was coming from. Three of the previous equations, each involving first derivatives of some of the field variables, had been differentiated and then added together. All the second derivatives canceled identically and most of the other terms were eliminated using other N-P equations, leaving equation (3.0.1).

The fact that the second derivatives all canceled should have been a warning to the authors. The mistake that they made was that they did not notice that they were simply recalculating one component of the Bianchi identities by adding together the appropriate derivatives of three of their curvature equations, and then simplifying the result by using some of their other equations, undifferentiated. The final result should have agreed with one of their derived Bianchi identities involving derivatives of the components of the conformal tensor, the Ψ_i functions, and should have given

$$n_1 + n_2 + n_3 \equiv 0. \tag{3.0.2}$$

In effect, they rediscovered one component of the identities, but with numerical errors. The real fault was the way the N-P formalism confuses the Bianchi identities with the derived equations involving derivatives of the Ψ_i variables.

Alan Thompson and myself were living in adjoining apartments, so I dashed next door and told him that their result was incorrect. Although it was totally unnecessary, we recalculated the first of the three terms, n_1, obtained a different result to the one in the preprint, and verified that Eq. (3.0.2) was now satisfied. Once this blockage was out of the way, I was then able to continue with what I had been doing and derive the metric and field equations for twisting algebraically special spaces. The coordinates I constructed turned out to be essentially the same as the ones given by Robinson and Trautman (14). This shows that they are the "natural" coordinates for this problem since the methods used by them were very different to those used by me. Ivor loathed the use of such things as N-P or rotation coefficients, and Andrzej and he had a nice way of proving the existence of their canonical complex coordinates ζ and $\bar{\zeta}$. I found this same result from one of the Cartan equations, as will be shown in the next section, but I have no doubt that their method is more elegant. Although Ivor explained it to me on more than one occasion I did not understand what he was saying until recently when I reread their 1964 paper, (14).

Soon afterwards I presented preliminary results at a monthly Relativity conference held at the Stevens Institute in Hoboken, N.J. When I told Edward Newman that Eq. (3.0.1) should have been identically zero, he said that they knew that the first coefficient n_1 was incorrect, but that the value for n_2 given in the preprint was a misprint and that Eq. (3.0.2) was still not satisfied. I replied that since the sum had to be zero the final term, n_3 must also be incorrect. Alan and I recalculated it that evening, confirming that Eq. (3.0.2) was satisfied.[1]

[1] Robinson and Trautman also doubted the original claim by Newman et al. since they had observed that the linearized equations had many solutions.

4. Algebraically special metrics with diverging rays

When I realized that the attempt by Newman et al. to find all rotating AS spaces had foundered and that Robinson and Trautman appeared to have stopped with the static ones, I rushed headlong into the search for these metrics.

Why was the problem so interesting to me? Schwarzschild, by far the most significant physical solution known at that time, has an event horizon. A spherically symmetric star that collapses inside this is forever lost to us, but it was not known whether angular momentum could stop this collapse to a black hole. Unfortunately, there was no known metric for a rotating star. Schwarzschild itself was a prime example of the Robinson-Trautman metrics, none of which could contain a rotating source as they were all hypersurface orthogonal. Many had tried to solve the Einstein equations assuming a stationary and axially symmetric metric, but none had succeeded in finding any physically significant rotating solutions. The equations for such metrics are complicated nonlinear PDEs in two variables. What was needed was some extra condition that would reduce these to ODEs, and it seemed to me that this might be the assumption that the metric is AS.

There were two competing formalisms being used around 1960, complex tetrads and spinors. Like Robinson and Trautman, I used the former, Newman et al. the latter. The derived equations are essentially identical, but each approach has some advantages. The use of spinors makes the the Petrov classification trivial, once it has been shown that a tensor with the symmetries of the conformal tensor is represented by a completely symmetric spinor, Ψ_{ABCD}. The standard notation for the components of this is

$$\Psi_0 = \Psi_{0000}, \quad \Psi_1 = \Psi_{0001}, \; \ldots \; \Psi_4 = \Psi_{1111}.$$

Now if ζ^A is an arbitrary spinor then the equation

$$\Psi_{ABCD}\zeta^A\zeta^B\zeta^C\zeta^D = 0$$

is a homogeneous quartic equation with four complex roots, $\{\zeta_i^A : i = 1\ldots4\}$. The related real null vectors, $Z_i^{\alpha\dot{\alpha}} = \zeta_i^{\alpha}\zeta_i^{\dot{\alpha}}$, are the four PNVs of Petrov. The spinor $\zeta^\alpha = \delta_0^\alpha$ is a PNV if $\Psi_0 = 0$. It is a repeated root and therefore it is the principal null vector of an AS spacetime precisely when $\Psi_1 = 0$ as well.

The main results of my calculations were published in a Physical Reviews Letter (16) but few details were given there. I was to spend many years trying to write up this research but, unfortunately, I could never decide whether to use spinors or a complex tetrad, and it did not get published until 1969 in a joint paper with my graduate student, George Debney (17). He also collaborated with Alfred Schild and myself on the Kerr-Schild metrics (18). The methods that I used to solve the equations for AS spaces are essentially those used by Stephani et al. in their monumental book on exact solutions in general relativity (19), culminating in their equation (27.27). I will try to use the same notation as in that book since it is almost identical to the one I used in 1963. The notation is explained in the appendix to this article.

We start with a null tetrad $(\mathbf{e}_a) = (\boldsymbol{m}, \bar{\boldsymbol{m}}, \boldsymbol{l}, \boldsymbol{k})$, a set of four null vectors where the first two are complex conjugates and the last two are real. The corresponding dual forms are $(\omega^a) = (\bar{m}, m, -k, -l)$ and the metric is

$$ds^2 = 2(m\bar{m} - kl) = 2(\omega^1\omega^2 - \omega^3\omega^4). \tag{4.0.1}$$

The vector $\boldsymbol{k}$ is a PNV with a uniquely defined direction but the other three basis vectors are far from unique. The form of the metric tensor in Eq. (4.0.1) is invariant under a combination of a null rotation (B) about $\boldsymbol{k}$, a rotation (C) in the $\boldsymbol{m} \wedge \bar{\boldsymbol{m}}$ plane and a Lorentz transformation (A) in the $\boldsymbol{l} \wedge \boldsymbol{k}$ plane,

$$\boldsymbol{k}' = \boldsymbol{k}, \qquad \boldsymbol{m}' = \boldsymbol{m} + B\boldsymbol{k}, \qquad \boldsymbol{l}' = \boldsymbol{l} + B\bar{\boldsymbol{m}} + \bar{B}\boldsymbol{m} + B\bar{B}\boldsymbol{k}, \tag{4.0.2a}$$

$$\boldsymbol{k}' = \boldsymbol{k}, \qquad \boldsymbol{m}' = e^{iC}\boldsymbol{m}, \qquad \boldsymbol{l}' = \boldsymbol{l}, \tag{4.0.2b}$$

$$\boldsymbol{k}' = A\boldsymbol{k}, \qquad \boldsymbol{m}' = \boldsymbol{m}, \qquad \boldsymbol{l}' = A^{-1}\boldsymbol{l}. \tag{4.0.2c}$$

The most important connection form (see appendix) is

$$\boldsymbol{\Gamma}_{41} = \Gamma_{41a}\omega^a = m^\alpha k_{\alpha;\beta}dx^\beta.$$

The optical scalars of Ray Sachs for $\boldsymbol{k}$ are just the components of this form with respect to the ω^a

$$\begin{aligned}\sigma &= \Gamma_{411} = \text{shear},\\ \rho &= \Gamma_{412} = \text{complex divergence},\\ \kappa &= \Gamma_{414} = \text{geodesy}.\end{aligned}$$

The fourth component, Γ_{413}, is not invariant under a null rotation about $\boldsymbol{k}$,

$$\Gamma'_{413} = \Gamma_{413} + B\rho,$$

and has no real geometric significance since it can be transformed to zero using an appropriate null rotation. Also, since $\boldsymbol{k}$ is geodesic and shearfree, both κ and σ are zero and therefore

$$\mathbf{\Gamma}_{41} = \rho\omega^2. \tag{4.0.3}$$

If we use the simplest field equations,

$$R_{44} = 2R_{4142} = 0, \quad R_{41} = R_{4112} - R_{4134} = 0, \quad R_{11} = 2R_{4113} = 0,$$

and the fact that the metric is AS,

$$\Psi_0 = -2R_{4141} = 0, \quad 2\Psi_1 = -R_{4112} - R_{4134} = 0,$$

then the most important of the second Cartan equations simplifies to

$$d\mathbf{\Gamma}_{41} - \mathbf{\Gamma}_{41} \wedge (\mathbf{\Gamma}_{12} + \mathbf{\Gamma}_{34}) = R_{41ab}\omega^a \wedge \omega^b = R_{4123}\omega^2 \wedge \omega^3. \tag{4.0.4}$$

Taking the wedge product of Eq. (4.0.4) with $\mathbf{\Gamma}_{41}$ and using (4.0.3),

$$\mathbf{\Gamma}_{41} \wedge d\mathbf{\Gamma}_{41} = 0. \tag{4.0.5}$$

This was the key step in my study of these metrics but this result was not found in quite such a simple way. At first, I stumbled around using individual component equations rather than differential forms to look for a useful coordinate system. It was only after I had found this that I realized that using differential forms from the start would have short-circuited the whole process.

Equation (4.0.5) is just the integrability condition for the existence of complex functions, ζ and Π, such that

$$\mathbf{\Gamma}_{41} = d\bar{\zeta}/\Pi, \quad \mathbf{\Gamma}_{42} = d\zeta/\bar{\Pi}.$$

The two functions ζ and its complex conjugate, $\bar{\zeta}$, were used as (complex) coordinates. They are not quite unique since ζ can always be replaced by an arbitrary analytic function $\Phi(\zeta)$.

Using the transformations in (4.0.2b) and (4.0.2c),

$$\mathbf{\Gamma}_{4'1'} = Ae^{iC}\mathbf{\Gamma}_{41} = Ae^{iC}d\bar{\zeta}/\Pi \quad \Rightarrow \quad \Pi' = A^{-1}e^{-iC}\Pi.$$

Π can therefore be eliminated entirely by choosing $Ae^{iC} = \Pi$, and that is what I did in 1963, but it is also common to just use the C-transformation to convert Π to a real function P

$$\mathbf{\Gamma}_{41} = \rho\omega^2 = d\bar{\zeta}/P\,. \tag{4.0.6}$$

This is the derivation for two of the coordinates used in 1963. Note that $\{\zeta, \bar{\zeta}\}$, are constant along the PNV since $\omega^1_\alpha k^\alpha = 0 \to \boldsymbol{k}(\zeta) = 0$.

The other two coordinates were very standard and were used by most people considering similar problems at that time. The simplest field equation is

$$R_{44} = 0 \quad \Rightarrow \quad \boldsymbol{k}\rho = \rho_{|4} = \rho^2,$$

so that the real part of $-\rho^{-1}$ is an affine parameter along the rays. This was the obvious choice for the third coordinate, r,

$$\rho^{-1} = -(r + i\Sigma).$$

There was no clear choice for the fourth coordinate, so u was chosen so that $l^\alpha u_{,\alpha} = 1,\ k^\alpha u_{,\alpha} = 0$, a pair of consistent equations.

Given these four coordinates, the basis forms are

$$\begin{aligned}
\omega^1 = m_\alpha dx^\alpha &= -d\zeta/P\bar{\rho} = (r - i\Sigma)d\zeta/P,\\
\omega^2 = \bar{m}_\alpha dx^\alpha &= -d\bar{\zeta}/P\rho = (r + i\Sigma)d\bar{\zeta}/P,\\
\omega^3 = k_\alpha dx^\alpha &= \ du + Ld\zeta + \bar{L}d\bar{\zeta},\\
\omega^4 = l_\alpha dx^\alpha &= \ dr + Wd\zeta + \bar{W}d\bar{\zeta} + H\omega^3.
\end{aligned}$$

where L is independent of R, and the coefficients Σ, W and H are still to be determined.

On substituting all this into the first Cartan equation, (10.0.21), and the simplest component of the second Cartan equation, (4.0.4), Σ and W were calculated as functions of L and its derivatives[1]

$$\begin{aligned}
2i\Sigma &= P^2(\bar{\partial}L - \partial\bar{L}), \quad \partial = \partial_\zeta - L\partial_u,\\
W &= -(r + i\Sigma)L_{,u} + i\partial\Sigma.
\end{aligned}$$

The remaining field equations, the "hard" ones, were more complicated, but still fairly straightforward to calculate. Two gave H as a function of a real "mass" function $m(u, \zeta, \bar{\zeta})$ and the higher derivatives of P and L,[2]

$$\begin{aligned}
H &= \tfrac{1}{2}K - r(lnP)_{,u} - \frac{mr + M\Sigma}{r^2 + \Sigma^2}.\\
M &= \Sigma K + P^2 Re[\partial\bar{\partial}\Sigma - 2\bar{L}_{,u}\partial\Sigma - \Sigma\partial_u\partial\bar{L}],\\
K &= 2P^{-2}Re[\partial(\bar{\partial}lnP - \bar{L}_{,u})],
\end{aligned}$$

Finally, the first derivatives of the mass function, m, are given by the rest of

[1] Ω, D and Δ were used instead of L, ∂ and Σ in the original letter but the results were the same, mutatis mutandis.

[2] This expression for M was first published by Robinson et al. (1969). The corresponding expression in Kerr (1963) is for the gauge when $P = 1$. The same is true for equation (4.0.7c).

the field equations, $R_{31} = 0$ and $R_{33} = 0$,

$$\partial(m + iM) = 3(m + iM)L_{,u}, \tag{4.0.7a}$$
$$\bar{\partial}(m - iM) = 3(m - iM)\bar{L}_{,u}, \tag{4.0.7b}$$
$$[P^{-3}(m + iM)]_{,u} = P[\partial + 2(\partial lnP - L_{,u}]\partial I, \tag{4.0.7c}$$

where

$$I = \bar{\partial}(\bar{\partial} lnP - \bar{L}_{,u}) + (\bar{\partial} lnP - \bar{L}_{,u})^2. \tag{4.0.8}$$

There are two natural choices that can be made to restrict the coordinates and simplify the final results. One is to rescale r so that $P = 1$ and L is complex, the other is to take L to be pure imaginary with $P \neq 1$. I chose to do the first since this gives the most concise form for M and the remaining field equations. It also gives the smallest group of permissable coordinate transformations, simplifying the task of finding all possible Killing vectors. The results for this gauge are

$$M = \mathrm{Im}(\bar{\partial}\bar{\partial}\partial L), \tag{4.0.9a}$$
$$\partial(m + iM) = 3(m + iM)L_{,u}, \tag{4.0.9b}$$
$$\bar{\partial}(m - iM) = 3(m - iM)\bar{L}_{,u}, \tag{4.0.9c}$$
$$\partial_u[m - \mathrm{Re}(\bar{\partial}\bar{\partial}\partial L)] = |\partial_u \partial L|^2. \tag{4.0.9d}$$

Since all derivatives of the real function m were known, the commutators were calculated to see whether the system was completely integrable. These derived equations gave m as a function of higher derivatives of L unless both $\Sigma_{,u}$ and $L_{,uu}$ were zero. As stated in Kerr (1963), if these are both zero then there is a coordinate system in which P and L are independent of u, and $m = cu + A(\zeta, \bar{\zeta})$, where c is a real constant. If this is zero then the metric is independent of u and is therefore stationary. The field equations in this special situation are

$$\nabla[\nabla(lnP)] = c, \qquad \nabla = P^2\partial^2/\partial\zeta\partial\bar{\zeta}, \tag{4.0.10a}$$
$$M = 2\Sigma\nabla(\ln P) + \nabla\Sigma, \qquad m = cu + A(\zeta, \bar{\zeta}), \tag{4.0.10b}$$
$$cL = (A + iM)_{\zeta}, \quad \Rightarrow \quad \nabla M = c\Sigma. \tag{4.0.10c}$$

These metrics were called quasi-stationary.

It was also stated that the solutions of these equations include the Kerr metric (for which $c = 0$). This is true but it is not how this solution was found. Furthermore, in spite of what many believe, its construction had nothing whatsoever to do with the Kerr-Schild ansatz.

5. Symmetries in algebraically special spaces

As had been expected, the field equations were so complicated that some extra assumptions were needed to reduce them to a more manageable form. I had been interested in the relationship between scalar invariants and groups of motion in a manifold so my next step in the hunt for physically interesting solutions was fairly obvious: assume that the metric is stationary and axisymmetric. Fortunately, I had some tricks that allowed me to find all possible Killing vectors without actually solving Killing's equation.

The key observation is that any Killing vector generates a 1-parameter group which must be a subgroup of the group $\mathcal{C}$ of coordinate transformations that preserve all imposed coordinate conditions.

Suppose that $\{x^{\star a}, \omega^{\star}_{a}\}$ is another set of coordinates and tetrad vectors that satisfy the conditions already imposed in the previous sections. If we restrict our coordinates to those that satisfy $P = 1$ then $\mathcal{C}$ is the group of transformations $x \to x^{\star}$ for which

$$\begin{aligned} \zeta^{\star} &= \Phi(\zeta), & \omega^{1\star} &= (|\Phi_{\zeta}|/\Phi_{\zeta})\omega^{1}, \\ u^{\star} &= |\Phi_{\zeta}|(u + S(\zeta,\bar{\zeta}), & \omega^{3\star} &= |\Phi_{\zeta}|^{-1}\omega^{3}, \\ r^{\star} &= |\Phi_{\zeta}|^{-1}r, & \omega^{4\star} &= |\Phi_{\zeta}|\omega^{4}, \end{aligned}$$

and the transformed metric functions, $L^{\star}$ and $m^{\star}$, are given by

$$L^{\star} = (|\Phi_{\zeta}|/\Phi_{\zeta})[L - S_{\zeta} - \tfrac{1}{2}(\Phi_{\zeta\zeta}/\Phi_{\zeta})(u + S(\zeta,\bar{\zeta})], \tag{5.0.1a}$$

$$m^{\star} = |\Phi_{\zeta}|^{-3}m. \tag{5.0.1b}$$

Let $\mathcal{S}$ be the identity component of the group of symmetries of our manifold. If these are interpreted as coordinate transformations, rather than point transformations, then $\mathcal{S}$ is the set of transformations $x \to x^{\star}$ for which

$$g^{\star}_{\alpha\beta}(x^{\star}) = g_{\alpha\beta}(x^{\star}).$$

For our AS metrics, $\mathcal{S}$ is precisely the subgroup of $\mathcal{C}$ for which[1]

$$m^\star(x^\star) = m(x^\star), \quad L^\star(x^\star) = L(x^\star).$$

Suppose now that $x \to x^\star(x,t)$ is a 1-parameter group of motions,

$$\begin{aligned} \zeta^\star &= \psi(\zeta;t), \\ u^\star &= |\psi_\zeta|(u + T(\zeta,\bar{\zeta};t), \\ r^\star &= |\psi_\zeta|^{-1} r. \end{aligned}$$

Since $x^\star(x;0) = x$, the initial values of ψ and T are

$$\psi(\zeta;0) = \zeta, \quad T(\zeta,\bar{\zeta};0) = 0.$$

The corresponding infinitesimal transformation, $\mathbf{K} = K^\mu \partial/\partial x^\mu$ is

$$K^\mu = \left[\frac{\partial x^{\star\mu}}{\partial t}\right]_{t=0}.$$

If we define

$$\alpha(\zeta) = \left[\frac{\partial \psi}{\partial t}\right]_{t=0}, \quad V(\zeta,\bar{\zeta}) = \left[\frac{\partial T}{\partial t}\right]_{t=0},$$

then the infinitesimal transformation is

$$\mathbf{K} = \alpha\partial_\zeta + \bar{\alpha}\partial_{\bar{\zeta}} + Re(\alpha_\zeta)(u\partial_u - r\partial_r) + V\partial_u. \tag{5.0.2}$$

Replacing $\Phi(\zeta)$ with $\psi(\zeta;t)$ in Eq. (5.0.1),

differentiating this with respect to t, and using the initial values for ψ and T, it follows that $\mathbf{K}$ is a Killing vector provided

$$\begin{gathered} V_\zeta + \tfrac{1}{2}\alpha_{\zeta\zeta} r + \mathbf{K}L + \tfrac{1}{2}(\alpha_\zeta - \bar{\alpha}_{\bar{\zeta}})L = 0, \\ \mathbf{K}m + 3Re(\alpha_\zeta)m = 0. \end{gathered}$$

The transformation rules for $\mathbf{K}$ under an element (Φ, S) of $\mathcal{C}$ are

$$\alpha^\star = \Phi_\zeta \alpha, \quad V^\star = |\Phi_\zeta|[V - Re(\alpha_\zeta)S + \mathbf{K}S].$$

Since α is itself analytic, if $\alpha \neq 0$ for a particular Killing vector then, Φ can be chosen so that $\alpha^\star = 1$ (or any other analytic function of ζ that one chooses), and then S can be chosen to make $V^\star = 0$. If $\alpha = 0$ then so is $\alpha^\star$, and $\mathbf{K}$ is already simple without the (Φ, S) transformation being used. There

[1] Note that this implies that all derivatives of these functions are also invariant, and so $g_{\alpha\beta}$ itself is invariant.

are therefore two canonical types for **K**,

$$\text{Type 1}: \underset{1}{\mathbf{K}} = V\partial_u, \quad \text{or} \quad \text{Type 2}: \underset{2}{\mathbf{K}} = \partial_\zeta + \partial_{\bar{\zeta}}. \tag{5.0.3}$$

These are asymptotically timelike and spacelike, respectively.

6. Stationary solutions

The first step in simplifying the field equations was to assume that the metric was stationary. The Type 2 Killing vectors are asymptotically spacelike and so $\mathfrak{E}$ was assumed to have a Type 1 Killing vector $\underset{1}{\mathbf{K}} = V\partial_u$. The coordinates used in the last section assumed $P = 1$ but it is more appropriate here to relax this condition. If we transform to coordinates where $P \neq 1$, using an A-transformation (4.0.2c) with associated change in the (r, u) variables,

$$\boldsymbol{k}' = A\boldsymbol{k}, \quad \boldsymbol{l}' = A^{-1}\boldsymbol{l}, r' = A^{-1}r, \quad u' = Au,$$
$$\underset{1}{\mathbf{K}} = V\partial_u = VA\partial_{u'} = \partial_{u'} \quad if\ VA = 1,$$

where A has been chosen to make $V = 1$ and the Killing vector a simple ∂_u. The metric can therefore be assumed independent of u, but P may not be a constant. The basic functions, L, P and m are functions of $(\zeta, \bar{\zeta})$ alone, and the metric simplifies to

$$ds^2 = ds_o^2 + 2mr/(r^2 + \Sigma^2)k^2, \tag{6.0.1}$$

where the "base" metric, is

$$(ds_0)^2 = 2(r^2 + \Sigma^2)P^{-2}d\zeta d\bar{\zeta} - 2l_0 k, \tag{6.0.2a}$$
$$l_0 = dr + i(\Sigma_{,\zeta}d\zeta - \Sigma_{,\bar{\zeta}}d\bar{\zeta}) + \left[\tfrac{1}{2}K - \frac{M\Sigma}{(r^2 + \Sigma^2)}\right]k. \tag{6.0.2b}$$

Although this is flat for Schwarzschild it is not so in general. Σ, K and M are all functions of the derivatives of L and P,

$$\Sigma = P^2\mathrm{Im}(L_{\bar{\zeta}}), \qquad K = 2\nabla^2 \ln P,$$
$$M = \Sigma K + \nabla^2\Sigma, \quad \nabla^2 = P^2\partial_\zeta\partial_{\bar{\zeta}}, \tag{6.0.3}$$

The mass function, m, and M are conjugate harmonic functions,

$$m_\zeta = -iM_\zeta, \quad m_{\bar{\zeta}} = +iM_{\bar{\zeta}}, \tag{6.0.4}$$

and the remaining field equations are

$$\nabla^2 K = \nabla^4 \ln P = 0, \quad \nabla^2 M = 0. \tag{6.0.5}$$

If m is a particular solution of these equations then so is $m + m_0$ where m_0 is an arbitrary constant. The most general situation where the metric splits in this way is when P, L and M are all independent of u but $m = cu + A(\zeta, \bar{\zeta})$. The field equations for these are given in Eq. (4.0.10) (and in Kerr (1963)).

Theorem 2 *If ds_0^2 is any stationary (diverging) algebraically special metric, or more generally a solution of (4.0.10), then so is*

$$ds_0^2 + \frac{2m_0 r}{r^2 + \Sigma^2} k^2,$$

where m_0 is an arbitrary constant. These are the most general diverging algebraically special spaces that split in this way.

These are all "generalized Kerr-Schild" metrics with base spaces ds_0^2 that are not necessarily flat.

These field equations for stationary AS metrics are certainly simpler than the original ones, (4.0.7), but they are still nonlinear PDE's, not ODE's. From the first equation in (6.0.5), the curvature $\nabla^2(\ln P)$ of the 2-metric $P^{-2} d\zeta d\bar{\zeta}$ is a harmonic function,

$$\nabla^2 \ln P = P^2 (\ln P)_{,\zeta\bar{\zeta}} = F(\zeta) + \bar{F}(\bar{\zeta}),$$

where F is analytic. If F is not a constant then it can be transformed to a constant multiple of ζ by the transformation $\zeta \to \Phi(\zeta)$. There is essentially only one known solution of the ensuing equation for P,

$$P = (\zeta + \bar{\zeta})^{\frac{3}{2}}, \quad \nabla^2 \ln P = -\frac{3}{2}(\zeta + \bar{\zeta}), \tag{6.0.6}$$

This does not lead to any asymptotically flat solutions. The second equation in (6.0.5) is a highly nonlinear PDE for the last of the basic metric functions, L.

7. Axial symmetry

We are getting close to Kerr. As was said early on, the best hope for finding a rotating solution was to look for an AS metric that was both stationary and axially symmetric. I should have revisited the Killing equations to look for any Killing vector (KV) that commutes with ∂_u. However, I knew that it could not also be[1] Type 1 and therefore it had to be Type 2. It seemed fairly clear that it could be transformed to the canonical form $i(\partial_\zeta - \partial_{\bar\zeta})$ ($= \partial_y$ where $\zeta = x + iy$) or equivalently $i(\zeta\partial_\zeta - \bar\zeta\partial_{\bar\zeta})$ ($= \partial_\phi$ in polar coordinates where $\zeta = Re^{i\phi}$) and I was getting quite eager at this point so I decided to just assume such a KV and see what turned up.[2]

If $\mathfrak{E}$ is to be the metric for a localized physical source then the null congruence should be asymptotically the same as Schwarzschild. The 2-curvature function $F(\zeta)$ must be regular everywhere, including at "infinity", and must therefore be constant. This must also be true for the analytic function $F(\zeta)$ of the last section.

$$\tfrac{1}{2}K = PP_{,\zeta\bar\zeta} - P_{,\zeta}P_{,\bar\zeta} = R_0 = \pm P_0^2, \quad \text{(say)}. \tag{7.0.1}$$

As was shown in Kerr and Debney (17), the appropriate Killing equations for a $\underset{2}{\mathbf{K}}$ that commutes with $\underset{1}{\mathbf{K}} = \partial_u$ are

$$\begin{aligned} \underset{2}{\mathbf{K}} &= \alpha\partial_\zeta + \bar\alpha\partial_{\bar\zeta}, \quad \alpha = \alpha(\zeta), \\ \underset{2}{\mathbf{K}}L &= -\alpha_\zeta L, \qquad \underset{2}{\mathbf{K}}\Sigma = 0, \\ \underset{2}{\mathbf{K}}P &= \mathrm{Re}(\alpha_\zeta)P, \quad \underset{2}{\mathbf{K}}m = 0. \end{aligned} \tag{7.0.2}$$

I do not remember the choice made for the canonical form for $\underset{2}{\mathbf{K}}$ in 1963, but it was probably ∂_y. The choice in Kerr and Debney (17) was

$$\alpha = i\zeta, \quad \Rightarrow \quad \underset{2}{\mathbf{K}} = i(\zeta\partial_\zeta - \bar\zeta\partial_{\bar\zeta}),$$

[1] No two distinct Killing vectors can be parallel.

[2] All possible symmetry groups were found for diverging AS spaces in George C. Debney's Ph.D. thesis. My 1963 expectations were confirmed there.

and that will be assumed here. For any function $f(\zeta, \bar{\zeta})$,

$$\underset{2}{\mathbf{K}} f = 0 \quad \Rightarrow \quad f(\zeta, \bar{\zeta}) = g(Z), \quad \text{where} \quad Z = \zeta\bar{\zeta}.$$

Now $\mathrm{Re}(\alpha_{,\zeta}) = 0$, and therefore

$$\underset{2}{\mathbf{K}} P = 0, \quad \Rightarrow \quad P = P(Z),$$

and

$$\tfrac{1}{2}K = P^2(\ln P)_{,\zeta\bar{\zeta}} = PP_{,\zeta\bar{\zeta}} - P_{,\zeta}P_{,\bar{\zeta}} = Z_0 \quad \Rightarrow \quad P = Z + Z_0,$$

after a $\Phi(\zeta)$-coordinate transformation. Note that the form of the metric is invariant under the transformation

$$r = A_0 r^*, \quad u = A_0^{-1} u^*, \quad \zeta = A_0 \zeta^*,$$
$$Z_0 = A_0^{-2} Z_0^*, \quad m_0 = A_0^{-3} m_0^*, \tag{7.0.3}$$

where A_0 is a constant, and therefore Z_0 is a disposable constant. We will choose it later.

The general solution of (7.0.2) for L and Σ is

$$L = i\bar{\zeta}P^{-2}B(Z), \quad \Sigma = ZB' - (1 - Z_0 P^{-1})B,$$

where $B' = dB/dZ$. The complex "mass", $m + iM$, is an analytic function of ζ from (6.0.4), and is also a function of Z from (7.0.2). It must therefore be a constant,

$$m + iM = \mu_0 = m_0 + iM_0.$$

Substituting this into (6.0.3), the equation for Σ,

$$\Sigma K + \nabla^2 \Sigma = M = M_0 \quad \longrightarrow$$
$$P^2[Z\Sigma'' + \Sigma'] + 2Z_0\Sigma = M_0.$$

The complete solution to this is

$$\Sigma = C_0 + \frac{Z - Z_0}{Z + Z_0}[-a + C_2 \ln Z],$$

where $\{C_0, a, C_2\}$ are arbitrary constants. This gave a four-parameter metric when these known functions are substituted into Eqs. (6.0.1),(6.0.2). However, if C_2 is nonzero then the final metric is singular at $R = 0$, and it was therefore omitted in Kerr (1963). The "imaginary mass" is then $M = 2Z_0C_0$ and so C_0 is a multiple of the NUT parameter. It was known in 1963 that the metric cannot be asymptotically flat if this is nonzero and so it was also omitted. The only constants retained were m_0, a and Z_0. When a is zero and Z_0 is positive the metric is that of Schwarzschild. It was not clear that the metric would be physically interesting when $a \neq 0$, but if it had not been so then this whole exercise would have been futile.

The curvature of the 2-metric $2P^{-2}d\zeta d\bar{\zeta}$ needs to have the same sign as Schwarzschild if the metric is asymptotically flat, and so $Z_0 = +P_0^2$. The basic functions in the metric are

$$Z_0 = P_0^2, \quad P = \zeta\bar{\zeta} + P_0^2, \quad m = m_0, \quad M = 0,$$
$$L = ia\bar{\zeta}P^{-2}, \quad \Sigma = -a\frac{\zeta\bar{\zeta} - Z_0}{\zeta\bar{\zeta} + Z_0}.$$

The metric was originally published in spherical polar coordinates. The relationship between these and the $(\zeta, \bar{\zeta})$ coordinates is

$$\zeta = P_0 \cot\tfrac{\theta}{2} e^{i\phi}.$$

At this point we choose A_0 in the transformation (7.0.3) so that

$$2P_0^2 = 1, \quad \Rightarrow \quad k = du + a\sin\theta d\phi$$

From equations (6.0.1) and (6.0.2),

$$ds^2 = ds_0^2 + 2mr/(r^2 + a^2\cos^2\theta)k^2 \tag{7.0.4}$$

where $m = m_0$, a constant, and

$$\begin{aligned} ds_0^2 =& (r^2 + a^2\cos^2\theta)(d\theta^2 + \sin^2\theta d\phi^2) \\ &-(2dr + du - a\sin^2\theta d\phi)(du + a\sin\theta d\phi). \end{aligned} \tag{7.0.5}$$

This is the original form of Kerr (1963), except that u has been replaced by $-u$ to agree with current conventions, and a has been replaced with its negative.[3]

At this point I had everything I could to find a physical solution. Assuming the metric was algebraically special had reduced the usual Einstein equations to PDEs with three independent variables, $\{u, \zeta, \bar{\zeta}\}$. The assumption that the metric was stationary, axially symmetric and asymptotically flat had eliminated two of these, leaving some ODEs that had fairly simple solutions with several arbitrary constants. Some of these were eliminated by the assumption that the metric was asymptotically flat, leaving Schwarzschild with one extra parameter. This did not seem much considering where it had started from.

Having found this fairly simple metric, I was desperate to see whether it was rotating. Fortunately, I knew that the curvature of the base metric, ds_0^2, was zero, and so it was only necessary to find coordinates where this metric

[3]We will see why later.

was manifestly Minkowskian. These were

$$(r + ia)e^{i\phi}\sin\theta = x + iy, \quad r\cos\theta = z, \quad r + u = -t.$$

This gave the Kerr-Schild form of the metric,

$$\begin{aligned} ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2mr^3}{r^4 + a^2z^2}[dt + \frac{z}{r}dz \\ + \frac{r}{r^2 + a^2}(xdx + ydy) - \frac{a}{r^2 + a^2}(xdx - ydy)]^2. \end{aligned} \quad (7.0.6)$$

where the surfaces of constant r are confocal ellipsoids of revolution about the z-axis,

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1. \quad (7.0.7)$$

Asymptotically, r is just the distance from the origin in the Minkowskian coordinates, and the metric is clearly asymptotically flat.

Angular momentum

The morning after the metric had been put into its Kerr-Schild form I went to Alfred Schild and told him I was about to calculate the angular momentum of the central body. He was just as eager as me to see whether this was nonzero, and so he joined me in my office while I computed. We were excessively heavy smokers at that time, so you can imagine what the atmosphere was like, Alfred puffing away at his pipe in an old arm chair, and myself chain-smoking cigarettes at my desk.

The Kerr-Schild form was ideal for calculating the physical parameters of the solution. As was said in the introduction, my PhD thesis at Cambridge was entitled "Equations of Motion in General Relativity." Because of this previous work I was well aware how to calculate the angular momentum in this new metric.

It was first expanded in powers of R^{-1}, where $R = x^2 + y^2 + z^2$ is the usual Euclidean distance from the origin, the center of the source,

$$\begin{aligned} ds^2 =& dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{R}(dt + dR)^2 \\ & - \frac{4ma}{R^3}(xdy - ydx)(dt + dR) + O(R^{-3}) \end{aligned} \quad (7.0.8)$$

Now, if $x^\mu \to x^\mu + a^\mu$ is an infinitesimal coordinate transformation, then

$ds^2 \to ds^2 + 2da_\mu dx^\mu$. If we choose

$$\begin{aligned} a_\mu dx^\mu &= -\frac{am}{R^2}(xdy - ydx) \quad \Rightarrow \\ 2da_\mu dx^\mu &= -4m\frac{4am}{R^3}(xdy - ydx)dR, \end{aligned}$$

then the approximation in (7.0.8) simplifies to

$$\begin{aligned} ds^2 =& dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{R}(dt + dR)^2 \\ & - \frac{4ma}{R^3}(xdy - ydx)dt + O(R^{-3}). \end{aligned} \tag{7.0.9}$$

The leading terms in the linear approximation for the gravitational field around a rotating body were well known at that time (for instance, see Papapetrou (1974) or Kerr (1960)). The contribution from the angular momentum vector, **J**, is

$$4R^{-3}\epsilon_{ijk}J^i x^j dx^k dt.$$

A comparison of the last two equations showed that the physical parameters were[4]

$$\text{Mass} = m, \quad \mathbf{J} = (0, 0, ma).$$

When I turned to Alfred Schild, who was still sitting in the arm-chair smoking away, and said "Its rotating!" he was even more excited than I was. I do not remember how we celebrated, but celebrate we did!

Robert Boyer subsequently calculated the angular momentum by comparing the known Lenze-Thirring results for frame dragging around a rotating object in linearized relativity with the frame dragging for a circular orbit in a Kerr metric. This was a very obtuse way of calculating the angular momentum since the approximation (7.0.9) was the basis for the calculations by Lenze and Thirring, but it did show that the sign was wrong in the original paper!

[4]Unfortunately, I was rather hurried when performing this calculation and got the sign wrong. This is why the sign of the parameter a in Kerr (1963) is different to that in all other publications, including this one. This way of calculating **J** was explained at the First Texas Symposium (see Ref. (20)) at the end of 1963.

8. Singularities and Topology

The first Texas Symposium on Relativistic Astrophysics was held in Dallas December 16-18, 1963, just a few months after the discovery of the rotating solution. It was organized by a combined group of Relativists and Astrophysicists and its purpose was to try to find an explanation for the newly discovered quasars. The source 3C273B had been observed in March and was thought to be about a million million times brighter than the sun.

It had been long known that a spherically symmetric body could collapse inside an event horizon to become what was to be later called a black hole by John Wheeler. However, the Schwarzschild solution was non-rotating and it was not known what would happen if rotation was present. I presented a paper called "Gravitational collapse and rotation" in which I outlined the Kerr solution and said that the topological and physical properties of the event horizon may change radically when rotation is taken into account. It was not known at that time that Kerr was the only possible stationary solution for such a rotating black hole and so I discussed it as an example of such an object and attempted to show that there were two event horizons for $a < m$.

Although this was not pointed out in the original letter, Kerr (1963), the geometry of this metric is even more complicated than the Kruskal extension of Schwarzschild. It is nonsingular everywhere, except for the ring

$$z = 0, \qquad x^2 + y^2 = a^2.$$

As we'll see in the next section on Kerr-Schild metrics, the Weyl scalar, $\Psi_2 \to \infty$ near these points and so the points on the ring are true singularities, not just coordinate ones. Furthermore, this ring behaves like a branch point in the complex plane. If one travels on a closed curve that threads the ring the initial and final metrics are different: r changes sign. Equation (7.0.7) has one nonnegative root for r^2, and therefore two real roots, $r_\pm$, for r. These coincide where $r^2 = 0$, i.e., on the disc D bounded by the ring singularity

$$\mathrm{D}: \quad z = 0, \quad x^2 + y^2 \le a^2.$$

The disc can be taken as a branch cut for the analytic function r. We have to take two spaces, E_1 and E_2 with the topology of R^4 less the disc D. The points above D in E_1 are joined to the points below D in E_2 and vice versa. In E_1 $r > 0$ and the mass is positive at infinity; in E_2 $r < 0$ and the mass is negative. The metric is then everywhere analytic except on the ring.

It was trivially obvious to everyone that if the parameter a is very much less than m then the Schwarzschild event horizon at $r = 2m$ will be modified slightly but cannot disappear. For instance, the light cones at $r = m$ in Kerr all point inwards for small a. Before I went to the meeting I had calculated the behavior of the time like geodesics up and down the axis of rotation and found that horizons occurred at the points on the axis in E_1 where

$$r^2 - 2mr + a^2 = 0, \quad |\zeta| = 0, \quad r = |z|.$$

but that there are no horizons in E_2 where the mass is negative. In effect, the ring singularity is "naked" in that sheet.

I made a rather hurried calculation of the two event horizons in E_1 before I went to the Dallas Symposium and claimed incorrectly there (20) that the equations for them were the two roots of

$$r^4 - 2mr^3 + a^2 z^2 = 0,$$

whereas z^2 should be replaced by r^2 in this and the true equation is

$$r^2 - 2mr + a^2 = 0.$$

This calculation was carried out using inappropriate coordinates and assuming that the equation would be: "$\psi(r, z)$ is null for some function of both r and z." I did not realize at the time that this function depended only on r. The Kerr-Schild coordinates are a generalization of the Eddington-Finkelstein coordinates for Schwarzschild. For the latter, future-pointing radial geodesics are well behaved but not those traveling to the past. Kruskal coordinates were designed to handle both. Similarly for Kerr, the coordinates given here only handle ingoing curves. This metric is known to be Type D and therefore it has another set of Debever-Penrose vectors and an associated coordinate system for which the outgoing geodesics are well behaved, but not the ingoing ones.

The metric in Kerr-Schild form consists of three blocks, outside the outer event horizon, between the two horizons and within the inner horizon (at least for $m < a$, which is probably true for all existing black holes). Just as Kruskal extends Schwarzschild by adding extra blocks, Boyer and Lindquist (1967) and Carter (1968) independently showed that the maximal extension of Kerr has a similar proliferation of blocks. However, the Kruskal extension has no application to a real black hole formed by the collapse of a spherically symmetric body and the same is true for Kerr. In fact, even what I call E_2, the sheet where the mass is negative, is probably irrelevant for the final state of a collapsing rotating object.

Ever since this metric was first discovered people have tried to fit an interior solution. One morning during the summer of 1964 Ray Sachs and myself decided that we would try to do so. Since the original form is useless and the Kerr-Schild form was clearly inappropriate we started by transforming to the canonical coordinates for stationary axisymmetric solutions.

Papapetrou (27) gave a very elegant treatment of stationary axisymmetric Einstein spaces. He shows that if there is a real non-singular axis of rotation then the canonical coordinates can be chosen so that there is only one off-diagonal component of the metric. Such a metric has been called quasi-diagonalisable. All cross terms between $\{dr, d\theta\}$ and $\{dt, d\phi\}$ can be eliminated by transformations of the type

$$dt' = dt + Adr + Bd\theta, \quad d\phi' = d\phi + Cdr + Dd\theta.$$

where the coefficients can be found algebraically. Papapetrou proved that dt' and $d\phi'$ are perfect differentials if the axis is regular.[1]

Ray and I calculated the coefficients $A \ldots D$, finding that

$$\begin{aligned} dt &\to dt + \frac{2mr}{\Delta}dr \\ d\phi &\to -d\phi + \frac{a}{\Delta}dr, \\ \Delta &= r^2 - 2mr + a^2, \end{aligned}$$

where, as before, $u = -(t + r)$. The right hand sides of the first two equations are clearly perfect differentials as the Papapetrou analysis showed. We then transformed the metric to the Boyer-Lindquist form,

$$\begin{aligned} ds^2 =& \frac{\Theta}{\Delta}dr^2 - \frac{\Delta}{\Theta}[dt - a\sin^2\theta d\phi]^2 + \\ & \Theta d\theta^2 + \frac{\sin^2\theta}{\Theta}[(r^2 + a^2)d\phi - adt]^2, \end{aligned} \tag{8.0.1}$$

where

$$\Theta = r^2 + a^2\cos^2\theta.$$

Having derived this canonical form, we studied the metric for a rather short time and then decided that we had no idea how to introduce a reasonable source into a metric of this form. Presumably those who have tried to solve this problem in the last 43 years have had similar reactions. Soon after this failed attempt Robert Boyer came to Austin. He said to me that he had found a new quasi-diagonalized form of the metric. I said "Yes. It is the one with the polynomial $r^2 - 2mr + a^2$" but for some reason he refused to believe that we had also found this form. Since it did not seem a "big deal" I

[1]It is shown in Kerr and Weir (1976) that if the metric is also algebraically special then it is quasi-diagonalisable precisely when it is Type D. These metrics include the NUT parameter generalization of Kerr.

did not pursue it further, but our relations were hardly cordial after that.

One of the main advantages of this form is that the event horizons can be easily calculated since the inverse metric is simple. If $f(r,\theta) = 0$ is a null surface then

$$\Delta(r) f_{,rr} + f_{,\theta\theta} = 0,$$

and therefore $\Delta \leq 0$. The two event horizons are the surfaces $r = r_\pm$ where the parameters $r_\pm$ are the roots of $\Delta = 0$,

$$\Delta = r^2 - 2mr + a^2 = (r - r_+)(r - r_-).$$

If $a < m$ there are two distinct horizons between which all time-like lines point inwards; if $a = m$ there is only one event horizon; and for larger a the singularity is bare! Presumably, any collapsing star can only form a black hole if the angular momentum is small enough, $a < m$. This seems to be saying that the body cannot rotate faster than light, if the final picture is that the mass is located on the ring radius a. However, it should be remembered that this radius is purely a coordinate radius, and that the final stage of such a collapse cannot have all the mass located at the singularity.

The reason for the last statement is that if the mass were to end on the ring there would be no way to avoid the second asymptotically flat sheet where $r < 0$ and the mass appears negative. I do not believe that the body opens up like this along the axis of rotation.

What I believe to be more likely is that the inner event horizon only forms asymptotically. As the body continues to collapse inside its event horizon it spins faster and faster so that the geometry in the region between its outer surface and the outer event horizon approaches that between the two event horizons for Kerr. The surface of the body will appear to be asymptotically null. Many theorems have been claimed stating that a singularity must exist if certain conditions are satisfied, but these include assumptions such as null geodesic completeness that may not be true for collapse to a black hole. Furthermore, these assumptions are often unstated or unrecognised, and the proofs are dependent on other claims/theorems that may not be correct.

This would be evidence supporting the Penrose conjecture that nature abhors naked singularities. When $a > m$ a classical rotating body will collapse until the centripetal forces balance the Newtonian (and other non-gravitational) ones. When the angular momentum is high the body will be in equilibrium before an event horizon forms, when it is low it will reach equilibrium after the formation. When the nonlinear effects are included this balance will be more complicated but the result will be the same, and no singularity can form.

The interior behaviour is still a mystery after more than four decades. It is also the main reason why I said at the end of Kerr (1963) that "It would be desirable to calculate an interior solution . . . ." This statement has been taken by some to mean that I thought the metric only represented a real rotating star. This is untrue and is an insult to all those relativists of that era who had been looking for such a metric to see whether the event horizon of Schwarzschild would generalize to rotating singularities.

The metric was known to be Type D with two distinct geodesic and shear-free congruences from the moment it was discovered. This means that if the other congruence is used instead of k then the metric must have the same form, i.e., it is invariant under a finite transformation that reverses "time" and possibly the axis of rotation in the appropriate coordinates. There had to be an extension that was similar to the Kruskal-Szekeres extension of Schwarzschild. Both Boyer and Lindquist (1967) and a fellow Australasian, Brandon Carter (1968), solved the problem of constructing the maximal extensions of Kerr, and even that for charged Kerr. These extensions are mathematically fascinating and the second paper is a particularly beautiful analysis of the problem, but the final result is of limited physical significance.

Brandon Carter's (1968) paper was one of the most significant papers on the Kerr metric during the mid sixties for another reason. He showed that there is an extra invariant for geodesic motion which is quadratic in the momentum components: $J = X_{ab}v^a v^b$ where X_{ab} is a Killing tensor, $X_{(ab;c)} = 0$. This gave a total of four invariants with the two Killing vector invariants and $|\mathbf{v}|^2$ itself, enough to generate a complete first integral of the geodesic equations. This has been used in countless papers on the motion of small bodies near a black hole.

The other significant development of the sixties was the proof that this is the only stationary metric with a simply connected bounded event horizon, i.e. the only possible black hole. The first paper on this was by another New Zealander, David Robinson (23).

9. Kerr-Schild metrics

Sometime around the time of the First Texas Symposium in December 1963, I tried to generalize the way that the field equations split for the Kerr metric by assuming the form

$$ds^2 = ds_0^2 + \frac{2mr}{r^2 + \Sigma^2} k^2,$$

where ds_0^2 is an algebraically special metric with $m = 0$. From an initial rough calculation ds_0^2 had to be flat. Also, it seemed that the canonical coordinates could be chosen so that $\partial L = L_\zeta - LL_u = 0$. The final metric depended on an arbitrary analytic function of the complex variable ζ, one which was simply $ia\zeta$ for Kerr. At this point I lost interest since the metric had to be singular at the poles of the analytic function unless this function was quadratic and the metric could then be rotated to Kerr.

Sometime after the Texas Symposium, probably during the Christmas break, Jerzy Plebanski visited Austin. Alfred Schild gave one of his many excellent parties for Jerzy during which I heard them talking about solutions of the Kerr-Schild type, $ds_0^2 + \lambda k^2$ where the first term is flat and k is any null vector. I commented that I thought that I knew of some algebraically special spaces that were of this type and that depended on an arbitrary function of a complex variable but that the result had not been checked.

At this point Alfred and I retired to his home office and calculated the simplest field equation, $R_{ab}k^a k^b = 0$. To our surprise this showed that the null vector had to be geodesic. We then calculated $k_{[a}R_{b]pq[c}k_{d]}k^p k^q$, found it to be zero and deduced that all metrics of this type had to be algebraically special and therefore might already be known. We checked my original calculations the next day and found them to be correct, so that all of these metrics are generated by a single analytic function.

As was stated in Theorem 2, m is a unique function of P and L unless there is a canonical coordinate system where m is linear in u and (L, P) are functions of $\zeta, \bar{\zeta}$ alone. If the base space is flat then $m_{,u} = c = 0$ and the metric is stationary. The way these metrics were found originally was by showing that in a coordinate system where $P = 1$ the canonical coordinates could be chosen so that $\partial L = 0$. Transforming from these coordinates to ones where $P \neq 1$ and ∂_u is a Killing vector

$$P_{,\zeta\zeta} = 0, \quad L = P^{-2}\bar{\phi}(\bar{\zeta}), \tag{9.0.1}$$

where $\phi(\zeta)$ is analytic. From the first of these P is a real bilinear function of

Y and therefore of $\bar{Y}$

$$P = p\zeta\bar{\zeta} + q\zeta + \bar{q}\bar{\zeta} + c.$$

This can be simplified to one of three canonical forms, $P = 1, 1 \pm \zeta\bar{\zeta}$ by a linear transformation on ζ. We will assume henceforth that

$$P = 1 + \zeta\bar{\zeta}.$$

The only problem was that this analysis depended on results for algebraically special metrics and these had not been published and would not be for several years. We had to derive the same results by a more direct method. The metric was written as

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + hk^2, \tag{9.0.2a}$$
$$k = (du + \bar{Y}d\zeta + Yd\bar{\zeta} + Y\bar{Y}dv)/(1 + Y\bar{Y}), \tag{9.0.2b}$$

where Y is the original coordinate ζ used in (9.0.1) and is essentially the ratio of the two components of the spinor corresponding to $\boldsymbol{k}$. Also[1]

$$u = z + t, \quad v = z - t, \quad \zeta = x + iy.$$

Each of these spaces has a symmetry which is also a translational symmetry for the base Minkowski space, ds_0^2. The most interesting metric is when this is time-like and so we will assume that the metric is independent of $t = \frac{1}{2}(u - v)$.

If $\phi(Y)$ is the same analytic function as in (9.0.1) then Y is determined as a function of the coordinates by

$$Y^2\bar{\zeta} + 2zY - \zeta + \phi(Y) = 0 \tag{9.0.3}$$

and the coefficient of k^2 in (9.0.2) is

$$h = 2m\mathrm{Re}(2Y_\zeta), \tag{9.0.4}$$

where m is a real constant. Differentiating (9.0.3) with respect to ζ gives

$$Y_\zeta = (2Y\bar{\zeta} + 2z + \phi')^{-1}. \tag{9.0.5}$$

Also, the Weyl spinor invariant is given by

$$\Psi_2 = c_0 m Y_\zeta^3,$$

where c_0 is some power of 2, and the metric is therefore singular precisely where Y is a repeated root of its defining equation (9.0.3).

[1] Note that certain factors of $\sqrt{2}$ have been omitted to simplify the results. This leads to an extra factor 2 appearing in (9.0.4).

If the k-lines are projected onto the Euclidean 3-space $t = 0$ with $\{x, y, z\}$ as coordinates so that $ds_E^2 = dx^2 + dy^2 + dz^2$, then the perpendicular from the origin meets the projected k-line at the point

$$F_0 : \quad \zeta = \frac{\phi - Y^2\bar{\phi}}{P^2}, \quad z = -\frac{\bar{Y}\phi + Y\bar{\phi}}{P^2},$$

and the distance of the line from the origin is

$$D = \frac{|\phi|}{1 + Y\bar{Y}},$$

a remarkably simple result. This was used by Kerr and Wilson (26) to prove that unless ϕ is quadratic the singularities are unbounded and the spaces are not asymptotically flat. The reason why I did not initially take the general Kerr-Schild metric seriously was that this was what I expected.

Another point that is easily calculated is Z_0 where the line meets the plane $z = 0$,

$$Z_0 : \quad \zeta = \frac{\phi + Y^2\bar{\phi}}{1 - (Y\bar{Y})^2}, \quad z = 0,$$

The original metric of this type is Kerr where

$$\phi(Y) = -2iaY, \quad D = \frac{2|a||Y|}{1 + |Y|^2} \le |a|.$$

If $\phi(Y)$ is any other quadratic function then it can be transformed to the same value by using an appropriate Euclidean rotation and translation about the t-axis. The points F_0 and Z_0 are the same for Kerr, so that F_0 lies in the z-plane and the line cuts this plane at a point inside the singular ring provided $|Y| \neq 1$. The lines where $|Y| = 1$ are the tangents to the singular ring lying entirely in the plane $z = 0$ outside the ring. When $a \to 0$ the metric becomes Schwarzschild and all the Y-lines pass through the origin.

When $\phi(Y) = -2iaY$ (9.0.3) becomes

$$Y^2\bar{\zeta} + 2(z - ia)Y - \zeta = 0.$$

There are two roots, Y_1 and Y_2 of this equation,

$$Y_1 = \frac{r\zeta}{(z + r)(r - ia)}, \quad 2Y_{1,\zeta} = +\frac{r^3 + iarz}{r^4 + a^2z^2}$$
$$Y_2 = \frac{r\zeta}{(z - r)(r + ia)}, \quad 2Y_{2,\zeta} = -\frac{r^3 + iarz}{r^4 + a^2z^2}$$

where r is a real root of (7.0.7). This is a quadratic equation for r^2 with only one nonnegative root and therefore two real roots differing only by sign, $\pm r$. When these are interchanged, $r \leftrightarrow -r$, the corresponding values for Y are also swapped, $Y_1 \leftrightarrow Y_2$.

When Y_2 is substituted into the metric then the same solution is returned except that the mass has changed sign. This is the other sheet where r has become negative. It is usually assumed that Y is the first of these roots, Y_1. The coefficient h of k^2 in the metric (9.0.2) is then

$$h = 2m\mathrm{Re}(2Y_{\zeta}) = \frac{2mr^3}{r^4 + a^2z^2}.$$

This gives the metric in its KS form (7.0.6).

The results were published in two places (24; 25). The first of these was a talk that Alfred gave at the Galileo Centennial in Italy; the second was an invited talk that I gave, but Alfred wrote, at the Symposium on Applied Mathematics of the American Mathematical Society, April 25, 1964. The manuscript had to be provided before the conference so that the participants had some chance of understanding results from distant fields. On page 205 we state

"Together with their graduate student, Mr. George Debney, the authors have examined solutions of the nonvacuum Einstein-Maxwell equations where the metric has the form (2.1).[2] Most of the results mentioned above apply to this more general case. This work is continuing."

Charged Kerr

What was this quote referring to? When we had finished with the Kerr-Schild metrics, we looked at the same problem with a nonzero electromagnetic field. The first stumbling block was that $R_{ab}k^ak^b = 0$ no longer implied that the k-lines are geodesic. The equations were quite intractable without this and so it had to be added as an additional assumption. It then followed that the principal null vectors were shearfree, so that the metrics had to be algebraically special. The general forms of the gravitational and electromagnetic fields were calculated from the easier field equations. The E-M field proved to depend on two functions called A and γ in Debney, Kerr and Schild (18).

When $\gamma = 0$ the remaining equations are linear and similar to those for the purely gravitational case. They were readily solved giving a charged generalization of the original Kerr-Schild metrics. The congruences are the same as for the uncharged metrics, but the coefficient of k^2 is

$$h = 2m\mathrm{Re}(2Y_{,\zeta}) - |\psi|^2|2Y_{,\zeta}|^2. \tag{9.0.6}$$

where $\psi(Y)$ is an extra analytic function generating the electromagnetic field.

[2] Eq. (9.0.2) in this paper. It refers to the usual Kerr-Schild ansatz.

The latter is best expressed through a potential,

$$f = \tfrac{1}{2}F_{\mu\nu}dx^\mu dx^\nu = -d\alpha,$$
$$\alpha = -P(\psi Z + \bar{\psi}\bar{Z})k - \tfrac{1}{2}(\chi d\bar{Y} + \bar{\chi}dY),$$

where

$$\chi = \int P^{-2}\psi(Y)dY,$$

$\bar{Y}$ being kept constant in this integration.

The most important member of this class is charged Kerr. For this,

$$h = \frac{2mr^3 - |\psi(Y)|^2 r^2}{r^4 + a^2z^2}. \tag{9.0.7}$$

Asymptotically, $r = R$, $k = dt - dR$ is a radial null-vector and $Y = \tan(\frac{1}{2}\theta)e^{i\phi}$. If the analytic function $\psi(Y)$ is nonconstant then it must be singular somewhere on the unit sphere and so the gravitational and electromagnetic fields will be also. The only physically significant charged Kerr-Schild is therefore when ψ is a complex constant, $e + ib$. The imaginary part, b, can be ignored as it gives a magnetic monopole, and so we are left with $\psi = e$, the electric charge,

$$\begin{aligned} ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2mr^3 - e^2r^2}{r^4 + a^2z^2}[dt + \frac{z}{r}dz \\ + \frac{r}{r^2+a^2}(xdx + ydy) - \frac{a}{r^2+a^2}(xdx - ydy)]^2, \end{aligned} \tag{9.0.8}$$

The electromagnetic potential is

$$\alpha = \frac{er^3}{r^4 + a^2z^2}\left[dt - \frac{a(xdy - ydx)}{r^2+a^2}\right],$$

where a pure gradient has been dropped. The electromagnetic field is

$$\begin{aligned} &(F_{xt} - iF_{yz}, F_{yt} - iF_{zx}, F_{zt} - iF_{xy}) \\ &\qquad = \frac{er^3}{(r^2 + iaz)^3}(x, y, z + ia). \end{aligned}$$

In the asymptotic region this field reduces to an electric field,

$$\mathbf{E} = \frac{e}{R^3}(x, y, z),$$

and a magnetic field,

$$\mathbf{H} = \frac{ea}{R^5}(3xz, 3yz, 3z^2 - R^2).$$

This is the electromagnetic field of a body with charge e and magnetic moment $(0, 0, ea)$. The gyromagnetic ratio is therefore $ma/ea = m/e$, the same as that for the Dirac electron. This was first noticed by Brandon Carter and was something that fascinated Alfred Schild.

This was the stage we had got to before March, 1964. We were unable to solve the equations where the function γ was nonzero so we enlisted the help of our graduate student, George Debney. Eventually we all realized that we were never going to solve the more general equations and so I suggested to George that he drop this investigation. By this time the only interesting member of the charged Kerr-Schild class, charged Kerr, had been announced by Newman et al. (29). George then tackled the problem of finding all possible groups of symmetries in diverging algebraically special spaces. He succeeded very well with this, solving many of the ensuing field equations for these metrics. This work formed the basis for his PhD thesis and was eventually published in 1970 (17).

10. The Kerr-Schild Ansatz Reprised

The Kerr solution was originally obtained by a systematic study of algebraically special vacuum solutions (16). Its physical properties were not obvious in the original coordinates but it was noticed that it split into two parts, (see 7.0.6),

$$ds^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \equiv (\eta_{\alpha\beta} + hk_\alpha k_\beta)\mathrm{d}x^\alpha \mathrm{d}x^\beta \,, \tag{10.0.1}$$

where $\eta_{\alpha\beta}$ is the metric for Minkowski space and k_α is a null vector,

$$\eta_{\alpha\beta}k^\alpha k^\beta = g_{\alpha\beta}k^\alpha k^\beta = 0, \qquad k^\alpha = \eta^{\alpha\beta}k_\beta = g^{\alpha\beta}k_\beta. \tag{10.0.2}$$

From the identity

$$(\eta_{\alpha\gamma} + hk_\alpha k_\gamma)(\eta^{\gamma\beta} - hk^\gamma k^\beta) = \delta^\beta_\alpha \quad \longrightarrow \quad |g_{\alpha\beta}| = |\eta_{\alpha\beta}| \,,$$

the inverse metric is linear in h,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - hk^\alpha k^\beta, \tag{10.0.3}$$

and the determinant of the metric is independent of h,

This simple form (7.0.6) was used to show that the metric is asymptotically flat and that the constants m and a are the total mass and specific angular momentum for a localised source. Note that the mass parameter m appears linearly in the metric. In particular, the vector $\boldsymbol{k}$ is independent of m: it is a function of a alone.

At the end of 1963 Alfred Schild and Roy Kerr looked for empty Einstein spaces whose metrics satisfy the ansatz in (10.0.1-10.0.3)[1]. From

$$R_{\alpha\beta}k^\alpha k^\beta = -\tfrac{1}{2}h^2\dot{k}_\alpha\dot{k}^\alpha = 8\pi T_{\alpha\beta}k^\alpha k^\beta = 0, \tag{10.0.4}$$

[1]This ansatz was first studied by Trautman (30). His idea was that a gravitational wave should have the ability to propagate information, and that this can be achieved if both the covariant and the contravariant components of the metric tensor depend linearly on the same function H of the coordinates.

where a "dot" is used to denote differentiation in the $\boldsymbol{k}$ direction,

$$\dot{f} = \boldsymbol{k}(f) = f_{,\alpha}k^{\alpha},$$

the vector k^{α} had to be geodesic. It was then shown that $R_{\alpha\beta\gamma\delta}k^{\beta}k^{\gamma} = (\dots)k_{\alpha}k_{\delta}$. Using the Goldberg-Sachs theorem, the metric is algebraically special and k^{α} is shearfree as well as geodesic. This led to the Kerr-Schild metrics (24; 25). One important property of these metrics is that the field equations are linear in h. They are *exact linear perturbations* of Minkowski space.

Early in 1964 Kerr and Schild looked for all metrics of this same type that satisfy the Einstein-Maxwell equations. Unlike the uncharged case it seems that the null-vector $\boldsymbol{k}$ does not have to be geodesic. They were unable to solve the field equations for non-geodesic $\boldsymbol{k}$ so they were forced to assume that $\dot{\boldsymbol{k}} = 0 \to T_{\alpha\beta}k^{\alpha}k^{\beta} = 0$ as an additional assumption. From this they deduced that $F_{\alpha\beta}k^{\beta} = 0$ and that $\boldsymbol{k}$ is shearfree. They were still unable to solve the field equations completely even then and had to make a further assumption that one of the functions that arose during the integration process was zero.

Given these two extra assumptions they obtained a straightforward generalization of the uncharged Kerr-Schild metrics with the same flat-space null congruences. This included the charged Kerr-Newman metric but with a more general electromagnetic field involving an arbitrary complex function. Without these assumptions the equations are still unsolved forty five years later. This may not matter since the known metrics include all black hole ones, whether charged or not. For these the congruence and metric are given by 9.0.2 with h as in 9.0.6,

$$h = 2m\mathrm{Re}(2Y_{,\zeta}) - |\psi|^2|2Y_{,\zeta}|^2. \tag{10.0.5}$$

Modified Ansatz

The congruence of k-lines in the Kerr-Newman metrics depends only on the rotation parameter a and not on the mass m or charge e. Furthermore, the electromagnetic field is linear in e and the gravitational metric is linear in m and e^2. These black hole metrics can be thought of as "exact perturbations" of the background Minkowski space. If they are going to be generalised to other situations then we believe that it is this linearity that will show us how. We already know that the Kerr-Schild ansatz was not enough by itself for Einstein-Maxwell fields. Two other assumptions had to be made before the charged solutions were found.

We will now present an alternative derivation of the Kerr and Kerr-Newman metrics based just on this linearity property. We will start with the uncharged case and then add the electromagnetic field.

The ϵ-expansion

Let ϵ be an arbitrary constant parameter, eventually to be se equal to 1,

$$g_{\alpha\beta} = \eta_{\alpha\beta} - 2\epsilon H k_\alpha k_\beta, \tag{10.0.6}$$

and suppose that coordinates are chosen so that the components $\eta_{\alpha\beta}$ are constants. The connexion is then quadratic in ϵ,

$$\begin{aligned}
\Gamma^\gamma_{\alpha\beta} =& \epsilon \underset{1}{\Gamma}{}^\gamma_{\alpha\beta} + \epsilon^2 \underset{2}{\Gamma}{}^\gamma_{\alpha\beta} \, . \\
\underset{1}{\Gamma}{}^\gamma_{\alpha\beta} =& [(Hk_\alpha k_\beta)_{,}{}^{\gamma} - (Hk_\alpha k^\gamma)_{,\beta} - (Hk_\beta k^\gamma)_{,\alpha}], \\
\underset{2}{\Gamma}{}^\gamma_{\alpha\beta} =& H[H(\dot{k}_\alpha k_\beta + \dot{k}_\beta k_\alpha) + \dot{H} k_\alpha k_\beta] k^\gamma \, ,
\end{aligned}$$

where indices are raised and lowered with the Minkowski metric. We will use an "index" 0 to denote contraction with $\boldsymbol{k}$,

$$\begin{aligned}
\Gamma^0_{\alpha\beta} =& \Gamma^\gamma_{\alpha\beta} k_\gamma = \quad \epsilon (Hk_\alpha k_\beta)_{,0} \, , \\
\Gamma^\gamma_{\alpha 0} =& \Gamma^\gamma_{\alpha\beta} k^\beta = -\epsilon (Hk_\alpha k^\gamma)_{,0} \, , \\
\Gamma^\gamma_{00} =& \Gamma^\gamma_{\alpha\beta} k^\alpha k^\beta = 0 \, .
\end{aligned}$$

The determinant of the full metric is independent of ϵ,

$$|g_{\alpha\beta}| = |\eta_{\alpha\beta} - 2\epsilon H k_\alpha k_\beta| = |\eta_{\alpha\beta}| \qquad \longrightarrow \qquad \Gamma^\beta_{\alpha\beta} = 0 \, ,$$

and the contracted Riemann tensor therefore reduces to

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta} = \Gamma^\gamma_{\alpha\beta,\gamma} - \Gamma^\gamma_{\alpha\delta}\Gamma^\delta_{\beta\gamma} \tag{10.0.7}$$

The simplest component is

$$\begin{aligned}
R_{\alpha\beta} k^\alpha k^\beta &= \Gamma^\gamma_{\alpha\beta,\gamma} k^\alpha k^\beta - \Gamma^\gamma_{\delta 0} \Gamma^\delta_{\gamma 0} = \Gamma^\gamma_{00,\gamma} - 2\Gamma^\gamma_{\alpha 0} k^\alpha_{,\gamma} \\
&= -2\epsilon H^2 \dot{k}_\alpha \dot{k}^\alpha.
\end{aligned} \tag{10.0.8}$$

If the L.H.S. is zero then $|\dot{\boldsymbol{k}}| = 0$ and so $\dot{\boldsymbol{k}}$ is a null-vector orthogonal to another null-vector, $\boldsymbol{k}$. It must be parallel to $\boldsymbol{k}$ and therefore $\boldsymbol{k}$ is a geodesic vector.

If the Riemann, Einstein and energy-momentum tensors are expanded as series in ϵ,

$$R_{\alpha\beta} = \epsilon \underset{1}{R}{}_{\alpha\beta} + \epsilon^2 \underset{2}{R}{}_{\alpha\beta} + \epsilon^3 \underset{3}{R}{}_{\alpha\beta} + \epsilon^4 \underset{4}{R}{}_{\alpha\beta} \, , \tag{10.0.9}$$

$$G_{\alpha\beta} = \epsilon \underset{1}{G}{}_{\alpha\beta} + \epsilon^2 \underset{2}{G}{}_{\alpha\beta} + \epsilon^3 \underset{3}{G}{}_{\alpha\beta} + \epsilon^4 \underset{4}{G}{}_{\alpha\beta} \, , \tag{10.0.10}$$

$$T_{\alpha\beta} = \epsilon \underset{1}{T}{}_{\alpha\beta} + \epsilon^2 \underset{2}{T}{}_{\alpha\beta} + \epsilon^3 \underset{3}{T}{}_{\alpha\beta} + \epsilon^4 \underset{4}{T}{}_{\alpha\beta} \, , \tag{10.0.11}$$

then the Einstein equations can be written as

$$\underset{n}{G}_{\alpha\beta} = \underset{n}{T}_{\alpha\beta}\,, \quad n = 1...4\,. \tag{10.0.12}$$

Note that for both Einstein and Einstein-Maxwell fields $T_{\alpha\beta}$ is traceless and so the curvature scalar R is zero.

The highest components of the expansion for the Riemann tensor are

$$\begin{aligned} \underset{4}{R}_{\alpha\beta} &= -\underset{2}{\Gamma}^{\rho}_{\alpha\sigma}\underset{2}{\Gamma}^{\sigma}_{\beta\rho} = -4\dot{k}_{(\alpha}k_{\sigma)}k^{\rho}\dot{k}_{(\beta}k_{\rho)}k^{\sigma} = 0\,, \\ \underset{3}{R}_{\alpha\beta} &= -\underset{1}{\Gamma}^{\rho}_{\alpha\sigma}\underset{2}{\Gamma}^{\sigma}_{\beta\rho} - \underset{2}{\Gamma}^{\rho}_{\alpha\sigma}\underset{1}{\Gamma}^{\sigma}_{\beta\rho} = -2H^{3}|\dot{\boldsymbol{k}}|^{2}k_{\alpha}k_{\beta}\,, \end{aligned}$$

This means that if the source tensor is linear in ϵ then $\boldsymbol{k}$ has to be geodesic and so the full tensor $T_{\alpha\beta}k^{\alpha}k^{\beta} = 0$. It is well known that this is very restrictive on the types of sources that are possible. For instance, it precludes perfect fluid sources.

If the source for the gravitational field is an electromagnetic field, $F_{\alpha\beta}$ satisfying the usual equations, and if $\mathcal{F}_{\alpha\beta}$ is the self-dual complex form

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + \tfrac{1}{2}i\eta_{\alpha\beta\rho\sigma}F^{\rho\sigma} \tag{10.0.13}$$

where $\eta_{\alpha\beta\rho\sigma}$ is completely antisymmetric and $\eta_{1234} = i$, then the electromagnetic field equations are

$$\mathcal{F}^{\alpha\beta}{}_{;\beta} = \mathcal{F}^{\alpha\beta}{}_{,\beta} + \mathcal{F}^{\alpha\rho}\Gamma^{\beta}_{\rho\beta} = \mathcal{F}^{\alpha\beta}{}_{,\beta} = 0\,. \tag{10.0.14}$$

These are independent of the gravitational field since the determinant of the metric is constant and therefore $\Gamma^{\beta}_{\rho\beta} = 0$.

If $\boldsymbol{k}$ is an eigenvalue of the field, as it is for all known charged Kerr-Schild metrics, then

$$\mathcal{F}^{\alpha\beta}k_{\beta} = 0 \quad \longrightarrow \quad \mathcal{F}^{\alpha\rho}g_{\rho\beta} = \mathcal{F}^{\alpha\rho}\eta_{\rho\beta}\,. \tag{10.0.15}$$

and so the indices of $\mathcal{F}^{\alpha\beta}$ can be raised and lowered with the Minkowski metric. If it is linear in $\sqrt{\epsilon}$ the the corresponding energy-momentum tensor is proportional to ϵ. More accurately, the source for the gravitational field is linear in the square of the charge.

The next component of $R_{\alpha\beta}$ is

$$\begin{aligned} \underset{2}{R}_{\alpha\beta} = \underset{2}{\Gamma}^{\rho}_{\alpha\beta,\rho} - \underset{1}{\Gamma}^{\rho}_{\alpha\sigma}\underset{1}{\Gamma}^{\sigma}_{\beta\rho} &= H[(Hk_{\alpha}k_{\beta})_{,00} + k^{\sigma}_{,\sigma}(Hk_{\alpha}k_{\beta})_{,0} \\ &\quad - 2H\dot{k}_{\alpha}\dot{k}_{\beta} - 4Hk_{(\alpha}k_{\beta),\sigma}\dot{k}^{\sigma} + 4Hk_{\alpha}k_{\beta}k_{[\rho,\sigma]}k^{[\rho,\sigma]}], \end{aligned} \tag{10.0.16}$$

If $\boldsymbol{k}$ is geodesic then it can be normalised so that $\dot{\boldsymbol{k}} = 0$ and $\underset{2}{R}_{\alpha\beta}$ simplifies to

$$\underset{2}{R}_{\alpha\beta} = AHk_\alpha k_\beta \; , \quad A = H_{,00} + k^\rho{}_{,\rho} H_{,0} + 4k_{[\rho,\sigma]}k^{[\rho,\sigma]} \; . \tag{10.0.17}$$

The final component of the Riemann tensor expansion is

$$\underset{1}{R}_{\alpha\beta} = (Hk_a k_b)^{,\sigma}{}_{,\sigma} - (Hk_a k^\sigma)_{,\sigma b} - (Hk_b k^\sigma)_{,\sigma a} \tag{10.0.18}$$

Program

We are evaluating the expanded field equations for known sources that give Kerr-Schild type metrics. We believe that all useful metrics of this type are asymptotically flat and stationary. It may be that there are some solutions where $\boldsymbol{k}$ is not geodesic and shearfree but it seems very doubtful that they can be calculated.

Any geodesic and shearfree congruence in flat space,

$$k = (du + \bar{Y}d\zeta + Yd\bar{\zeta} + Y\bar{Y}dv)$$

must satisfy the Kerr Theorem, i.e. Y is a root of an analytic equation,

$$0 = F(Y, \bar{\zeta}Y + u, vY + \zeta) \; ,$$

where F is an arbitrary function analytic in the three complex variables Y, $\bar{\zeta}Y + u$ and $vY + \zeta$.

The hardest thing to prove for the original Kerr-Schild metrics was that the metrics were stationary and that this equation simplified to

$$Y^2\bar{\zeta} + 2zY - \zeta + \phi(Y) = 0$$

In fact this gives all such stationary congruences. We would like to see whether this result can be obtained in a simpler fashion and whether these metrics can be extended to other sources.

Appendix: Standard Notation

Let $\{\mathbf{e}_a\}$ and $\{\omega^a\}$ be dual bases for tangent vectors and linear 1-forms, respectively, i.e., $\omega^a(\mathbf{e}_b) = \delta^a_b$. Also let g_{ab} be the components of the metric tensor,

$$ds^2 = g_{ab}\omega^a\omega^a, \quad g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b.$$

The components of the connection in this frame are the Ricci rotation coefficients,

$$\Gamma^a{}_{bc} = -\omega^a{}_{\mu;\nu} e_b{}^\mu e_c{}^\nu, \qquad \Gamma_{abc} = g_{as}\Gamma^s{}_{bc},$$

The commutator coefficients $D^a{}_{bc} = -D^a{}_{cb}$ are defined by

$$[\mathbf{e}_b, \mathbf{e}_c] = D^a{}_{bc}\mathbf{e}_a, \quad \text{where} \quad [\mathbf{u}, \mathbf{v}](\mathrm{f}) = \mathbf{u}(\mathbf{v}(\mathrm{f})) - \mathbf{v}(\mathbf{u}(\mathrm{f}))$$

or equivalently by

$$d\omega^a = D^a{}_{bc}\omega^b \wedge \omega^c. \tag{10.0.19}$$

Since the connection is symmetric, $D^a{}_{bc} = -2\Gamma^a{}_{[bc]}$, and since it is metrical

$$\Gamma_{abc} = \tfrac{1}{2}(g_{ab|c} + g_{ac|b} - g_{bc|a} + D_{bac} + D_{cab} - D_{abc}),$$

$$\Gamma_{abc} = g_{am}\Gamma^m{}_{bc}, \qquad D_{abc} = g_{am}D^m{}_{bc}.$$

If it is assumed that the g_{ab} are constant, then the connection components are determined solely by the commutator coefficients and therefore by the exterior derivatives of the tetrad vectors,

$$\Gamma_{abc} = \tfrac{1}{2}(D_{bac} + D_{cab} - D_{abc}).$$

The components of the curvature tensor are

$$\Theta^a{}_{bcd} \equiv \Gamma^a{}_{bd|c} - \Gamma^a{}_{bc|d} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed} - D^e{}_{cd}\Gamma^a{}_{be}. \tag{10.0.20}$$

We must distinguish between the expressions on the right, the $\Theta^a{}_{bcd}$, and the curvature components, $R^a{}_{bcd}$, which the N-P formalism treat as extra variables, their (Ψ_i).

A crucial factor in the discovery of the spinning black hole solutions was the use of differential forms and the Cartan equations. The connection 1-forms $\mathbf{\Gamma}^a{}_b$ are defined as

$$\mathbf{\Gamma}^a{}_b = \Gamma^a{}_{bc}\omega^c.$$

These are skew-symmetric when $g_{ab|c} = 0$,

$$\mathbf{\Gamma}_{ba} = -\mathbf{\Gamma}_{ab}, \quad \mathbf{\Gamma}_{ab} = g_{ac}\mathbf{\Gamma}^{c}{}_{a}.$$

The first Cartan equation follows from (10.0.19),

$$d\omega^a + \mathbf{\Gamma}^a{}_b\omega^b = 0. \tag{10.0.21}$$

The curvature 2-forms are defined from the second Cartan equations,

$$\mathbf{\Theta}^a{}_b \equiv d\mathbf{\Gamma}^a{}_b + \mathbf{\Gamma}^a{}_c \wedge \mathbf{\Gamma}^c{}_b = \tfrac{1}{2}R^a{}_{bcd}\omega^c\omega^d. \tag{10.0.22}$$

The exterior derivative of (10.0.21) gives

$$\mathbf{\Theta}^a{}_b \wedge \omega^b = 0 \quad \Rightarrow \quad \Theta^a{}_{[bcd]} = 0,$$

which is just the triple identity for the Riemann tensor,

$$R^a{}_{[bcd]} = 0. \tag{10.0.23}$$

Similarly, from the exterior derivative of (10.0.22),

$$d\mathbf{\Theta}^a{}_b - \mathbf{\Theta}^a{}_f \wedge \mathbf{\Gamma}^f{}_b + \mathbf{\Gamma}^a{}_f \wedge \mathbf{\Theta}^f{}_b = 0,$$

that is

$$\mathbf{\Theta}^a{}_{b[cd;e]} \equiv 0, \quad \rightarrow \quad R^a{}_{b[cd;e]} = 0.$$

This equation says nothing about the Riemann tensor, $R^a{}_{bcd}$ directly. It says that certain combinations of the derivatives of the expressions on the right hand side of ((10.0.20)) are linear combinations of these same expressions.

$$\Theta_{ab[cd|e]} + D^s{}_{[cd}\Theta_{e]sab} - \Gamma^s{}_{a[c}\Theta_{de]sb} - \Gamma^s{}_{b[c}\Theta_{de]as} \equiv 0. \tag{10.0.24}$$

These are the true Bianchi identities. A consequence of this is that if the components of the Riemann tensor are thought of as variables, along with the components of the metric and the base forms, then these variables have to satisfy

$$R_{ab[cd|e]} = -2R^a_{be[c}\Gamma^e_{df]}. \tag{10.0.25}$$

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Bibliography

[1] Einstein, A., L. Infeld and B. Hoffmann (1938). The gravitational equations and the problem of motion, *Ann. Math.* **39**, 65–100.

[2] Einstein, A., and L. Infeld (1949). On the motion of particles in general relativity theory, *Can. J. Math.* **1**, 209.

[3] Kerr, R.P. (1960). On the quasi-static approximation method in general relativity, *Il Nuovo Cim.*, **X16**, 26.

[4] Kerr, R.P. (1958). The Lorentz-covariant approximation method in general relativity I, *Il Nuovo Cim.*, **X13**, 469.

[5] Kerr, R.P. (1958). The Lorentz-covariant approximation method in general relativity II, the second approximation, *Il Nuovo Cim.*, **X13**, 492.

[6] Kerr, R.P. (1959). The Lorentz-covariant approximation method in general relativity III, Einstein-Maxwell Fields, *Il Nuovo Cim.*, **X13**, 469.

[7] Petrov, A.Z. (1954). The classification of spaces defining gravitational fields, *Scientific Proceedings of Kazan State University* **114** 55, Translation by J. Jezierski and M.A.H. MacCallum, with introduction, by M.A.H. MacCallum, (2000) *Gen. Rel. Grav.* **32**, 1661.

[8] Pirani, F.A.E. (1957). Invariant formulation of gravitational radiation theory, *Phys. Rev.* **105**, 1089.

[9] Bondi, H., M.G.J. van der Burg and A.W.K. Metzner (1962). Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems. textitProc. Roy. Soc. Lond. A **269**, 21-52.

[10] Goldberg, J.N. and Kerr, R.P. (1961). Some applications of the infinitesimal-holonomy group to the Petrov classification of Einstein spaces *J. Math. Phys.* **2**, 327.

[11] Kerr, R.P. and Goldberg, J.N. (1961). Einstein spaces with four-parameter holonomy groups, *J. Math. Phys.* **2**, 332

[12] Goldberg, J.N. and Sachs, R.K. (1962) A theorem on Petrov Types, *Acta Phys. Polon., Suppl.* **22**, 13.

[13] Robinson, I. and Trautman, A. (1962) Some spherical gravitational waves in general relativity, *Proc. Roy. Soc. Lond.* **A 265**, 463.

[14] Robinson, I. and Trautman, A. (1964) Exact Degenerate Solutions of Einstein's Equations, in *Proceedings on Theory of Gravitation, Conference in Warszawa and Jablonna, 1962* ed. L. Infeld (Gauthier-Villars, Paris and PWN, Warszawa), 107–114

[15] Newman, E.T., Tambourino, L.A., and Unti, T. (1963). Empty-space generalization of the Schwarzschild metric. *J. Math. Phys.* **4**, p. 915.

[16] Kerr, R.P. (1963). Gravitational field of a spinning mass as an example of algebraically special metrics, *Phys. Rev. Lett.* **11**, 26.

[17] Kerr, R.P. and Debney, G.C. (1970). Einstein spaces with symmetry groups, *J. Math. Phys.* **11**, 2807.

[18] Debney, G.C., Kerr, R.P. and Schild, A. (1969). Solutions of the Einstein and Einstein-Maxwell equations, *J. Math. Phys.* **10**, 1842.

[19] Stephani, H.,Kramer, D., MacCallum, M., Hoenselaers, C. and Herlt, E. (2003). *Exact Solutions to Einstein's Field Equations*, Second Edition Cambridge University Press, Cambridge UK.

[20] Kerr, R.P., (1965). Gravitational collapse and rotation, *Quasi-stellar Sources and Gravitational Collapse, including the Proceedings of the First Texas Symposium on Relativistic Astrophysics*, Edited by I. Robinson, A. Schild and E.L. Schucking. The University of Chicago Press, Chicago and London, 99-109.

[21] Boyer, R. H. and Lindquist, R. W. (1967). Maximal Analytic Extension of the Kerr Metric, *J. Math. Phys.* **8**, 265–280.

[22] Carter, B. (1968). Global Structure of the Kerr Family of Gravitational Fields, *Phys. Rev.* **174**, 1559-1571.

[23] Robinson, D. C. (1975). Uniqueness of the Kerr black hole, *P*hys. Rev. Lett., **3**4, 905–906.

[24] Kerr, R.P. and Schild, A. (1965). A new class of vacuum solutions of the Einstein field equations, *Atti del Congregno Sulla Relativita Generale: Galileo Centenario*.

[25] Kerr, R.P. and Schild, A. (1965). Some algebraically degenerate solutions of Einstein's gravitational field equations, *Proc. Sym. in Applied Math.* **XVII**, Edited by R. Finn. American Math. Soc, Providence, Rhode Island, 199–209.

[26] Kerr, R.P. and Wilson, W.B. (1978). Singularities in the Kerr-Schild Metrics, *Gen. Rel. Grav.* **10**, 273.

[27] Papapetrou, A. (1974) *Lectures on General Relativity*, D.Reidel, Dordrecht and Boston, 116.

[28] Penrose, R. (1960). A spinor approach to general relativity, *Ann. Phys. (USA)* **10**, 171.

[29] Newman, E.T., Couch E., Chinnapared K., Exton A., Prakash A., and Torrence, R. (1965) Metric of a Rotating, Charged Mass. *J. Math. Phys.* **6**, 918–919.

[30] Trautman, A., On the propagation of information by waves, in *Recent developments in general relativity*, p. 459 (Pergamon Press, Oxford, 1962).

A. The Kerr-Schild ansatz revised

We include here the published paper "The Kerr-Schild ansatz revised," by D. Bini, A. Geralico, R. P. Kerr, submitted, 2009.

An alternative derivation of Kerr solution is presented by treating Kerr-Schild metrics as *exact linear perturbations* of Minkowski spacetime. In fact they have been introduced as a linear superposition of the flat spacetime metric and a squared null vector field $\boldsymbol{k}$ multiplied by a scalar function H. In the case of Kerr solution the vector $\boldsymbol{k}$ is geodesic and shearfree and it is independent of the mass parameter $\mathcal{M}$, which enters instead the definition of H linearly. This linearity property allows one to solve the field equations order by order in powers of H in complete generality, i.e. without any assumption on the null congruence $\boldsymbol{k}$. The Ricci tensor turns out to consist of three different contributions. Third order equations all imply that $\boldsymbol{k}$ must be geodesic; it must be also shearfree as a consequence of first order equations, whereas the solution for H comes from second order equations too.

A.1. Abstract

Kerr-Schild metrics have been introduced as a linear superposition of the flat spacetime metric and a squared null vector field, say $\boldsymbol{k}$, multiplied by some scalar function, say H. The basic assumption which led to Kerr solution was that $\boldsymbol{k}$ be both geodesic and shearfree. This condition is relaxed here and Kerr-Schild ansatz is revised by treating Kerr-Schild metrics as *exact linear perturbations* of Minkowski spacetime. The scalar function H is taken as the perturbing function, so that Einstein's field equations are solved order by order in powers of H. It turns out that the congruence must be geodesic and shearfree as a consequence of third and second order equations, leading to an alternative derivation of Kerr solution.

A.2. Introduction

Kerr-Schild metrics have the form [1, 2]

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \equiv (\eta_{\alpha\beta} - 2Hk_\alpha k_\beta)\mathrm{d}x^\alpha \mathrm{d}x^\beta \,, \qquad \text{(A.2.1)}$$

where $\eta_{\alpha\beta}$ is the metric for Minkowski space and k_α is a null vector

$$\eta_{\alpha\beta}k^\alpha k^\beta = g_{\alpha\beta}k^\alpha k^\beta = 0, \qquad k^\alpha = \eta^{\alpha\beta}k_\beta = g^{\alpha\beta}k_\beta \,. \tag{A.2.2}$$

The inverse metric is also linear in H

$$g^{\alpha\beta} = \eta^{\alpha\beta} + 2Hk^\alpha k^\beta \,, \tag{A.2.3}$$

and so the determinant of the metric is independent of H

$$(\eta_{\alpha\gamma} - 2Hk_\alpha k_\gamma)(\eta^{\gamma\beta} + 2Hk^\gamma k^\beta) = \delta^\beta_\alpha \quad \longrightarrow \quad |g_{\alpha\beta}| = |\eta_{\alpha\beta}| \,. \tag{A.2.4}$$

Within this class of general metrics the Kerr solution was obtained in 1963 by a systematic study of algebraically special vacuum solutions [3]. If $(x^0 = t, x^1 = x, x^2 = y, x^3 = z)$ are the standard Cartesian coordinates for Minkowski spacetime with $\eta_{\alpha\beta} = \text{diag}[-1,1,1,1]$, then for Kerr metric we have

$$-k_\alpha \mathrm{d}x^\alpha = \mathrm{d}t + \frac{(rx+ay)\mathrm{d}x + (ry-ax)\mathrm{d}y}{r^2+a^2} + \frac{z}{r}\mathrm{d}z \,, \tag{A.2.5}$$

where r and H are defined implicitly by

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1 \,, \qquad H = -\frac{\mathcal{M}r^3}{r^4+a^2z^2} \,. \tag{A.2.6}$$

Kerr solution is asymptotically flat and the constants $\mathcal{M}$ and a are the total mass and specific angular momentum for a localized source. They both have the dimension of a length in geometrized units. The vector $\boldsymbol{k}$ is geodesic and shearfree, implying that Kerr metric is algebraically special according to the Goldberg-Sachs theorem [4]. Moreover, $\boldsymbol{k}$ is independent of $\mathcal{M}$ and hence a function of a alone. Note that the mass parameter $\mathcal{M}$ appears linearly in the metric, i.e. in H.

In this paper we consider Kerr-Schild metrics (A.2.1) as *exact linear perturbations* of Minkowski space and solve Einstein's field equations order by order in powers of H. The results of this analysis will be that $\boldsymbol{k}$ must be geodesic and shearfree as a consequence of third and second order equations, leading to an alternative derivation of Kerr solution.

A.3. Modified ansatz

Let ϵ be an arbitrary constant parameter, eventually to be set equal to 1, so that the Kerr-Schild metric (A.2.1) reads

$$g_{\alpha\beta} = \eta_{\alpha\beta} - 2\epsilon Hk_\alpha k_\beta \,, \tag{A.3.1}$$

with inverse

$$g^{\alpha\beta} = \eta^{\alpha\beta} + 2\epsilon H k^\alpha k^\beta \,, \tag{A.3.2}$$

and suppose that coordinates are chosen so that the components $\eta_{\alpha\beta}$ are constants, but not necessarily of the form $\eta_{\alpha\beta} = \text{diag}[-1, 1, 1, 1]$. The connection is then quadratic in ϵ

$$\Gamma^\gamma{}_{\alpha\beta} = \epsilon \underset{1}{\Gamma}{}^\gamma{}_{\alpha\beta} + \epsilon^2 \underset{2}{\Gamma}{}^\gamma{}_{\alpha\beta} \,, \tag{A.3.3}$$

where

$$\begin{aligned}
\underset{1}{\Gamma}{}^\gamma{}_{\alpha\beta} &= -(Hk_\alpha k^\gamma)_{,\beta} - (Hk_\beta k^\gamma)_{,\alpha} + (Hk_\alpha k_\beta)_{,\lambda}\eta^{\lambda\gamma} \,, \\
\underset{2}{\Gamma}{}^\gamma{}_{\alpha\beta} &= 2H[H(\dot{k}_\alpha k_\beta + \dot{k}_\beta k_\alpha) + \dot{H} k_\alpha k_\beta] k^\gamma \equiv 2Hk^\gamma (Hk_\alpha k^\beta)^{\cdot} \,,
\end{aligned} \tag{A.3.4}$$

a "dot" denoting differentiation in the $\boldsymbol{k}$ direction, i.e. $\dot{f} = \boldsymbol{k}(f) = f_{,\alpha}k^\alpha$. Note that only the indices of $\boldsymbol{k}$ can be raised and lowered with the Minkowski metric. Hereafter we will use an "index" 0 to denote contraction with $\boldsymbol{k}$, i.e.

$$\begin{aligned}
\Gamma^0{}_{\alpha\beta} &= \Gamma^\gamma{}_{\alpha\beta} k_\gamma = \epsilon (Hk_\alpha k_\beta)^{\cdot} \,, \\
\Gamma^\gamma{}_{\alpha 0} &= \Gamma^\gamma{}_{\alpha\beta} k^\beta = -\epsilon (Hk_\alpha k^\gamma)^{\cdot} \,, \\
\Gamma^\gamma{}_{00} &= \Gamma^\gamma{}_{\alpha\beta} k^\alpha k^\beta = 0 \,, \\
\Gamma^0{}_{\alpha 0} &= \Gamma^\gamma{}_{\alpha\beta} k^\beta k_\gamma = 0 \,.
\end{aligned} \tag{A.3.5}$$

The determinant of the full metric is independent of ϵ

$$|g_{\alpha\beta}| = |\eta_{\alpha\beta} - 2\epsilon H k_\alpha k_\beta| = |\eta_{\alpha\beta}| = \text{const.} \qquad \longrightarrow \qquad \Gamma^\beta{}_{\alpha\beta} = 0 \,, \tag{A.3.6}$$

and the contracted Riemann tensor therefore reduces to

$$R_{\alpha\beta} = R^\gamma{}_{\alpha\gamma\beta} = \Gamma^\gamma{}_{\alpha\beta,\gamma} - \Gamma^\gamma{}_{\alpha\delta}\Gamma^\delta{}_{\beta\gamma} \,. \tag{A.3.7}$$

The simplest component is

$$\begin{aligned}
R_{\alpha\beta}k^\alpha k^\beta &= \Gamma^\gamma{}_{\alpha\beta,\gamma} k^\alpha k^\beta - \Gamma^\gamma{}_{\delta 0}\Gamma^\delta{}_{\gamma 0} = \Gamma^\gamma{}_{00,\gamma} - 2\Gamma^\gamma{}_{\alpha 0} k^\alpha{}_{,\gamma} \\
&= 2\epsilon H ||\dot{\boldsymbol{k}}||^2 \,.
\end{aligned} \tag{A.3.8}$$

In vacuum the LHS is zero, then $||\dot{\boldsymbol{k}}|| = 0$ and so $\dot{\boldsymbol{k}}$ is a null-vector orthogonal to another null-vector, $\boldsymbol{k}$. Hence $\dot{\boldsymbol{k}}$ must be parallel to $\boldsymbol{k}$ and therefore $\boldsymbol{k}$ is a geodesic vector.

The Ricci tensor expanded as series in ϵ is given by

$$R_{\alpha\beta} = \epsilon \underset{1}{R}{}_{\alpha\beta} + \epsilon^2 \underset{2}{R}{}_{\alpha\beta} + \epsilon^3 \underset{3}{R}{}_{\alpha\beta} + \epsilon^4 \underset{4}{R}{}_{\alpha\beta} \,. \tag{A.3.9}$$

The vacuum Einstein's equations $R_{\alpha\beta} = 0$ imply that contributions of all orders must vanish. Let us evaluate all such components.

The highest components of the expansion for the Ricci tensor are

$$\underset{4}{R}_{\alpha\beta} = -\underset{2}{\Gamma}^{\rho}{}_{\alpha\sigma}\underset{2}{\Gamma}^{\sigma}{}_{\beta\rho} = 0\,, \tag{A.3.10}$$

$$\underset{3}{R}_{\alpha\beta} = -\underset{1}{\Gamma}^{\rho}{}_{\alpha\sigma}\underset{2}{\Gamma}^{\sigma}{}_{\beta\rho} - \underset{2}{\Gamma}^{\rho}{}_{\alpha\sigma}\underset{1}{\Gamma}^{\sigma}{}_{\beta\rho} = 4H^3||\dot{\boldsymbol{k}}||^2 k_\alpha k_\beta\,. \tag{A.3.11}$$

The next component of $R_{\alpha\beta}$ is

$$\begin{aligned}\underset{2}{R}_{\alpha\beta} &= \underset{2}{\Gamma}^{\rho}{}_{\alpha\beta,\rho} - \underset{1}{\Gamma}^{\rho}{}_{\alpha\sigma}\underset{1}{\Gamma}^{\sigma}{}_{\beta\rho}\\ &= 2H\left[(Hk_\alpha k_\beta)^{\cdot\cdot} + k^{\sigma}{}_{,\sigma}(Hk_\alpha k_\beta)^{\cdot} - H\dot{k}_\alpha \dot{k}_\beta\right]\\ &\quad - H^2\Phi k_\alpha k_\beta - 2Hk_{(\alpha}\psi_{\beta)}\,,\end{aligned} \tag{A.3.12}$$

where

$$\Phi = 4\eta^{\gamma\lambda}\eta^{\delta\mu}k_{[\lambda,\delta]}k_{[\mu,\gamma]}\,, \qquad \psi_\alpha = 2\dot{k}^{\gamma}(Hk_\alpha)_{,\gamma}\,. \tag{A.3.13}$$

Finally, the first component of the Ricci tensor expansion is

$$\begin{aligned}\underset{1}{R}_{\alpha\beta} &= \underset{1}{\Gamma}^{\gamma}{}_{\alpha\beta,\gamma}\\ &= Ak_\alpha k_\beta + 2k_{(\alpha}B_{\beta)} + X_{\alpha\beta}\,,\end{aligned} \tag{A.3.14}$$

where

$$\begin{aligned}A &= \eta^{\lambda\gamma}H_{,\lambda\gamma}\,,\\ B_\beta &= -(Hk^{\gamma})_{,\gamma\beta} + \frac{1}{H}\eta^{\lambda\gamma}(H^2k_{\beta,\gamma})_{,\lambda}\,,\\ X_{\alpha\beta} &= -2H\left[(k_{(\alpha,\beta)}k^{\gamma})_{,\gamma} + k_{(\alpha,|\gamma|}k^{\gamma}{}_{,\beta)} - \eta^{\lambda\gamma}k_{\alpha,\gamma}k_{\beta,\lambda}\right]\\ &\quad - 2k^{\gamma}\left[H_{,(\alpha}k_{\beta),\gamma} + H_{,\gamma}k_{(\alpha,\beta)}\right]\\ &= -2H\left[\dot{k}_{(\alpha,\beta)} + k^{\gamma}{}_{,\gamma}k_{(\alpha,\beta)} - \eta^{\lambda\gamma}k_{\alpha,\gamma}k_{\beta,\lambda}\right]\\ &\quad - 2\dot{H}k_{(\alpha,\beta)} - 2H_{,(\alpha}\dot{k}_{\beta)}\,.\end{aligned} \tag{A.3.15}$$

A.3.1. Kinematical properties of the congruence $\boldsymbol{k}$

Taking the covariant derivative of $\boldsymbol{k}$ gives

$$\nabla_\alpha k_\beta = k_{\beta,\alpha} - \epsilon(Hk_\alpha k_\beta)^{\cdot}\,, \tag{A.3.16}$$

so that its 4-acceleration is simply

$$a(k)_\beta = k^{\mu}\nabla_\mu k_\beta = \dot{k}_\beta\,. \tag{A.3.17}$$

The other optical scalars of interest are the expansion

$$\theta = \frac{1}{2}k^{\alpha}{}_{;\alpha} = \frac{1}{2}\eta^{\alpha\beta}k_{\beta,\alpha} = \frac{1}{2}k^{\alpha}{}_{,\alpha} \,, \tag{A.3.18}$$

the vorticity

$$\omega^2 = \frac{1}{2}k_{[\alpha;\beta]}k^{\alpha;\beta} = \frac{1}{2}k_{[\beta,\alpha]}\left(\eta^{\alpha\mu}\eta^{\beta\nu}k_{\mu,\nu} - 2\epsilon H\dot{k}^{\alpha}k^{\beta}\right) \,, \tag{A.3.19}$$

and the shear, implicitly defined by

$$\theta^2 + |\sigma|^2 = \frac{1}{2}k_{(\alpha;\beta)}k^{\alpha;\beta} = \frac{1}{2}k_{(\beta,\alpha)}\eta^{\alpha\mu}\eta^{\beta\nu}k_{\mu,\nu} - \frac{1}{2}\epsilon H||\dot{\boldsymbol{k}}||^2 \,. \tag{A.3.20}$$

A.3.2. First result: $\boldsymbol{k}$ be geodesic

The third order field equations (A.3.11) imply that $\boldsymbol{k}$ be geodesic. Then it can be normalized so that $\dot{\boldsymbol{k}} = 0$. The optical scalars (A.3.19) and (A.3.20) thus become

$$\begin{aligned} \omega^2 &= \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu}k_{[\beta,\alpha]}k_{\mu,\nu} \,, \\ \theta^2 + |\sigma|^2 &= \frac{1}{2}\eta^{\alpha\mu}\eta^{\beta\nu}k_{(\beta,\alpha)}k_{\mu,\nu} \,. \end{aligned} \tag{A.3.21}$$

The second order Ricci tensor (A.3.12) simplifies to

$$\underset{2}{R}_{\alpha\beta} = 2H\mathcal{D}k_{\alpha}k_{\beta} \,, \qquad \mathcal{D} = \ddot{H} + 2\theta\dot{H} + 4H\omega^2 \,, \tag{A.3.22}$$

leading to the condition $\mathcal{D} = 0$, which gives the following equation for H

$$0 = \ddot{H} + 2\theta\dot{H} + 4H\omega^2 \,. \tag{A.3.23}$$

Finally, the first order Ricci tensor (A.3.14)–(A.3.15) becomes

$$\begin{aligned} \underset{1}{R}_{\alpha\beta} &= \eta^{\lambda\gamma}H_{,\lambda\gamma}k_{\alpha}k_{\beta} + 2k_{(\alpha}B_{\beta)} \\ &\quad - 2\left[(\dot{H} + 2\theta H)k_{(\alpha,\beta)} - \eta^{\lambda\gamma}Hk_{\alpha,\gamma}k_{\beta,\lambda}\right] \,, \end{aligned} \tag{A.3.24}$$

with

$$B_{\beta} = -(\dot{H} + 2\theta H)_{,\beta} + \eta^{\lambda\gamma}(2H_{,\lambda}k_{\beta,\gamma} + Hk_{\beta,\gamma\lambda}) \,. \tag{A.3.25}$$

The vector $\boldsymbol{k}$ is an eigenvalue of the Ricci tensor, i.e.

$$R_{\alpha\sigma}k^{\sigma} = (B_{\sigma}k^{\sigma})k_{\alpha} \,. \tag{A.3.26}$$

It proves easier to handle with the remaining set of first order field equations by specifying a general field of real null direction in Minkowski space together with an adapted tetrad frame, then setting to zero each individual frame component of the first order Ricci tensor.

A.3.3. Simplified tetrad procedure

Following [5, 6] introduce the set of null coordinates in Minkowski space $(u, v, \zeta, \bar{\zeta})$, which are related to the standard Cartesian coordinates (t, x, y, z) by

$$u = \frac{1}{\sqrt{2}}(t - z)\,, \qquad v = \frac{1}{\sqrt{2}}(t + z)\,,$$
$$\zeta = \frac{1}{\sqrt{2}}(x + iy)\,, \qquad \bar{\zeta} = \frac{1}{\sqrt{2}}(x - iy)\,. \tag{A.3.27}$$

The metric (A.3.1) becomes

$$\mathrm{d}s^2 = 2(\mathrm{d}\zeta \mathrm{d}\bar{\zeta} - \mathrm{d}u\mathrm{d}v) - 2\epsilon H k_\alpha k_\beta \mathrm{d}x^\alpha \mathrm{d}x^\beta\,. \tag{A.3.28}$$

A general field of real null directions in Minkowski space is given by

$$k = -[\mathrm{d}u + Y\bar{Y}\mathrm{d}v + \bar{Y}\mathrm{d}\zeta + Y\mathrm{d}\bar{\zeta}]\,, \qquad \boldsymbol{k} = Y\bar{Y}\partial_u + \partial_v - Y\partial_\zeta - \bar{Y}\partial_{\bar{\zeta}}\,, \tag{A.3.29}$$

where Y is an arbitrary complex function of coordinates. In fact the independent components of $\boldsymbol{k}$ reduce to two real functions of the coordinates, due to the two conditions 1) $\boldsymbol{k}$ forms a lightlike world line and 2) $\boldsymbol{k}$ has an arbitrary parametrization. In Eq. (A.3.29) these two real functions of the coordinates collapsed in a single complex function Y, namely $\boldsymbol{k} = \boldsymbol{k}(Y, \bar{Y})$.

We introduce the following frame

$$\omega^1 = \mathrm{d}\zeta + Y\mathrm{d}v\,, \quad \omega^2 = \mathrm{d}\bar{\zeta} + \bar{Y}\mathrm{d}v\,, \quad \omega^3 = -k\,, \quad \omega^4 = \mathrm{d}v + \epsilon H\omega^3\,, \tag{A.3.30}$$

so that

$$\mathrm{d}s^2 = 2\omega^1\omega^2 - 2\omega^3\omega^4\,. \tag{A.3.31}$$

The dual frame is

$$\mathbf{e}_1 = \partial_\zeta - \bar{Y}\partial_u\,, \quad \mathbf{e}_2 = \partial_{\bar{\zeta}} - Y\partial_u\,, \quad \mathbf{e}_3 = \partial_u - \epsilon H\boldsymbol{k}\,, \quad \mathbf{e}_4 = \boldsymbol{k}\,. \tag{A.3.32}$$

The connection coefficients are given by

$$\Gamma_{cab} = -e_c{}^\mu e_{a\,\mu;\nu} e_b{}^\nu\,. \tag{A.3.33}$$

Note that $\omega^1_\mu = -k_{\mu,\bar{Y}}$ and $\omega^2_\mu = -k_{\mu,Y}$, trivially implying $\omega^1(\boldsymbol{k}) = 0 =$

$\omega^2(\boldsymbol{k})$, because

$$\boldsymbol{k}\cdot\omega^1 = \eta^{\alpha\beta}k_\alpha\omega^1_\beta = -\eta^{\alpha\beta}k_\alpha k_{\beta,\bar{Y}} = -k^\alpha k_{\alpha,\bar{Y}} = 0\,. \tag{A.3.34}$$

Similarly $\boldsymbol{k}\cdot\omega^2 = 0$.

The derivative of $\boldsymbol{k}$ is quite simple

$$k_{\mu,\nu} = k_{\mu,\bar{Y}}\bar{Y}_{,\nu} + k_{\mu,\bar{Y}}\bar{Y}_{,\nu} = -\omega^1_\mu\bar{Y}_{,\nu} - \omega^2_\mu Y_{,\nu}\,. \tag{A.3.35}$$

Next introduce the following standard notation for the directional derivatives along the frame vectors

$$\begin{aligned} D \equiv \nabla_{\boldsymbol{k}} &= \partial_v + Y\bar{Y}\partial_u - Y\partial_\zeta - \bar{Y}\partial_{\bar{\zeta}}\,,\\ \Delta \equiv \nabla_{\mathbf{e}_3} &= \partial_u - \epsilon H D\,,\\ \delta \equiv \nabla_{\mathbf{e}_1} &= \partial_\zeta - \bar{Y}\partial_u\,. \end{aligned} \tag{A.3.36}$$

The geodesic curvature κ, complex expansion ρ and shear σ of the null congruence $\boldsymbol{k}$ are given by

$$\begin{aligned} \kappa &\equiv -\Gamma_{414} = -k^\alpha De_{1\alpha} = D\bar{Y}\,,\\ \rho &\equiv -\Gamma_{412} = -k^\alpha\bar{\delta}e_{1\alpha} = \bar{\delta}\bar{Y}\,,\\ \sigma &\equiv -\Gamma_{411} = -k^\alpha\delta e_{1\alpha} = \delta\bar{Y}\,, \end{aligned} \tag{A.3.37}$$

respectively. It is also useful to introduce the quantity

$$\tau \equiv -\Gamma_{413} = -k^\alpha\Delta e_{1\alpha} = \partial_u\bar{Y}\,. \tag{A.3.38}$$

If the principal null vector $\boldsymbol{k}$ is geodesic, then $\kappa = 0$, i.e.

$$0 = D\bar{Y} = \bar{Y}_{,v} + Y\bar{Y}\bar{Y}_{,u} - Y\bar{Y}_{,\zeta} - \bar{Y}\bar{Y}_{,\bar{\zeta}}\,. \tag{A.3.39}$$

If it is also shearfree, then $\sigma = 0$, i.e.

$$0 = \delta\bar{Y} = \bar{Y}_{,\zeta} - \bar{Y}\bar{Y}_{,u}\,, \qquad \rightarrow \qquad \text{(c.c.)} \qquad 0 = Y_{,\bar{\zeta}} - YY_{,u}\,, \tag{A.3.40}$$

where "c.c." stands for "complex conjugate." Substituting it into Eq. (A.3.39) then yields

$$0 = \bar{Y}_{,v} - \bar{Y}\bar{Y}_{,\bar{\zeta}}\,, \qquad \rightarrow \qquad \text{(c.c.)} \qquad 0 = Y_{,v} - YY_{,\zeta}\,. \tag{A.3.41}$$

The conditions (A.3.40) and (A.3.41) thus give

$$Y_{,\bar{\zeta}} = YY_{,u}\,, \qquad Y_{,v} = YY_{,\zeta}\,, \tag{A.3.42}$$

whence if $\Box_0$ is the flat-space wave operator, then

$$\Box_0 Y \equiv \eta^{\alpha\beta} Y_{,\alpha\beta} = 2Y_{,\bar{\zeta}\zeta} - 2Y_{,uv} = (Y^2)_{,u\zeta} - (Y^2)_{,\zeta u} = 0\,, \tag{A.3.43}$$

and therefore Y is a solution of the wave equation in Minkowski space whenever the congruence is geodesic and shearfree. They also show that the congruence $\boldsymbol{k}$ must satisfy the Kerr Theorem, i.e. Y is a root of an analytic equation

$$0 = F(Y, \bar{\zeta}Y + u, vY + \zeta)\,, \tag{A.3.44}$$

where F is an arbitrary function analytic in the three complex variables Y, $\bar{\zeta}Y + u$ and $vY + \zeta$.

A.3.4. Completion of the solution

In terms of the connection coefficients previously introduced the optical scalars write as

$$\theta = -\frac{1}{2}(\rho + \bar{\rho})\,, \qquad \omega^2 = -\frac{1}{4}(\rho - \bar{\rho})^2\,, \tag{A.3.45}$$

so that the single equation (A.3.23) coming from the vanishing of second order Ricci tensor reads

$$0 = \ddot{H} - (\rho + \bar{\rho})\dot{H} - (\rho - \bar{\rho})^2 H\,. \tag{A.3.46}$$

The nonvanishing relevant frame components of the first order Ricci tensor (A.3.24) are given by

$$\underset{1}{R}_{11} = 2\sigma[\dot{H} - (\bar{\rho} - \rho)H]\,, \tag{A.3.47a}$$

$$\underset{1}{R}_{12} = (\rho + \bar{\rho})\dot{H} - (\rho^2 + \bar{\rho}^2 - 2\sigma\bar{\sigma})H\,, \tag{A.3.47b}$$

$$\underset{1}{R}_{13} = \delta\dot{H} + (\rho - \bar{\rho})\delta H + 2\sigma\bar{\delta}H - \tau\dot{H} - (\delta\bar{\rho} + 2\bar{\tau}\sigma + 2\tau\rho - \bar{\delta}\sigma)H\,, \tag{A.3.47c}$$

$$\underset{1}{R}_{33} = 2\left[\delta\bar{\delta}H - (\rho_{,u} + \bar{\rho}_{,u})H - \tau\bar{\delta}H - \bar{\tau}\delta H - \rho H_{,u}\right]\,, \tag{A.3.47d}$$

$$\underset{1}{R}_{34} = \ddot{H} - (\rho + \bar{\rho})\dot{H} - (\rho - \bar{\rho})^2 H\,, \tag{A.3.47e}$$

since $\underset{1}{R}_{22}$ and $\underset{1}{R}_{23}$ are c.c. of $\underset{1}{R}_{11}$ and $\underset{1}{R}_{13}$ respectively. The identities

$$\begin{aligned} \bar{\delta}\tau &= \rho_{,u} + \tau\bar{\tau}\,, & \delta\tau &= \sigma_{,u} + \tau^2\,, \\ \delta\rho &= \bar{\delta}\sigma + \tau(\rho - \bar{\rho})\,, & D\rho &= \sigma\bar{\sigma} + \rho^2\,, \\ D\tau &= \bar{\tau}\sigma + \tau\rho\,, & D\sigma &= \sigma(\rho + \bar{\rho})\,, \end{aligned} \tag{A.3.48}$$

as well as the commutation relations

$$\begin{aligned} \partial_u D - D\partial_u &= -\bar{\tau}\delta - \tau\bar{\delta}\,, & \delta D - D\delta &= -\bar{\rho}\delta - \sigma\bar{\delta}\,, \\ \delta\partial_u - \partial_u\delta &= \tau\partial_u\,, & \bar{\delta}\delta - \delta\bar{\delta} &= -(\rho - \bar{\rho})\partial_u\,, \end{aligned} \tag{A.3.49}$$

have been used here to simplify the expressions involving frame derivatives of H. Setting to zero each component of Eqs. (A.3.47a)–(A.3.47e) gives a set of first order equations. Note that the condition coming from Eq. (A.3.47e) is the same as Eq. (A.3.46).

Equation (A.3.47a) implies $\sigma = 0$, i.e. the congruence $\boldsymbol{k}$ must be shearfree. The remaining first order equations thus simplify as

$$0 = (\rho + \bar{\rho})\dot{H} - (\rho^2 + \bar{\rho}^2)H\,, \tag{A.3.50a}$$
$$0 = \delta\dot{H} + (\rho - \bar{\rho})\delta H - \tau\dot{H} - (\delta\bar{\rho} + 2\tau\rho)H\,, \tag{A.3.50b}$$
$$0 = \delta\bar{\delta}H - (\rho_{,u} + \bar{\rho}_{,u})H - \tau\bar{\delta}H - \bar{\tau}\delta H - \rho H_{,u}\,, \tag{A.3.50c}$$

and the identities (A.3.48) become

$$\begin{aligned} \bar{\delta}\tau &= \rho_{,u} + \tau\bar{\tau}\,, & \delta\tau &= \tau^2\,, \\ \delta\rho &= \tau(\rho - \bar{\rho})\,, & \dot{\rho} &= \rho^2\,, \\ \dot{\tau} &= \tau\rho\,. \end{aligned} \tag{A.3.51}$$

Taking the δ derivative of Eq. (A.3.50a) together with Eq. (A.3.50b) gives rise to the following compatibility condition

$$\rho(\rho + \bar{\rho})\delta H = [\tau\rho(\rho + \bar{\rho}) + \tau\bar{\rho}(\rho + 3\bar{\rho}) + \rho\delta\bar{\rho}]H\,. \tag{A.3.52}$$

Take the complex conjugate of this equation and then its δ derivative; Eq. (A.3.50c) thus gives rise to a second compatibility condition

$$(\rho + \bar{\rho})H_{,u} = \left[3\left(\frac{\rho\bar{\tau}}{\bar{\rho}^2}\delta\bar{\rho} + \frac{\bar{\rho}\tau}{\rho^2}\bar{\delta}\rho\right) + (\bar{\rho} + 3\rho)\frac{\bar{\rho}_{,u}}{\bar{\rho}} + (\rho - 3\bar{\rho})\frac{\rho_{,u}}{\rho} + 6\frac{\tau\bar{\tau}}{\bar{\rho}}\right]H\,. \tag{A.3.53}$$

By using Eq. (A.3.50a), Eq. (A.3.46) rewrites as

$$\ddot{H} = 2(\rho + \bar{\rho})\dot{H} - 2\rho\bar{\rho}H\,. \tag{A.3.54}$$

Let the complex expansion be nonzero, i.e. $\rho \neq 0$. It is easy to check that $\rho\bar{\rho}$ and $\rho + \bar{\rho}$ are particular solutions, and therefore the general solution is

$$H = \frac{1}{2}M(\rho + \bar{\rho}) + B\rho\bar{\rho}\,, \qquad \dot{M} = \dot{B} = 0\,, \tag{A.3.55}$$

where $M(Y, \bar{Y})$ and $B(Y, \bar{Y})$ are real functions of Y and $\bar{Y}$. Substituting the general solution (A.3.55) for H into Eq. (A.3.50a) one easily gets $B = 0$, by using the relation $\dot{\rho} = \rho^2$, so that

$$H = \frac{1}{2} M(\rho + \bar{\rho}) \ . \tag{A.3.56}$$

Substituting now this solution for H into Eq. (A.3.52) leads to

$$\delta M = \frac{3M}{\rho} \tau \bar{\rho} \ . \tag{A.3.57}$$

But $M = M(Y, \bar{Y})$, so that $\delta M = M_{,Y} \bar{\rho}$, implying that

$$M_{,Y} = \frac{3M}{\rho} \tau \ , \qquad M_{,\bar{Y}} = \frac{3M}{\bar{\rho}} \bar{\tau} \ . \tag{A.3.58}$$

The second compatibility condition (A.3.53) then yields

$$\frac{\bar{\rho}}{\rho} \left(\bar{\delta}\tau - \frac{\tau}{\rho} \bar{\delta}\rho \right) - \text{c.c.} = 0 \ , \tag{A.3.59}$$

where the relation

$$M_{,u} = 3M\tau\bar{\tau} \left(\frac{1}{\rho} + \frac{1}{\bar{\rho}} \right) \tag{A.3.60}$$

has been used. Equation (A.3.59) is an additional equation for Y and $\bar{Y}$ which we will discuss later.

Following the original work [5] we now introduce $P = (M/m)^{-1/3}$, where m is a real constant. The first equation of (A.3.58) thus becomes

$$P^{-1} P_{,Y} = -\frac{\tau}{\rho} \ . \tag{A.3.61}$$

By taking δ of both sides we then find

$$-\bar{\rho} P^{-2} (P_{,Y})^2 + \bar{\rho} P^{-1} P_{,YY} = -\frac{\tau^2}{\rho^2} \bar{\rho} = -\bar{\rho} P^{-2} (P_{,Y})^2 \ , \tag{A.3.62}$$

since $\delta P = \bar{\rho} P_{,Y}$ and $\delta P_{,Y} = \bar{\rho} P_{,YY}$, and the identities (A.3.51) have been used to replace $\delta\rho$ and $\delta\tau$ on the RHS. Equation (A.3.62) thus implies $P_{,YY} = 0$, whose solution is

$$P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c \ , \tag{A.3.63}$$

where p and c are real constants and q is a complex constant.

Let us turn to the remaining compatibility condition (A.3.59). First note that it can be equivalently rewritten as

$$\bar{\rho}\bar{\delta} \left(\frac{\tau}{\rho} \right) - \text{c.c.} = 0 \ . \tag{A.3.64}$$

By using Eq. (A.3.61) we have

$$\bar{\rho}\bar{\delta}\left(\frac{\tau}{\rho}\right) = \rho\bar{\rho}P^{-1}[P^{-1}P_{,Y}P_{,\bar{Y}} - P_{,Y\bar{Y}}] \,. \tag{A.3.65}$$

Take the complex conjugate of this expression taking into account that P is real; substituting then into Eq. (A.3.64) we find that it is identically satisfied.

Finally, taking the exterior derivative of Y gives

$$\begin{aligned} \mathrm{d}Y &= \delta Y\omega^1 + Y_u\omega^3 = P^{-1}\bar{\rho}[P\omega^1 - P_{,\bar{Y}}\omega^3] \\ &= P^{-1}\bar{\rho}[(qY+c)(\mathrm{d}\zeta + Y\mathrm{d}v) - (pY+\bar{q})(\mathrm{d}u + Y\mathrm{d}\bar{\zeta})] \,, \end{aligned} \tag{A.3.66}$$

whose general solution is

$$0 = F \equiv \phi(Y) + (qY+c)(\zeta + Yv) - (pY+\bar{q})(u + Y\bar{\zeta}) \,, \tag{A.3.67}$$

according to Eq. (A.3.44), with ϕ an arbitrary analytic function of the complex variable Y. In fact, differentiating Eq. (A.3.67) leads to

$$F_{,Y}\mathrm{d}Y = \mathrm{d}F = F_{,\alpha}\mathrm{d}x^\alpha = (qY+c)(\mathrm{d}\zeta + Y\mathrm{d}v) - (pY+\bar{q})(\mathrm{d}u + Y\mathrm{d}\bar{\zeta}) \,. \tag{A.3.68}$$

Furthermore, taking the δ derivative of F, i.e.

$$\bar{\rho}F_{,Y} = \delta F = (\partial_\zeta - \bar{Y}\partial_u)F = P \,, \tag{A.3.69}$$

implies that the complex expansion of the null vector $\boldsymbol{k}$ is given by

$$\bar{\rho} = P{F_{,Y}}^{-1} \,. \tag{A.3.70}$$

Equation (A.3.66) then immediately follows.

Summarizing, the solution is given by

$$\mathrm{d}s^2 = 2(\mathrm{d}\zeta\mathrm{d}\bar{\zeta} - \mathrm{d}u\mathrm{d}v) - \frac{m}{P^3}(\rho+\bar{\rho})[\mathrm{d}u + Y\bar{Y}\mathrm{d}v + \bar{Y}\mathrm{d}\zeta + Y\mathrm{d}\bar{\zeta}]^2 \,, \tag{A.3.71}$$

with

$$P = pY\bar{Y} + qY + \bar{q}\bar{Y} + c \,, \qquad \bar{\rho} = P{F_{,Y}}^{-1} \,. \tag{A.3.72}$$

The main properties of such a family of solutions are listed below (see e.g. [7]):

1. They are all algebraically special, with $\boldsymbol{k}$ shearfree and geodesic.
2. They all admit at least a one-parameter group of motions with Killing vector

   $$\boldsymbol{\xi} = c\partial_u + p\partial_v + \bar{q}\partial_\zeta + q\partial_{\bar{\zeta}} \,, \tag{A.3.73}$$

   which is simultaneously a Killing vector of flat spacetime. The solutions can be simplified by performing a Lorentz transformation. One can thus assume that if

a) $\eta_{\alpha\beta}\xi^\alpha\xi^\beta < 0$, then $P = (1 + Y\bar{Y})/\sqrt{2}$, i.e. with $\boldsymbol{\xi}$ pointing along the $u + v$ (or t) direction ($p = c = 1/\sqrt{2}$, $q = 0$);

b) $\eta_{\alpha\beta}\xi^\alpha\xi^\beta > 0$, then $P = (1 - Y\bar{Y})/\sqrt{2}$, i.e. with $\boldsymbol{\xi}$ pointing along the $v - u$ (or z) direction ($-p = c = 1/\sqrt{2}$, $q = 0$);

c) $\eta_{\alpha\beta}\xi^\alpha\xi^\beta = 0$, then $P = 1$, i.e. with $\boldsymbol{\xi}$ pointing along the u direction ($p = q = 0, c = 1$).

3. For a timelike Killing vector $\boldsymbol{\xi}$, the particular case $\phi = -iaY$, with $m = \mathcal{M}$, leads to the Kerr solution (A.2.1)–(A.2.6), once written in Kerr-Schild coordinates.

A.4. Concluding remarks

We have presented an alternative derivation of Kerr solution by treating Kerr-Schild metrics as *exact linear perturbations* of Minkowski spacetime. In fact they have been introduced as a linear superposition of the flat spacetime metric and a squared null vector field $\boldsymbol{k}$ multiplied by a scalar function H.

In the case of Kerr solution the vector $\boldsymbol{k}$ is geodesic and shearfree and it is independent of the mass parameter $\mathcal{M}$, which enters instead the definition of H linearly. This linearity property allows one to solve the field equations order by order in powers of H in complete generality, i.e. without any assumption on the null congruence $\boldsymbol{k}$. The Ricci tensor turns out to consist of three different contributions. Third order equations all imply that $\boldsymbol{k}$ must be geodesic; it must be also shearfree as a consequence of first order equations, whereas the solution for H comes from second order equations too.

The present treatment can be generalized to include also the electromagnetic field, i.e. to the case of Kerr-Newman. In fact, even in the charged Kerr solution the congruence of $\boldsymbol{k}$-lines depend only on the rotation parameter a and not on the mass $\mathcal{M}$ or charge Q. Furthermore, the electromagnetic field is linear in Q and the metric is linear in $\mathcal{M}$ and Q^2, since the function H is obtained simply by replacing $\mathcal{M} \to \mathcal{M} - Q^2/(2r)$.

A.5. References

1. R. P. Kerr and A. Schild, A new class of vacuum solutions of the Einstein field equations, in *Atti del convegno sulla relatività generale; problemi dell'energia e onde gravitazionali*, ed. G. Barbera (Firenze, 1965) p. 173.

2. R. P. Kerr and A. Schild, *Proc. Symp. Appl. Math.* **17** (1965), 199.

3. R. P. Kerr, *Phys. Rev. Lett.* **11** (1963), 237.

4. J. N. Goldberg and R. K. Sachs, *Acta Phys. Polon.* **22** *Suppl.* (1962), 13.
5. G. Debney, R. P. Kerr and A. Schild, *J. Math. Phys.* **10** (1969), 1842.
6. R. P. Kerr and W. B. Wilson, *Gen. Rel. Grav.* **10** (1979), 273.
7. H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd ed. (Cambridge Univ. Press, Cambridge, 2003).