Crossover scaling functions in the asymmetric avalanche process

(A. Trofimova joint with Alexander Povolotsky)
arXiv: 2109.06318

Recent Advances in Quantum Integrable Systems, Lyon
dedicated to the 60th birthday of Nikita Slavnov

August 30, 2022

Outline

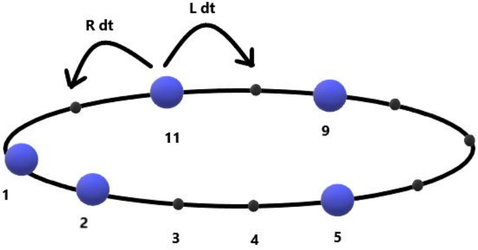
- Model and Motivation
- Stationary state analysis results
- Review of exact formulas for mean particle current and diffusion coefficient
- Their behaviour in the thermodynamic limit

Asymmetric Avalanche Process on a ring

(Priezzhev, Ivashkevich, Povolotsky, Hu, 2001)

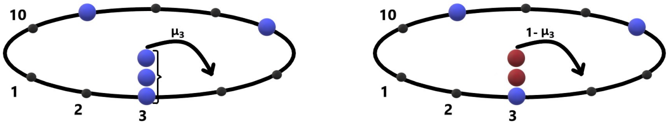
is a one dimensional stochastic process on a ring evolving in continuous time

- p particles, N sites; state $x(t) = (1, 2, 5, 9, 11)$
- Evolution:
  - all particles occupy different sites: jump randomly and independently having waited for $\mathbb{P}(t(x_k) < T) = 1 - e^{-T}$ with probabilities L to the left or R to the right$(R + L = 1)$
  - particle comes to already occupied site the avalanche dynamics starts



Avalanche dynamics

- with probability μ_n, n particles go to the site $x+1$;
- with probability $1-\mu_n$, $n-1$ particles go to the site $x+1$ and one particle stays at the current site x.
- occurs instantly



(a) (b)

Figure: Totally asymmertic avalanche hopping with probabilities

Master equation

$P_t(\boldsymbol{x}) := \mathbb{P}(\mathbf{x}(t) = \mathbf{x})$ - probability to be at state $\boldsymbol{x}$ at time t.

Given an initial distribution $P_0(\boldsymbol{x})$, $P_t(\boldsymbol{x})$ satisfies forward Kolmogorov equation

$$\partial_t P_t(\boldsymbol{x}) = \mathcal{L} P_t(\boldsymbol{x}),$$

$$\mathcal{L} P_t(\boldsymbol{x}) = \sum_{\boldsymbol{x}'} (t(\boldsymbol{x}' \to \boldsymbol{x}) P_t(\boldsymbol{x}') - t(\boldsymbol{x} \to \boldsymbol{x}') P_t(\boldsymbol{x}))$$

,

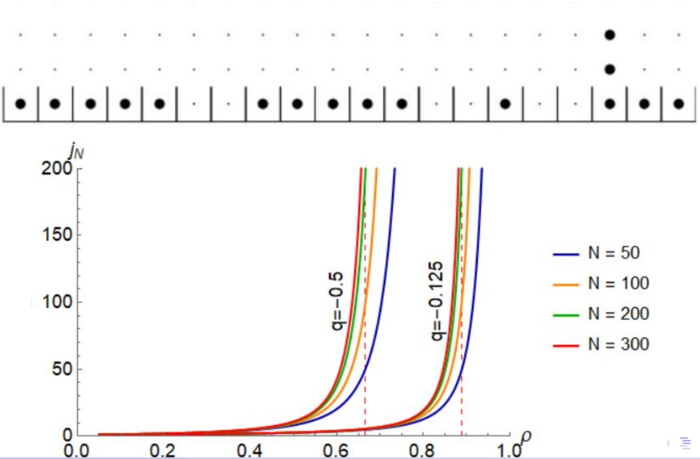
$$t(\boldsymbol{x}' \to \boldsymbol{x}) \text{ - transition rate}$$

Bethe ansatz integrability condition + positivity of rates (Priezzhev, Ivashkevich, Povolotsky, Hu, 2001)

$$\mu_n = 1 - [n]_q = 1 - \frac{1 - q^n}{1 - q}, \quad -1 < q < 0$$

Why this process is interesting ?

- Unstable states may appear randomly
- specific transition into a totally unstable state, when ρ approaches ρ_c and an avalanche never stops in the thermodynamic limit
- unusual universal scaling behaviour



Stationary probability measure

$$\boldsymbol{P}_{st}(\boldsymbol{x}) = \frac{1}{C_N^p}.$$

is extremely simple

Analysis of discretized AAP stationary measure reveals the structure of avalanches resulting in

$$j_N = \frac{(1-q)}{C_N^p} \oint \frac{(1+z)^N}{z^p} \Big[R g'(zq) - L g'(z) \Big] \frac{dz}{2\pi i} =$$

$$= \frac{(1-q)}{C_N^p} \sum_{m=0}^{p-1} (m+1) \frac{(-1)^m C_N^{p-m-1}}{1-q^{m+1}} (Rq^m - L).$$

in terms of

$$g(z) = -\sum_{k=0}^{\infty} \frac{(-z)^{k+1}}{1-q^{k+1}} = \sum_{k=0}^{\infty} \frac{q^i z}{1+q^i z}.$$

(it has poles $z_i = -q^i, i \geq 0$)

Research Problem

is to investigate the current in the model

- to develop a technique which allows to **analyse higher cumulants of the current** with attention to transition point $\rho = \rho_c$
- to use the connection to random growth interface problems (the common point here is **the universal behaviour at large scales**, the important problem is testing the universality, study the scaling behaviour of the models in the Edwards-Wilkinson and Kardar-Parisi-Zhang universality classes and beyond).
- to find and analyse the scaling exponents and scaling functions for particle current and diffusion coefficient

Total distance Y_t

$Y_0 = 0$,
$Y_t : \Omega \times \mathbb{R} \to \mathbb{Z}_{\geq 0}$ - random variable of total number of jumps made by all particles

$$Y_t \to Y_t + \Delta Y_t,\ \Delta Y_t \in \{1, -1, n \leq p\}$$

The behaviour of moment generating function in the large time limit $t \to \infty$ is dominated by the largest eigenvalue $\lambda(\gamma)$ of the deformed model generator

$$\lambda(\gamma) = \lim_{t\to\infty} \frac{\ln \mathbb{E} e^{\gamma Y_t}}{t} = \sum_{n=1}^{\infty} c_n \frac{\gamma^n}{n!},$$

First and second scaled cumulants:

$$J := c_1 = \lim_{t\to\infty} \frac{\mathbb{E}(Y_t)}{t}, \quad \Delta := c_2 = \lim_{t\to\infty} \frac{\mathbb{E}(Y_t^2) - \mathbb{E}(Y_t)^2}{t},$$

(Bethe anzatz, Baxter's TQ-equation, (Baxter, 1972, Prolhac, Mallick, 2008))

Mean particle current

Introducing *normalized differential*

$$D_{N,p}(t) := \frac{dz}{2\pi \mathrm{i}} \frac{1}{C_N^p} \frac{(1+t)^N}{t^{p+1}}.$$

we reproduce the stationary state result

$$j_N = Rj_N^R - Lj_N^L$$

$$j_N^R = (1-q) \oint D_{N,p}(z) z g'(zq), \qquad j_N^L = (1-q) \oint D_{N,p}(z) z g'(z).$$

The group diffusion coefficient is

$\Delta = R\Delta^R - L\Delta^L$, where both right and left parts are given by the formula

$$\Delta^I = \epsilon(I)pNj_N^I + 2N^2 \sum_{i=0}^{\infty} \oint \oint D_{N,p}(t)D_{N,p}(y)ty\frac{a^I(y)}{t-q^iy}$$
$$+2N^2 \sum_{i=1}^{\infty} \oint \oint D_{N,p}(t)D_{N,p}(y)ty\frac{q^ia^I(q^iy)}{t-q^iy}$$

for $I \in \{R, L\}$, where function $\epsilon(R) = 1, \epsilon(L) = -1$ stands for sign and functions

$$a^R(y) = (1-q)g'(qy) - \frac{j_N^R}{\rho(1+y)},$$
$$a^L(y) = (1-q)g'(y) - \frac{j_N^L}{\rho(1+y)}.$$

Asymptotic analysis in the thermodynamic limit $p, N \to \infty, p/N = \rho$

The critical density is a point of model phase transition $\rho_c = \frac{1}{1-q}$.

$$j_N(\rho) \simeq \begin{cases} \frac{\rho(1-\rho)(R\rho_c+(1-\rho_c)L)}{(\rho-\rho_c)^2} + j_\infty^{\text{reg}}(\rho), & \rho < \rho_c, \\ N(R\rho_c + L(1-\rho_c)), & \rho = \rho_c, \\ N^{3/2} e^{Ns(\rho|\rho_c)} \frac{\sqrt{2\pi\rho(1-\rho)}}{\rho_c(1-\rho_c)}(\rho-\rho_c)(\rho_c R + (1-\rho_c)L), & \rho > \rho_c, \end{cases}$$

where

$$j_\infty^{\text{reg}}(\rho) = \frac{\rho_c R + (1-\rho_c)L}{\rho_c(1-\rho_c)} \sum_{k=1}^{\infty} k \frac{\left[\frac{(\rho_c-1)^2}{\rho-1}\frac{\rho}{\rho_c^2}\right]^k}{1-\left[\frac{\rho_c-1}{\rho_c}\right]^k} - \frac{L\rho(1-\rho)}{\rho_c}$$

$$s(\rho|\rho_c) = (1-\rho)\ln\left(\frac{1-\rho}{1-\rho_c}\right) + \rho\ln\left(\frac{\rho}{\rho_c}\right)$$

Crossover function for j_N

Result 2: Under the scaling of

$$\beta = \frac{\sqrt{N}(\rho_c - \rho)}{\sqrt{\rho_c(1-\rho_c)}}$$

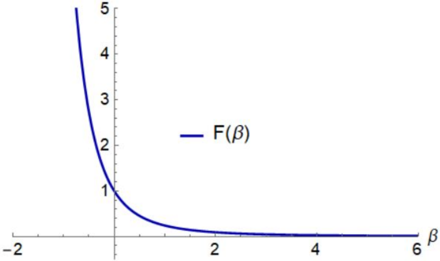
the particle current is described by

$$j_N(\rho) = N(R\rho_c + L(1-\rho_c))\mathcal{F}(\beta) + O(N^{\frac{1}{2}}),$$

where

$$\mathcal{F}(\beta) = 1 - \sqrt{\frac{\pi}{2}}\beta \,\mathrm{erfc}\left(\frac{\beta}{\sqrt{2}}\right) e^{\frac{\beta^2}{2}}.$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt.$$



Asymptotic analysis in the thermodynamic limit $p, N \to \infty, p/N = \rho$

$$\Delta_N(\rho) \simeq \begin{cases} N^{3/2}\left(\frac{f(\rho)}{2(\rho-\rho_c)^4} + \Delta_\infty^{reg}(\rho)\right), & \rho < \rho_c \\ N^{7/2}(R\rho_c + L(1-\rho_c))\sqrt{\pi\rho_c(1-\rho_c)}, & \rho = \rho_c \\ N^4 e^{2Ns(\rho|\rho_c)} 4\pi(\rho-\rho_c)(R\rho_c + L(1-\rho_c))\frac{\rho(1-\rho)}{\rho_c(1-\rho_c)}, & \rho > \rho_c \end{cases}$$

where

$$f(\rho) = \sqrt{\pi}(R\rho_c + L(1-\rho_c))(\rho_c(1-\rho_c))^{3/2}(\rho_c^2 - 2\rho_c(1-\rho) - \rho)$$

$$\Delta_\infty^{\mathrm{reg}}(\rho) \simeq \frac{\sqrt{\pi}(R\rho_c + L(1-\rho_c))}{4\sqrt{\rho(1-\rho)}\rho_c(1-\rho_c)} \sum_{k=1}^{\infty} \frac{\left[\frac{(\rho_c-1)^2}{\rho-1}\frac{\rho}{\rho_c^2}\right]^k}{1-\left[\frac{\rho_c-1}{\rho_c}\right]^k}\left(k^2(1-2\rho) - k^3\right) - \frac{\sqrt{\pi}(\rho(1-\rho))^{3/2}}{4\rho_c}L.$$

Result 3: Under the scaling of

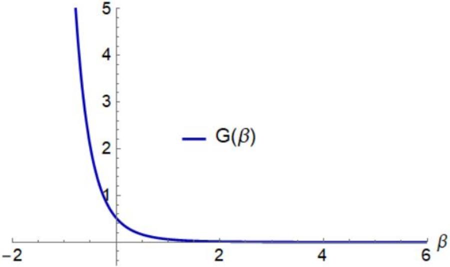
$$\beta = \frac{\sqrt{N}(\rho_c - \rho)}{\sqrt{\rho_c(1-\rho_c)}}$$

the group diffusion coefficient is

$$\Delta_N(\rho) = N^{\frac{7}{2}} \left(R\rho_c + L(1-\rho_c)\right) \sqrt{\rho_c(1-\rho_c)}\, \mathcal{G}(\beta) + O(N^3)$$

where the crossover function is

$$\mathcal{G}(\beta) = \sqrt{\pi}(2\mathcal{F}(\sqrt{2}\beta) - \mathcal{F}(\beta))$$



Current cumulats and statistics of avalanches

Consider particle current as a sum of signed avalanche sizes with the number of avalanches given by the Poisson process $\mathfrak{N}_t(p)$ with the arrival rate p

$$Y_t = \sum_{i=1}^{\mathfrak{N}_t(p)} S_i,$$

$$J = \lim_{t\to\infty} \frac{\mathbb{E} Y_t}{t} = \lim_{t\to\infty} \frac{1}{t} \mathbb{E} \sum_{i=1}^{\mathfrak{N}_t(p)} \mathbb{E}\left(S_i | \mathbf{n}(t_i); \mathfrak{N}_\tau(p), \tau \in [0,t]\right) = p\mathbb{E}_{st} S,$$

$$\Delta = \lim_{t\to\infty} \frac{\mathbb{E}(Y_t^2) - \mathbb{E}(Y_t)^2}{t} = \lim_{t\to\infty} \frac{1}{t} \mathbb{E} \sum_{i=1}^{\mathfrak{N}_t(p)} \sum_{j=1}^{\mathfrak{N}_t(p)} Cov(S_i, S_j),$$

At low densities $Cov(S_i, S_j), i \neq j$ change the asymptotic behaviour of Δ. At high densities the avalanches become large and the greatest contribution comes from $Var(S_i)$.

From AAP to Ornstein-Uhlenbeck process

Consider AAP with $L = 0, R = 1$ in the scaling limit the avalanche size

$$S = \sum_{k=1}^{T} \chi(k).$$

where $\chi(k)$ is a biased random walk with steps $-1, 0, 1$ (Povolotsky, Priezzhev, Hu, 2003) performed till the first return to the origin with transition probabilities

$$\mathbb{P}_{b|a} \simeq \begin{cases} \left(1 - \left(\rho - \frac{a}{N}\right)\right)(1 - \mu_a), & b = a - 1, \\ \left(1 - \left(\rho - \frac{a}{N}\right)\right)\mu_a + \left(\rho - \frac{a}{N}\right)(1 - \mu_a), & b = a, \\ \left(\rho - \frac{a}{N}\right)\mu_a, & b = a + 1. \end{cases}$$

for $a, b > 1$ and the $\lim_{a\to\infty} \mu_a = (1 - \rho_c)$. Introducing

$$X_t^N = (\rho_c(1 - \rho_c)N)^{-1/2}\chi([tN]).$$

$$dX_t = -(\beta + X_t)dt + \sqrt{2}dW_t,$$

where W_t is the standard Wiener process.

First passage area for Ornstein-Uhlenbeck process (Kearney, Martin, 2021)

$$dX_t = -(\beta + X_t)dt + \sqrt{2}dW_t, \quad X_0 = \alpha > 0$$

with the time of stop $\tau = \inf(t \in \mathbb{R}_{\geq 0} : X_t = 0)$ is the rescaled avalanche size. Thus, we are interested in evaluation of the area under the trajectory of the process X_t

$$\mathcal{A}(\alpha) = \int_0^\tau X_t dt.$$

The key idea is to introduce the generating function

$$\tilde{P}(s|\alpha) = \mathbb{E}e^{-s\mathcal{A}(\alpha)} = \sum_{n=0}^{\infty} \frac{(-s)^n \mathcal{A}_n(\alpha)}{n!}$$

that satisfy the following ODE

$$\left[\frac{d^2}{d\alpha^2} - (\beta + \alpha)\frac{d}{d\alpha} - s\alpha\right] \tilde{P}(s|\alpha) = 0$$

with $\tilde{P}(s|0) = 1$ and $\lim_{\alpha\to\infty} \tilde{P}(s|\alpha) = 0$.

Differentiating the last ODE in s and setting $s = 0$

$$\left[\frac{d^2}{d\alpha^2} - (\beta+\alpha)\frac{d}{d\alpha}\right]\mathcal{A}_n(\alpha) = -n\alpha\mathcal{A}_{n-1}(\alpha); \quad \mathcal{A}_0(\alpha) \equiv 1.$$

subject to initial conditions $\mathcal{A}_n(0) = 0$. It is solved by the recursion

$$\mathcal{A}_n(\alpha) = n\int_0^\alpha e^{\frac{1}{2}(z+\beta)^2}\int_z^\infty z' e^{-\frac{1}{2}(z'+\beta)^2}\mathcal{A}_{n-1}(z')dz'dz$$

that yields the following integral expressions

$$\begin{aligned}
\mathcal{A}_1(\alpha) &= \int_0^\alpha e^{\frac{1}{2}(z+\beta)^2}\int_z^\infty z' e^{-\frac{1}{2}(z'+\beta)^2}dz'dz \\
\mathcal{A}_2(\alpha) &= 2\int_0^\alpha dz_1 e^{\frac{1}{2}(z_1+\beta)^2}\int_{z_1}^\infty dz_2 z_2 e^{-\frac{1}{2}(z_2+\beta)^2} \\
&\times \int_0^{z_2} dz_3 e^{\frac{1}{2}(z_3+\beta)^2}\int_{z_3}^\infty dz_4 z_4 e^{-\frac{1}{2}(z_4+\beta)^2}.
\end{aligned}$$

For the moments of the avalanche size we should rescale $\mathcal{A}_n(\alpha) \to \left(N^{3/2}\sqrt{\rho_c(1-\rho_c)}\right)^n \mathcal{A}_n(\alpha)$ and set $\alpha = 1/\sqrt{\rho_c(1-\rho_c)N}$. Then to the leading order in $1/\sqrt{N}$ we obtain

$$
\begin{aligned}
\mathbb{E}S &\simeq N\mathcal{F}(\beta) \\
\mathbb{E}S^2 &\simeq N^{5/2}\sqrt{\rho_c(1-\rho_c)} \\
&\times \left(\frac{2(1-\mathcal{F}(\beta))}{\beta} - 4\beta\mathcal{F}(\beta) + e^{\frac{\beta^2}{2}}\pi\beta^2 \int_\beta^\infty e^{\frac{x^2}{2}} \left(\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)\right)^2 dx\right)
\end{aligned}
$$

The result for avalanche size agrees exactly with the crossover function $\mathcal{F}(\beta)$ while the results for dispersion are agreed only in the dominant terms.

Thank you!