Distinguishing between quantum many-body integrability and chaos: Insights from quantum cellular automata

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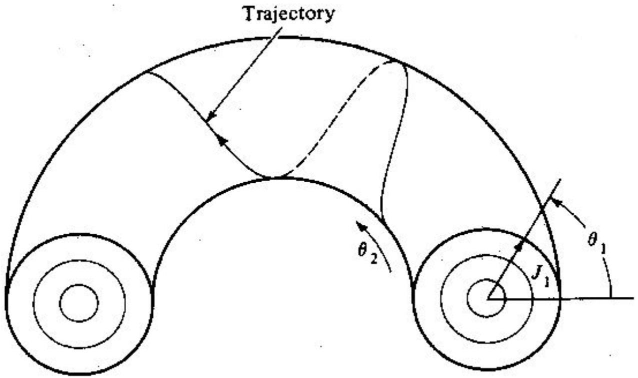
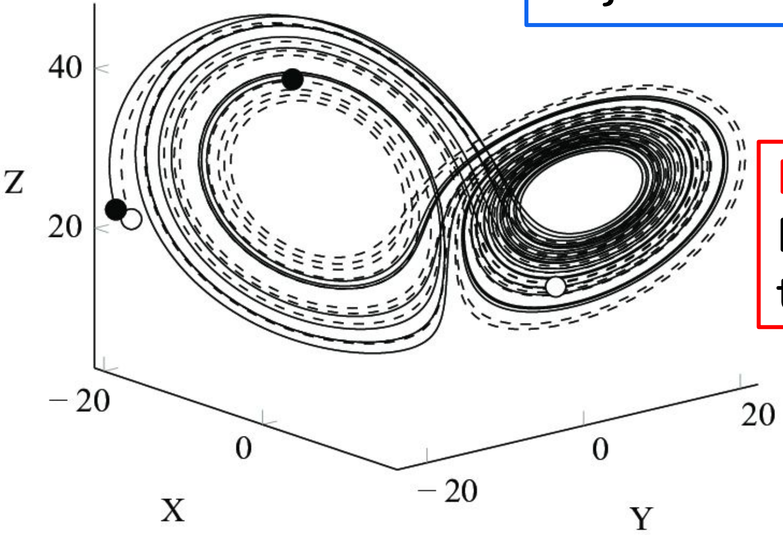
Integrability vs chaos
In classical systems

Classical integrability with few degrees of freedom:
Mapping to the action-angle coordinates - trajectories stay close by





Nonintegrable systems:
Fast separation of close by trajectories $\sim \exp(\lambda t)$

Conservation laws in integrable systems
Is there a difference in complexity

Integrability vs chaos in classical systems

Distinguishing features:

- Complexity of dynamics

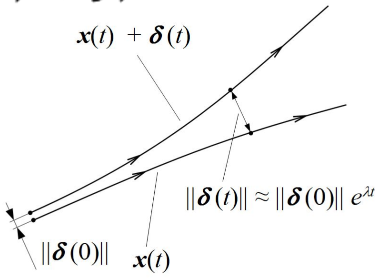
  Integrable systems (linear in t) $$|\delta \underline{x}(t)| < \varepsilon \rightarrow |\delta \underline{x}(\Delta t)| < \frac{\varepsilon C}{t}$$

  Chaotic systems (exponential in t) $$|\delta \underline{x}(t)| < \varepsilon \rightarrow |\delta \underline{x}(\Delta t)| < \frac{\varepsilon C}{\exp(\lambda t)}$$

- Lyapunov exponents

  Leading Lyapunov exponent $\lambda = \lim_{t\rightarrow\infty} \lim_{|\delta_x(0)|\rightarrow 0} \frac{1}{t}\log\left(\frac{|\delta \underline{x}(t)|}{|\delta x(0)|}\right)$

  Single particle $\frac{\partial x(t)}{\partial x(0)} = \{x(t), p\} \rightarrow \lambda = \lim_{t\rightarrow\infty} \frac{1}{t}\log(\{x(t), p\})$



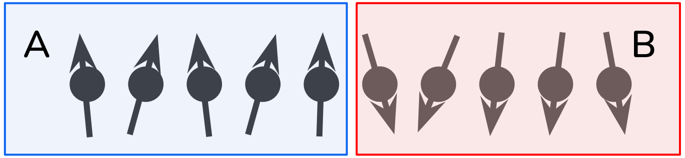
Integrability vs chaos in quantum systems

A natural measure of complexity of local evolution in quantum systems is the Operator space entanglement entropy (OSEE).

OSEE is the entropy of the operator after the operator in the channel-state transformation

$$O = \sum_{S_1 S_2 \cdots S_N} a_{S_1 S_2 \cdots S_N} S_1 \otimes S_2 \otimes \cdots \otimes S_N \to |O\rangle = \sum_{S_1 S_2 \cdots S_N} a_{S_1 S_2 \cdots S_N} |S_1 S_2 \cdots S_N\rangle$$

$$\frac{O}{\sqrt{\mathrm{tr}(OO^\dagger)}} = \sum_i \sqrt{\lambda_i} O_i^{(A)} O_i^{(B)}; \ \mathrm{tr}(O_i^{(A/B)} O_j^{(A/B)}) = \delta_{ij} \quad S^{(A,B)} = -\sum_i \lambda_i \log(\lambda_i)$$



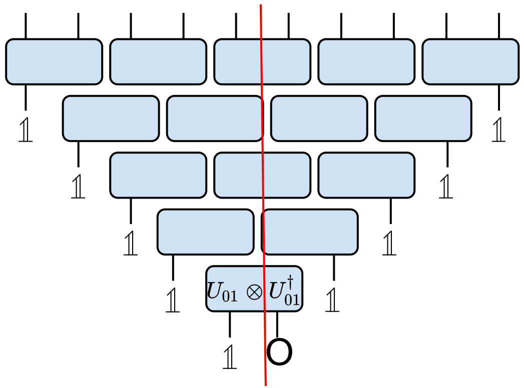
Information about the complexity of the MPO simulations

Operator space entanglement

Lieb-Robinson bound:

In locally interacting systems operators spread with a finite velocity up to the exponentially small corrections

For simplicity we can imagine a local quantum circuit with a strict light-cone and consider OSEE along the central cut:



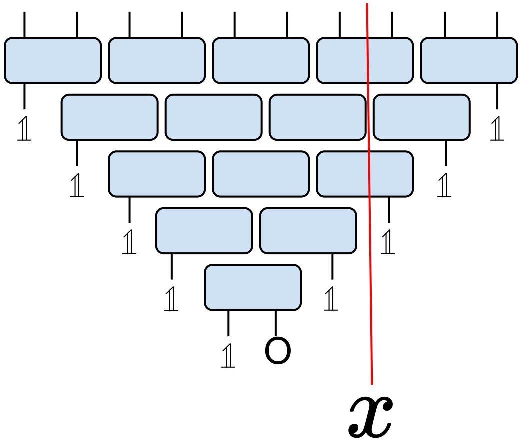
The dimension of the basis on each side of the cut increases exponentially with time:

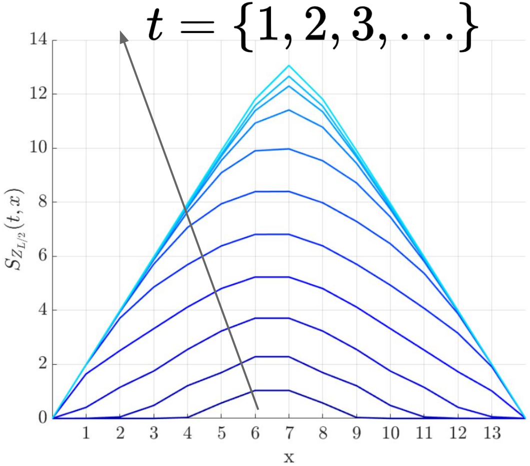
In general OSEE increases linearly with time t

Operator space entanglement in chaotic systems

Example of the circuits with Haar random unitary gates U_{ii+1}

Jonay, C., Huse, D. A., & Nahum, A. (2018). Coarse-grained dynamics of operator and state entanglement. *arXiv preprint arXiv:1803.00089*.





Quantum cellular automata and operator dynamics

Motivation for studying reversible cellular automata:

Obtaining exact and closed form solution of dynamics in unitary interacting systems

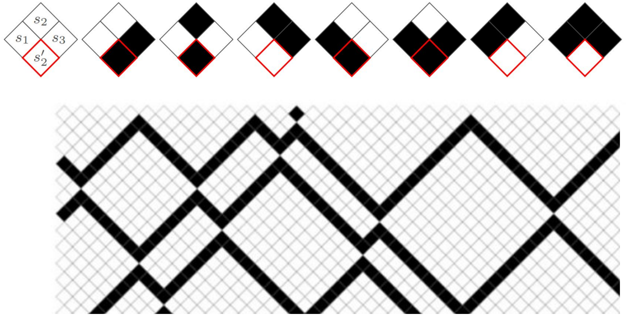
Classical reversible cellular automata:

- Deterministic local maps between the pointer states
- The dynamics looks the same in both time directions

Example 1: Rule 54

Bobenko, A., et al. (1993). *CMP 158*(1), 127-134.

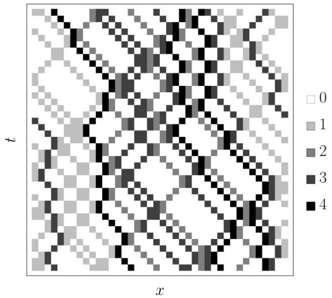
$$\chi(s_1, s_2, s_3) = s_1 + s_2 + s_3 + s_1 s_3 \pmod 2$$



Example 2: Hard-core interacting gases

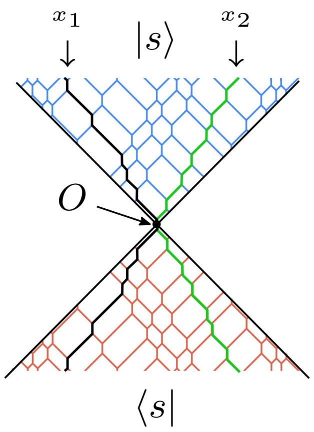
Medenjak, M., Klobas, K. and Prosen, T., 2017. PRL, *119*(11), p.110603.

$$U_{12}\,|p_1 p_2\rangle = (1-\delta_{p_1,0})(1-\delta_{p_2,0})\,|p_1 p_2\rangle + (\delta_{p_1,0} + \delta_{p_2,0} - \delta_{p_1,0}\delta_{p_2,0})\,|p_2 p_1\rangle$$



Quantum cellular automata and operator dynamics

Cellular automata define unitary maps U, which give rise to quantum time evolution



- Solving the local operator dynamics in terms of the matrix product ansatz

$$E_x^{ij} = \cdots \otimes \mathbb{1} \otimes \underbrace{|i\rangle\langle j|}_{\text{site x}} \otimes \mathbb{1} \otimes \cdots$$

$$O(t) = \sum_{\underline{i},\underline{j}} \langle L_{i_{-t+1},j_{-t+1}}(t)| A_{i_{-t+2},j_{-t+2}}(t) \cdots B_{i_{t-1},j_{t-1}}(t) A_{i_t,j_t}(t) |R(t)\rangle E_{-t+1}^{i_{-t+1}j_{-t+1}} \ldots E_t^{i_t j_t}$$

- Exact solution $\longrightarrow$ upper bound on the OSEE:

MPA gives us the decomposition:

$$O(t) = \sum_{\underline{i}_L,\underline{j}_L,\underline{i}_R,\underline{j}_R} c_{\underline{i}_L,\underline{j}_L,\underline{i}_R,\underline{j}_R} E_{-t+1}^{i_{-t+1}j_{-t+1}} \cdots E_{-t}^{i_t j_t}$$

There exists a rotation of the basis, which brings the operator in the normal form $\frac{O}{\sqrt{\mathrm{tr}(OO^\dagger)}} = \sum_i \sqrt{\lambda_i} O_i^{(A)} O_i^{(B)}$; $\mathrm{tr}(O_i^{(A/B)} O_j^{(A/B)}) = \delta_{ij}$ obtained by oing the SVD decomposition of the matrix $[C]_{\underline{i}_L\underline{j}_L,\underline{i}_R\underline{j}_R} = c_{\underline{i}_L,\underline{j}_L,\underline{i}_R,\underline{j}_R} E_{-t+1}^{i_{-t+1}j_{-t+1}} \ldots E_{-t}^{i_t j_t}$

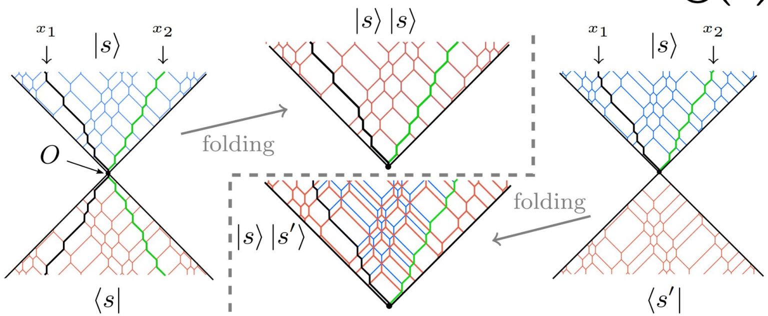
Maximal number of non-zero λ is upper bounded by the dimension D of the matrix C (and consequently MPA matrices) $\longrightarrow$ upper bound on OSEE

$$S < \log(D)$$

Upper bounds on OSEE in quantum cellular automata

Rule 54:

- Does the configuration emerge from the soliton in the origin
- MPA counts the number of colisions
- MPA dimension $t^2 \rightarrow S < 2\log(t)$



Klobas, K., Medenjak, M., Prosen, T., & Vanicat, M. (2019). CMP *371*(2), 651-688.

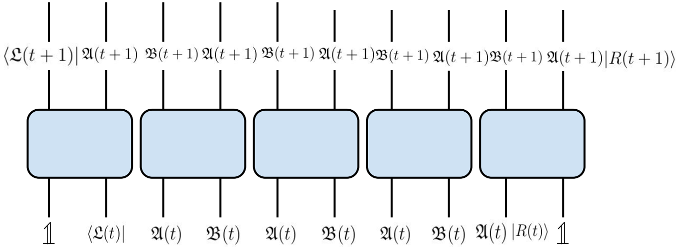
Alba, V., Dubail, J., & Medenjak, M. (2019), PRL *122*(25), 250603.

HC gas:

$$\mathfrak{A}(t) = \sum_{i,j} A_{ij}(t) E^{ij}, \quad \mathfrak{B}(t) = \sum_{i,j} B_{ij}(t) E^{ij}, \quad \langle \mathfrak{L}(t)| = \sum_{i,j} \langle L_{ij}(t)|\, E^{ij},$$

$$O(t) = \langle \mathfrak{L}_{i_{-t+1},j_{-t+1}}(t)|\, \mathfrak{A}_{i_{-t+2},j_{-t+2}}(t) \cdots \mathfrak{B}_{i_{t-1},j_{t-1}}(t) \mathfrak{A}_{i_t,j_t}(t)\, |\mathfrak{R}(t)\rangle$$

- Bulk $\quad U(\mathfrak{A}(t) \otimes \mathfrak{B}(t))U^\dagger = \mathfrak{B}(t+1) \otimes \mathfrak{A}(t+1)$
- Boundary $\quad \langle \mathfrak{L}(t+1)|_{-t}\, \mathfrak{A}(t+1)_{-t+1} = U_{-t,-t+1}(\mathbb{1} \otimes \langle \mathfrak{L}(t)|_{-t+1})U^\dagger_{-t,-t+1}, \quad \mathbb{1}\, |R(t)\rangle = \mathfrak{B}(t)\, |R(t+1)\rangle\,,$
- Initial condition $\quad O(0) = \langle \mathfrak{L}(0)|R(0)\rangle$

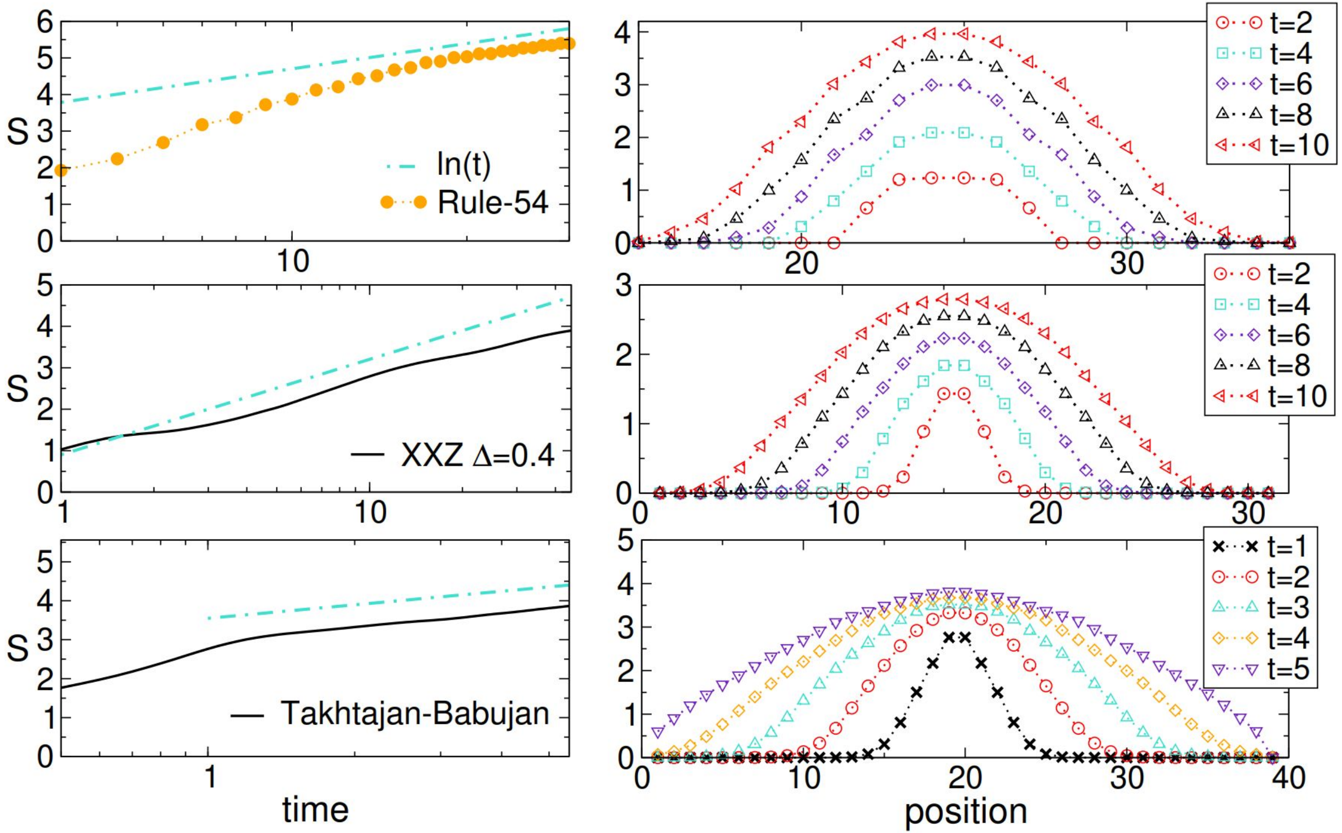


- MPA dimension $t \rightarrow S < \log(t)$

Medenjak, M., Popkov, V., Prosen, T., Ragoucy, E., & Vanicat, M. (2019). *SciPost physics*, *6*(6), 074.

M. Medenjak, arXiv:2201.00395,

OSEE in generic integrable systems



Quantum Lyapunov exponents: Out-of-time ordered correlation functions

Reminder: classical Lyapunov exponent $\lambda = \lim_{t\to\infty} \lim_{|\delta_x(0)|\to 0} \frac{1}{t}\log\left(\frac{|\delta \underline{x}(t)|}{|\delta x(0)|}\right)$

$$\frac{\partial x(t)}{\partial x(0)} = \{x(t), p\} \to \lambda = \lim_{t\to\infty} \frac{1}{t}\log(\{x(t), p\})$$

Naive quantization: $\lim_{t\to\infty} \frac{1}{t}\log(\{A(t), B\}) \to \lim_{t\to\infty} \frac{1}{t}\log(\|[A(t), B]\|)$

$$\|[A(t), B]\|^2 = \frac{\mathrm{tr}([A(t),B][A(t),B]^\dagger)}{\mathrm{tr}(\mathbf{1})}$$

OTOCs in chaotic systems

Results for OTOCs in quantum systems

- Bound on chaos $\lambda \leq 2\pi \frac{k_B T}{\hbar}$

Maldacena, J., Shenker, S. H., & Stanford, D. (2016). JHEP, *2016*.

- Finite local Hilbert-space dimension: no increase of the norm with time

$$\|[A(t), B]\| \leq \|[A(t), B]\|^*$$

- OTOCs in chaotic 2d CFTs

Roberts, Daniel A., and Douglas Stanford. PRL 115.13 (2015): 131603.

- What happens with integrable systems in the limit of a large HS dimension?

OTOCs in HC gasses

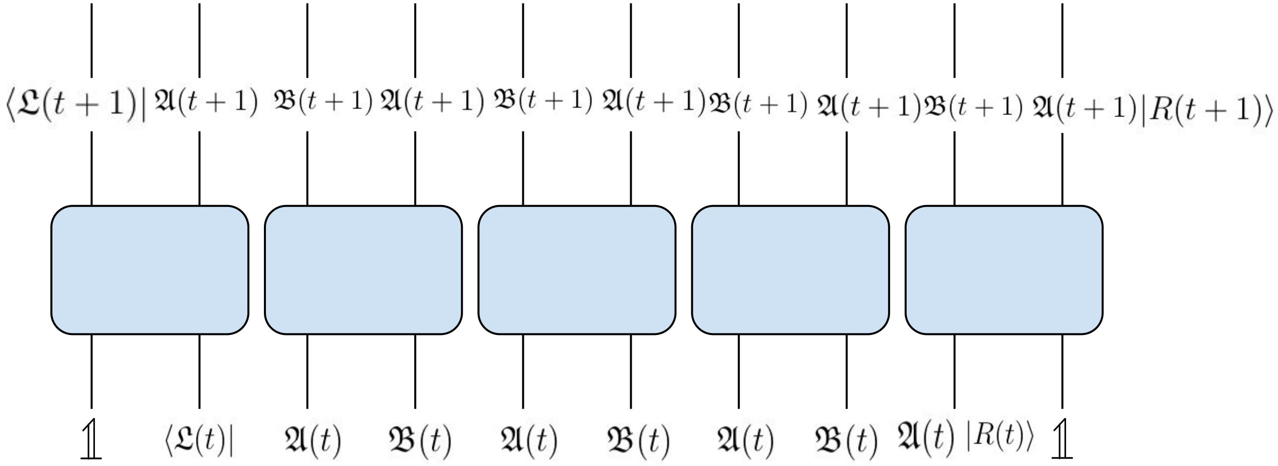
- Bulk condition
- Boundary conditions
- Initial condition

$$U(\mathfrak{A}(t) \otimes \mathfrak{B}(t))U^\dagger = \mathfrak{B}(t+1) \otimes \mathfrak{A}(t+1)$$

$$\langle \mathfrak{L}(t+1)|_{-t}\ \mathfrak{A}(t+1)_{-t+1} = U_{-t,-t+1}(\mathbb{1} \otimes \langle \mathfrak{L}(t)|_{-t+1})U^\dagger_{-t,-t+1},$$

$$\mathbb{1}\,|R(t)\rangle = \mathfrak{B}(t)\,|R(t+1)\rangle\,,$$

$$O(0) = \langle \mathfrak{L}(0)|R(0)\rangle$$



Local physics from algebraic conditions

- Example: diagonal observables

$$\mathfrak{A} = \sum_s A_s E^{ss}$$
$$\mathfrak{B} = \sum_s B_s E^{ss}$$

$$O(t) = \sum_{\underline{i},\underline{j}} \langle L_{i_{-t+1},j_{-t+1}}(t)| A_{i_{-t+2},j_{-t+2}}(t) \cdots B_{i_{t-1},j_{t-1}}(t) A_{i_t,j_t}(t) |R(t)\rangle E^{i_{-t+1}j_{-t+1}}_{-t+1} \dots E^{i_t j_t}_t$$

- Bulk conditions

$$A_s B_{s'} = B_s A_{s'}; \quad s, s' \neq 0,$$
$$A_s B_0 = B_0 A_s,$$
$$A_0 B_s = B_s A_0,$$

- Boundary/initial conditions

$$\langle L_s| A_{s'} = \langle L_{s'}|; \quad s \neq 0,$$
$$\langle L_0| A_{s'} = \langle L_0|,$$
$$\langle L_s| A_0 = \langle L_s|,$$
$$B_s |R\rangle = |R\rangle,$$
$$\langle L_s| R\rangle = \delta_{s,z}.$$

Out-of-time ordered correlation functions

- Spreading of OTOCs

$$C(i,t) = \frac{\mathrm{tr}([E_i^{xy}(0), E_1^{zw}(t)][E_i^{xy}(0), E_1^{zw}(t)]^\dagger)}{\mathrm{tr}(\mathbb{1})}$$

- If z=w

$$C(i,t) = \frac{(q-1)(\delta_{z,x}+\delta_{z,y}-2\delta_{z,x}\delta_{z,y})}{q} \sum_{l}^{\min(k-1,t-k)} \binom{k-1}{l}\binom{t-k}{l}(1-1/q)^{2l} q^{-(t-2l-1)}$$

- Infinite local Hilbert-space limit:

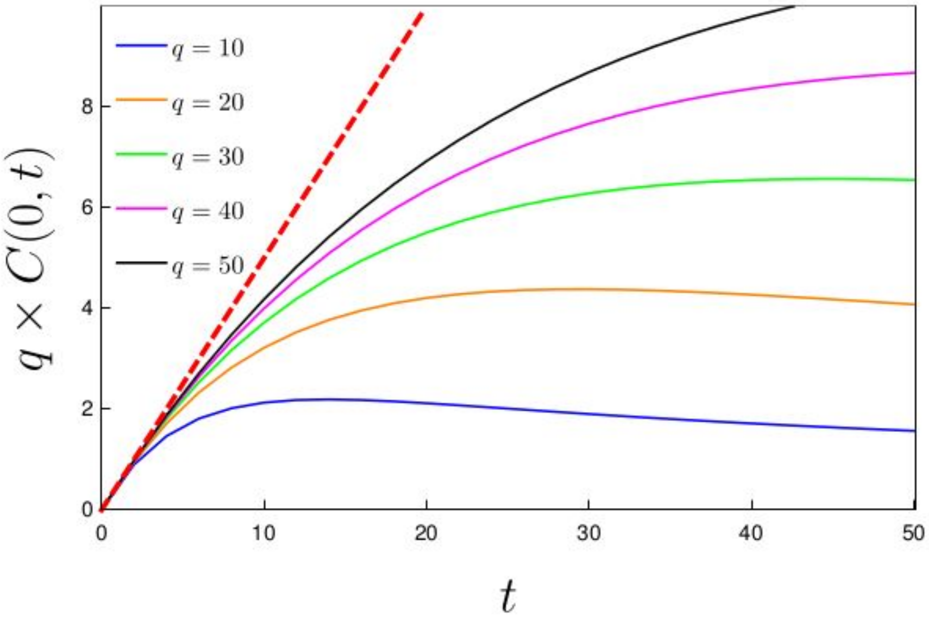
$$\lim_{q\to\infty} q \times C(i,t) = \begin{cases} \frac{t}{2}; & i = \{0,2\}, \\ 0; & \text{Otherwise.} \end{cases}$$

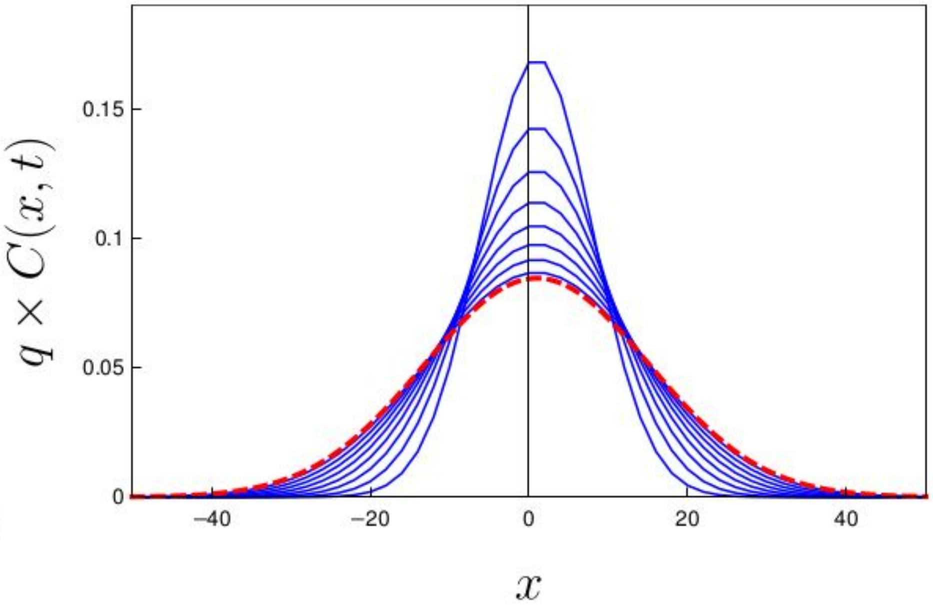
Hydrodynamics:

$$C(x\sqrt{t},t) = \frac{1-1/q}{\sqrt{2\pi t/q(1-1/q)}} \exp\left(-\frac{(q-1)x^2}{2}\right) + \mathcal{O}(1/t)$$

Out-of-time ordered correlation functions

- Spreading of OTOCs





Open problems

1. When can algebraic conditions be solved
2. Proving that OSEE is logarithmic in all integrable systems
3. Proving KPZ in integrable systems
4. Can more complicated many-body integrable systems exhibit exponential increase of OTOCs