Ising correlations; open questions

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Introduction

The goal of this talk is to bring to attention several of the outstanding unsolved problems of the correlation functions of the anisotropic Ising model on the square lattice. Some of these problems go back for decades. More recent work is

BMM and J.M. Maillard, The anisotropic Ising correlations as ellipic integrals: duality and differential equations, J. Phys. A 40 (2016) 434004 (arXiv:1606.08796v4)

S. Boukraa, J.M. Maillard and BMM,The Ising correlation $C(M,N)$ for $\nu = -k$, J. Phys. A 53 (2020), 465202 (arXiv:2204.10096v1)

S Boukraa, C. Cosgrove, J.M. Maillard and BMM, Factorization of Ising correlations $C(M,N)$ for $\nu = -k$ and $M+N$ odd, $M \leq N, T < T_C$ and their lambda extensions, arXiv:2204.10096v1

Outline

Background

1. Nonlinear equations in the scaling limit
2. Nonlinear equations for $C(N, N)$ on the lattice

Open questions

3. Determinants versus difference equations
4. The determinant for $C(N-1, N)$
5. The case $\nu = -k$ for $T < T_c$
6. Painlevé VI for $C(M, N)$ for $\nu = -k$
7. PVI for 2 factors $M + N$ odd
8. PVI for 4 factors of $C(0, N)$ N odd
9. More determinants

Very open questions

1. Scaling limit

The Ising model has the interaction energy

$$‘E = -\sum_{j,k}\{E_v \sigma_{j,k}\sigma_{j+1,k} + E_h \sigma_{j,k}\sigma_{j,k+1}\}$$

where $\sigma_{j,k} = \pm 1$ is the spin at row j and column k and the sum is over all lattice sites and we set

$$k_{\mp} = (\sinh 2E_v/k_BT \ \sinh 2E_h/k_BT)^{\mp 1}, \quad \nu = \frac{\sinh 2E_h/k_BT}{\sinh 2E_v/k_BT}.$$

At T_c (the critical temperature) $k_{\pm} = 1$ and the correlation function $C(M,N) = \langle \sigma_{0,0}\sigma_{M,N}\rangle$ is constant on the ellipses

$$\nu^{1/2}M^2 + \nu^{-1/2}N^2 = R^2(\sinh 2E_h/k_BT_c + \sinh 2E_v/k_BT_c)$$

for large R which for $M = N$ reduces to $R = N$.

The scaling limit is $R \to \infty, \ \ k_{\pm} \to \infty$ with

$r = R(1 - k_{\pm})$ fixed

The scaling function is defined as

$$G_{\pm}(r) = \lim(1 - k_{\pm}^2)^{-1/4} C(M,N)$$

We use the convention $0 \leq M \leq N$

Scaling function as Painlevé

In 1976 it was found by Wu, McCoy, Tracy and Barouch

$G_{\pm}(r) =$

$\frac{1}{2}[1 \pm \eta(r/2)]\eta(r/2)^{-1/2} \exp \int_{r/2}^{\infty} d\theta \frac{1}{4}\theta\eta^{-2} \left[(1-\eta^2)^2 - (\eta')^2\right]$

where $\eta(\theta)$ satisfies the Painlevé III equation

$\frac{d^2\eta}{d\theta^2} = \frac{1}{\eta}\left(\frac{d\eta}{d\theta}\right)^2 - \frac{1}{\theta}\frac{d\eta}{d\theta} + \eta^3 - \eta^{-1}$

with the boundary conditions

$\eta(\theta) \sim 1 - 2\lambda K_0(2\theta)$ as $\theta \rightarrow \infty$ where $\lambda = 1/\pi$

and $K_0(2\theta)$ is the modied Bessel function.

Equivalently for $\zeta = r\frac{d\ln G_{\pm}}{dr}$

$(r\zeta'')^2 = 4(r\zeta' - \zeta)^2 - 4(\zeta')^2(r\zeta' - \zeta - 1/4)$

which is a form of Painlevé V.

Comments

1.The only place where the anisotropy enters in the scaling limit is the determination of the elliptical symmetry. The anisotropy does not enter in the PIII or the PV equation,

2. By definition the Painleve equations have the property that the locations of branch points is independent of the boundary conditions; however the behaviour at these branch points will depend on boundary conditions. The location of poles, however, will depend on boundary conditions. It is important to note that the Painlevé property is a property of the equation and is not a property of the solution so the term Painlevé function is misleading. It is not known if the Painlevé property has any intrinsic relation to integrability.

2. Painlevé VI for $C(N,N)$

In 1980 Jimbo and Miwa found for $T < T_c$

$$\sigma_N(t) = t(t-1)\frac{d\ln C_-(N,N)}{dt} - \frac{t}{4}$$

with $t = k_-^2$

and for $T > T_c$

$$\sigma_N(t) = t(t-1)\frac{d\ln C_-(N,N)}{dt} - \frac{1}{4}$$

with $t = k_+^2$

that for both $T < T_c$ and $T > T_c$

$$\left(t(t-1)\frac{d^2\sigma}{dt^2}\right)^2 =$$

$$N^2\left((t-1)\frac{d\sigma}{dt} - \sigma\right)^2 - 4\frac{d\sigma}{dt}\left((t-1)\frac{d\sigma}{dt} - \sigma - \frac{1}{4}\right)\left(t\frac{d\sigma}{dt} - \sigma\right)$$

which is a form of Painlevé VI

Boundary conditions

For $T < T_c$ the boundary condition at $t = 0$ is
$$C(N,N;t) = (1-t)^{1/4}\left[1 + \lambda^2 \frac{(1/2)_N\,(3/2)_N}{4[(N+1)!]^2} t^{N+1}(1 + O(t))\right],$$
with $\lambda = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$ and $(a)_0 = 1$.

For $T > T_c$ the boundary condition at $t = 0$ is
$$C(N,N;t) = (1-t)^{1/4} t^{N/2}\left[\frac{(1/2)_N}{N!}\,{}_2F_1\left([\tfrac{1}{2},\ N+\tfrac{1}{2}],\ [N+1],\ t\right)\right.$$
$$\left. + \lambda^2\ \frac{(1/2)_N\ ((3/2)_N)^2}{16\ (N+1)!\,(N+2)!} t^{N+2}(1 + O(t))\right],$$
with $\lambda = 1$.

These expressions satisfy the PVI equation for all λ. The reason why $\lambda = 1$ gives the Ising correlation is an open question.

Comments

This PVI for $C(N,N)$ does not depend on the anisotropy ν. The reason is that on the diagonal transfer matrices for different values of ν commute so ν is a spectral variable for the diagonal transfer matrix. All other $C(M,N)$ will depend on ν. As an example for $T < T_c$

$$C(1,1) = \tilde{E}(k) = \frac{2}{\pi}\int_0^{\pi/2} d\phi (1 - k^2 \sin^2\phi)^{1/2}$$

and

$$C(0,1) = \sqrt{1+\nu k}\left[\left(1+\frac{k}{\nu}\right)\tilde{\Pi}(-\nu k,\, k) \; - \frac{k}{\nu}\tilde{K}(k)\right]$$
$$= \sqrt{1+\nu k}\frac{2}{\pi}\int_0^{\pi/2} d\theta \frac{(1-k^2\sin^2\theta)^{1/2}}{1+k\nu\sin^2\theta}.$$

where the three elliptic integrals

$$\tilde{K}(k) = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}}$$
$$\tilde{\Pi}(-k\nu,\, k) = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{(1+k\,\nu\,\sin^2\theta)\,(1-k^2\sin^2\theta)^{1/2}}$$

are normalized to 1 at $k = n = 0$

3. Determinants

All correlation functions $C(M,N)$ can be represented as determinants by drawing paths on the lattice (or on the diagonal) which connect $(0,0)$ with (M,N). Any path connecting these points can be used. For $C(0,N)$ the straight line path of N links gives the $N\times N$ Toeplitz determinant

$$C(0,N)=\begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix}$$

$$a_n=\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{(1-\alpha_1\,e^{i\theta})\,(1-\alpha_2\,e^{-i\theta})}{(1-\alpha_1\,e^{-i\theta})\,(1-\alpha_2\,e^{i\theta})}\right]^{1/2}e^{in\theta}\;d\theta$$

$$\alpha_1=e^{-2E_v/k_BT}\tanh E_h/k_BT,\quad \alpha_2=e^{-2E_v/k_BT}\coth E_h/k_BT$$

For $C(N,N)$ a diagonal path of N steps gives the same $N\times N$ determinant but with $\alpha_1=0$ and $\alpha_2=k$

Relations of variables

For $T < T_c$

$k = \frac{\alpha_2 - \alpha_1}{1 - \alpha_1 \alpha_2}, \quad \nu = \frac{4\alpha_1\alpha_2}{(\alpha_2 - \alpha_1)(1 - \alpha_1\alpha_2)}$

Thus

$k\nu = \frac{4\alpha_1\alpha_2}{(1-\alpha_1\alpha_2)^2}, \quad (\alpha_1\alpha_2)^{\pm 1} = 1 + \frac{2}{k\nu} \mp \frac{2}{k\nu}\sqrt{1+k\nu}$

and

$$\alpha_{2,1} = \frac{1}{\nu}[\sqrt{1+k\nu} - 1]$$
$$(+,-)\frac{1}{\nu}\sqrt{[\nu/k + 1 - \sqrt{1+k\nu}]^2 + \nu^2(1-k^{-2})}$$

Note that if $\nu = 0$ then $\alpha_2 = k, \ \alpha_1 = 0$

if $k = 1$ then $\alpha_2 = 1$

if $k = 0$ then $\alpha_2 = \alpha_1 = 0$

If $N \to \infty$ then $C(N,N)$ decays to $(1-k^2)^{1/4}$ as k^{2N}/N^2 and $C(0,N)$ decays to $(1-k^2)^{1/4}$ as α_2^{2N}/N^2.

Difference equations

We thus have two very different representations of $C(0,1)$ in terms of either $\alpha_1,\ \alpha_2$ or $k\ \nu$ and the relation of the two is NOT seen by a simple change of variables. Instead a nontrivial identity must be used. It is an open question how to generalize this identity to $C(0,N)$. In 2016 Witte proposed the existence of partial differential equations for $C(0,N)$ in terms of the $\alpha_1,\ \alpha_2$ variables. However, no expression for the general case of $C(M,N)$ in terms of α_1 and α_2 is known.

To overcome this limitation we have computed $C(M,N)$ using the quadratic difference equations obtained by Wu, Perk and BMM in 1980-1981 which allows the recursive computation of all $C(M,N)$ starting with the known expressions of $C(0,1)$ and $C(N,N)$ in terms of the three complete elliptic integrals using the variables k and ν (and not $\alpha_1,\ \alpha_2$).

Examples

$$C(0,2) = \frac{1}{k\nu}\{(1+\frac{k}{\nu})(1+k\nu)(2k^2+k\nu+\frac{k}{\nu})\tilde{\Pi}^2$$
$$-2\frac{k}{\nu}(\frac{k}{\nu}+1)(k\nu+1)^2\tilde{K}\tilde{\Pi}$$
$$+(k^2\frac{k}{\nu}+2k^2+\frac{k^2}{\nu^2}+\frac{k}{\nu}-1)\tilde{K}^2+2(1-k^2)\tilde{E}\tilde{K}-\tilde{E}^2\}$$

$$C(1,2) = \frac{(1+k\nu)^{1/2}}{k\nu}\{\frac{k}{\nu}(k^2-1)\tilde{K}^2+(\frac{k}{\nu}-1)\tilde{E}\tilde{K}+\tilde{E}^2$$
$$+(k\nu-1)(\frac{k}{\nu}+1)\tilde{E}\tilde{\Pi}-(k^2-1)(\frac{k}{\nu}+1)\tilde{K}\tilde{\Pi}\}$$

$C(M,N)$ is a homogeneous polynomial of degree N in $\tilde{E},\ \tilde{K},\ \tilde{\Pi}$

The highest power of $\tilde{\Pi}$ is $N-M$

For odd $N-M$ there is a factor of $(1+k\nu)^{1/2}$

If $\nu=-k$ then $C(M,N)$ is a function of $\tilde{K},\ \tilde{E}$ only because the coefficient of all terms with $\tilde{\Pi}$ vanishes.

4. $C(N-1,N)$

In 1987 AuYang and Perk derived

$$C(N-1,N) = \begin{vmatrix} a_0 & a_1 & \cdots & a_{N-2} & b_{N-1} \\ a_{-1} & a_0 & \cdots & a_{N-3} & b_{N-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{-N+2} & a_{-N+3} & \cdots & a_0 & b_1 \\ a_{-N+1} & a_{-N+2} & \cdots & a_{-1} & b_0 \end{vmatrix}$$

where

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{in\theta}\sqrt{\frac{1-ke^{i\theta}}{1-ke^{-i\theta}}}$$

$$b_n = \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}z^n\{\sqrt{1+k\nu}\sqrt{\frac{1-kz}{1-kz^{-1}}}\frac{1}{1+z\nu} + \frac{\sqrt{1+k/\nu}}{1+1/z\nu}\}$$
$$= \sqrt{1+k\nu}\frac{1}{\pi}\int_0^k \frac{dz}{z}z^n\sqrt{\frac{1-kz}{k/z-1}}\frac{1}{1+\nu z}$$

We note that $b_0 = C(0,1)$

and at $\nu = 0$ we have $C(N-1,N) = C(N,N)$

Can the AuYang-Perk result be generalized to $C(N-n,N)$?

5. The case $\nu = -k$ for $T < T_c$

In the special case $\nu = -k$ the elliptic integral $\tilde{\Pi}$ does not appear. For example with $t = k^2$

$$C(0,2) \;=\; \frac{1}{t} \cdot \left[\tilde{E}^2 \,-2\,(1-t) \cdot \tilde{E}\tilde{K} \,+ (1-t) \cdot \tilde{K}^2 \right]$$

$$C(0,3) \;=\; -\frac{4\,\sqrt{1-t}}{t^2} \cdot \tilde{E} \cdot (\tilde{E} \,-\tilde{K}) \cdot (\tilde{E} \,+ (t\,-1) \cdot \tilde{K})$$

$$C(1,3) \;=\; \frac{4}{3\,t^2} \cdot \left[(2\,-t) \cdot \tilde{E}^3 \,-5 \cdot (1\,-t) \cdot \tilde{E}^2\,\tilde{K} \right.$$

$$\left. +(1\,-t) \cdot (2-t) \cdot \tilde{E}\tilde{K}^2 \,-(1\,-t)^2 \cdot \tilde{K}^3 \right]$$

$$C(0,5) \;=\; \frac{256\sqrt{1\,-t}}{81\,t^6} \cdot \left[(1\,+t) \cdot \tilde{E} + (t-1) \cdot \tilde{K} \right] \cdot \left[(t\,-2) \cdot \tilde{E} \,+ 2\,(1\,-t) \cdot \tilde{K} \right]$$

$$\times \left[(2t\,-1) \cdot \tilde{E} \,+(1\,-t) \cdot \tilde{K} \right] \cdot \left[3\,\tilde{E}^2 \,+(2t\,-4) \cdot \tilde{E}\,\tilde{K} \,+(1\,-t) \cdot \tilde{K}^2 \right].$$

$$C(1,2) = -\frac{\sqrt{1-k^2}}{k^2}\{\tilde{E}^2 - 2\tilde{E}\tilde{K} + (1-k^2)\tilde{K}^2$$

$$= -\frac{\sqrt{1-k^2}}{k^2}\{\tilde{E} - \tilde{K} - k\tilde{K}\}\{\tilde{E} - \tilde{K} + k\tilde{K}\}$$

$$C(2,3) = \frac{4}{9}\,\frac{\sqrt{1-t}}{t^2}\,\left(3\tilde{E}^2 + (t-5)\tilde{E}\tilde{K} - 2(t-1)\tilde{K}^2\right)\left((t+1)\tilde{E} + (t-1)\tilde{K}\right)$$

Factorization for $M + N$ odd

1. For $M + N$ odd the correlations $C(M, N)$ with $M \neq 0$ factorize into two homogeneous polynomials in $\tilde{K},\ \tilde{E}$.
For N odd the polynomials are in the variable $t = k^2$ which are analytic at $t = 0$. One factor is of degree $(N - 1)/2$ and the other is of degree $(N + 1)/2$,
For N even the polynomials are in the variable k which are analytic at $k = 0$ and both factors are of degree $N/2$.

2. $C(0, N)$ for N odd has three factors for $N = 3$ and four factors for $N \geq 5$ which are homogeneous polynomials in t. For $N = 4n + 1$ there are three factors of degree n and one of degree $n + 1$. For $N = 4n - 1$ there are three factors of degree n and one factor of degree $n - 1$. The explanation for the existence of 4 factors is not known.

6. Painlevé VI

The Gambier form is

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$
$$+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right).$$

The Okamoto form is

$$y'\{t(t-1)y''\}^2 + \{y'(2y-(2t-1)y') + n_1n_2n_3n_4\}^2$$
$$-(y'+n_1^2)(y'+n_2^2)(y'+n_3^2)(y'+n_4^2) = 0$$

The Cosgrove-Scoufis form is

$$x^2(x-1)^2y''^2 + 4y'(xy'-y)((x-1)y'-y)$$
$$+c_5(xy'-y)^2 + c_6y'(xy'-y) + c_7(y')^2$$
$$+c_8(xy'-y) + c_9y' + c_{10} = 0.$$

The Cosgrove-Scoufis equation preserves its form under the linear shift

$$y \longrightarrow y + A + Bx.$$

Relations

The three forms are related to each other.

The 4 parameter Gambier form is birationally equivalent to the 4 parameter Okamoto form.

The linear change of variable of the Cosgrove-Scoufis equation may be used to set $c_5 = c_6 = 0$ and then the Cosgrove-Scoufis equation reduces to the Okamoto equation once it is expanded and a factor of y' is cancelled where then

$$c_7 = -(n_1^2 + n_2^2 + n_3^2 + n_4^2), \quad c_8 = -4n_1n_2n_3n_4,$$

$$c_9 = -(n_1^2n_2^2 + n_1^2n_3^2 + n_1^2n_4^2 + n_2^2n_3^2 + n_2^2n_4^2 + n_3^2n_4^2 - 2\,n_1n_2n_3n_4),$$

$$c_{10} = -(n_1^2n_2^2n_3^2 + n_1^2n_2^2n_4^2 + n_1^2n_3^3n_4^2 + n_2^2n_3^2n_4^2)$$

lambda extensions

The correlations $C(M,N)$ are all analytic at $k^2 = t = 0$. The solutions of Okamoto's equation which are analytic at $t = 0$ are in general unique. However this uniqueness fails for the following 4 cases

1. $n_1 + n_2 - n_3 - n_4 \pm n = 0$
2. $n_1 - n_2 - n_3 + n_4 \pm n = 0$
3. $n_1 - n_2 + n_3 - n_4 \pm n = 0$
4. $n_1 + n_2 + n_3 + n_4 \pm n = 0$

where n_i are the four Okamoto parameters and n is an integer. In this case the coefficient of t^{n+1} is arbitrary. All our $C(M,N)$ are of this form with a specific value of the arbitrary constant which serves as a boundary condition on the PVI equation. The general solution with the constant arbitrary we call lambda extensions.

PVI for $C(M,N)$ at $\nu = -k$

To obtain a nonlinear equation characterizing $C(M,N)$ with $\nu = -k$ we set

$$\sigma(M,N) = t(t-1)\frac{d\ln C(M,N)}{dt} - \frac{t}{4}$$

and using the homogeneous polynomials for $C(M,N)$ in terms of $\tilde{K}(k)$ and $\tilde{E}(k)$ we expand $\sigma(M,N)$ in a power series in t and then use the program of Jay Pantone called *guessfunc* to produce a nonlinear equation for $\sigma(M,N)$. We have done this for many values of M, N. The output is an equation of third order. There exists an integrating factor for this third order equation and we find an equation of the Cosgrove-Scoufis form with

$$c_5 = -M^2, \;\; c_6 = M^2 + N^2 - \tfrac{1}{2}(1 + (-1)^{M+N}), \;\; c_7 = -N^2,$$
$$c_8 = c_9 = c_{10} = 0$$

Okamoto for $C(M, N)$

To convert the Cosgrove-Scoufis form to an Okamoto equation for Painlevé VI we set

$\sigma = \tilde{\sigma} + At + B$

with

$A = \frac{M^2}{4}, \quad B = \frac{1}{8}(N^2 - M^2 - \frac{1+(-1)^{M+N}}{2})$

and find Okamoto parameters (unique up to permutations and the change of an even number of signs)

$n_1 = \frac{1}{2}(N + \frac{1+(-1)^{M+N}}{2}) \quad n_2 = \frac{1}{2}(N - \frac{1+(-1)^{M+N}}{2})$

$n_3 = \frac{M}{2}, \quad n_4 = -\frac{M}{2}$

Note that $n_1 + n_2 + n_3 + n_4 = N$ so the coefficient (called λ^2) of t^{N+1} in the expansion about $t = 0$ is arbitrary and must be chosen such that $C(M, N)$ is a homogeneous polynomial in $\tilde{K}(k)$ and $\tilde{E}(k)$.

7. PVI for factors $M+N$ odd

For $M+N$ we write the factorization as

$(1-t)^{-1/4}C(M,N) = g_+(M,N;t)g_-(M,N;t)$

Then defining

$\sigma_\pm(M,N;t) = t(t-1)\frac{\ln g_\pm(M,N;t)}{dt}$

there is an additive decomposition of the sigma functions

$\sigma(M,N;t) = \sigma_+(M,N;t) + \sigma_-(M,N;t)$

where $\sigma(M,N;t)$ is the sigma function for $C(M,N;t)$ previously found.

Nonlinear equations

We apply Pantones program *guessfunc* to $\sigma_\pm$ and use an integrating factor to find that both σ_+ and σ_- satisfy the *same* nonlinear equation but with *different* boundary conditions at $t = 0$.

$$32t^3(t-1)^2\sigma''^2 + 4t^2(t-1)\Big(8\sigma - 8(t+1)\sigma' + M^2 - N^2\Big)\sigma''$$
$$-\Big(\sigma - 16t\sigma' + M^2t - N^2 + 1 - t\Big)$$
$$\times\Big(8t(t-1)\sigma'^2 - 16t\sigma\sigma' + 8\sigma^2 + (M^2-N^2)\sigma\Big) = 0$$

In contrast with the $\sigma(M,N)$ for $C(M,N)$ this equation is NOT of the Cosgrove-Scoufis form.

Landen transformation

We are able to reduce the nonlinear equation for $\sigma_\pm$ to an Okamoto form of PVI by making a Landen change of the independent variable

$$k^2 = t = \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2,$$

and redefining the dependent variable from $\sigma(t)$ to $h(x)$ where

$$\sigma(t) = \tilde{\sigma}(x) = \frac{2}{\pi}\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}(h(x) - h_0(x))$$

$$h_0(x) = \frac{M^2-3N^2+1}{16} - \frac{M^2-N^2+1}{16}x + \frac{M^2-N^2}{16}x\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)$$

$h(x)$ satisfies the Okamoto equation with parameters

$$n_1 = \frac{M+N+1}{4}, \quad n_2 = \frac{M+N-1}{4}, \quad n_3 = \frac{N-M+1}{4}, \quad n_4 = \frac{N-M-1}{4}.$$

Boundary conditions

To specify the boundary conditions on the Okamoto equation for $h(x)$ we note for class 4 boundary conditions
$n_1 + n_2 + n_3 + n_4 = N$
and thus the coefficient of x^{N+1} in the expansion of $h(x)$ at $x = 0$ is arbitrary.

To proceed further we extend the recursive analysis beyond the term x^{N+1}. We find that the coefficients of x^n for $(N+1) \leq n \leq (2N+1)$ depend only on c_{N+1}, the coefficient of x^{N+1} but that starting with $c_{2(N+1)}$ the coefficients depend on c_{N+1}^2 as well as c_{N+1}.

$\lambda_+ = -\lambda_-$ for $C(M,N)$

Thus we obtain

$\sigma_\pm(M,N;t;\lambda_\pm) = \sum_{n=1}^{\infty} (\lambda_\pm t^{(N+1)/2})^n B_n(M,N;t)$,

where the $B_n(M,N;t)$'s are power series in t, analytic at $t=0$.

In order for the additive decomposition to hold we need

$\lambda_+ = -\lambda_-$

which gives a one parameter family of factorizable lambda extensions of $C(M,N)$. The specific values of λ_+ needed to reduce to the homogeneous polynomials in $\tilde{K}(k)$, $\tilde{E}(k)$ have been computed recursively as

$$\lambda_+ = \frac{(N+M)!(N-M)!}{2^{2N+1} N!((N+M-1)/2)!((N-M-1)/2)!}.$$

8. $C(0, N)$ with N odd

$C(0,N)$ with N odd factors into 4 terms

$C(0,N) = (1-t)^{1/4} g_1(0,N) g_2(0,N) g_3(0,N) g_3(0,N)$

Setting

$g_i(0,N) = (1-t)^{N/16} t^{-N/8} \tilde{g}_i(0,N) \quad i = 1,2$

$g_i(0,N) = (1-t)^{-N/16} t^{N/8} \tilde{g}_i(0,N) \quad i = 3,4$

we have for example with $N = 5$

$$\tilde{g}_1(0,5) = \tfrac{2}{3}(1-t)^{-3/8} t^{-1} \left((2t-1)\tilde{E} - (t-1)\tilde{K} \right),$$

$$\tilde{g}_2(0,5) = \tfrac{2}{3}(1-t)^{-1/8} t^{-1} \left((t+1)\tilde{E} + (t-1)\tilde{K} \right),$$

$$\tilde{g}_3(0,5) = -\tfrac{8}{3}(1-t)^{1/4} t^{-2} \left((t-2)\tilde{E} - 2\cdot(t-1)\tilde{K} \right)$$

$$\tilde{g}_4(0,5) = -\tfrac{8}{3}(1-t)^{1/2} t^{-2} \left(3\tilde{E}^2 + 2(t-2)\tilde{E}\tilde{K} - (t-1)\tilde{K}^2 \right).$$

$\sigma(0, N)$ for N odd

We define

$\sigma_j = t(t-1)\frac{d \ln g_j(t)}{dt}$

so the factorization of $C(0, N)$ becomes the additive decomposition

$$\sigma(0,N) = \sigma_1(0,N) + \sigma_2(0,N) + \sigma_3(0,N) + \sigma_4(0,N)$$

$$\sigma_1(0,5) = \frac{5}{8} - \frac{5}{16}t - \frac{5}{2^6}t^2 - \frac{5\cdot 11}{2^{10}}t^3 - \frac{5}{2^7}t^4 - \frac{3^2\cdot 5\cdot 43}{2^{16}}t^5 - \frac{5\cdot 4817}{2^{20}}t^6 - \frac{5\cdot 241\cdot 509}{2^{25}}t^7 - \frac{5\cdot 397811}{2^{27}}t^8 - \frac{3\cdot 5\cdot 13\cdot 134401}{2^{31}}t^9 + \cdots$$

$$\sigma_2(0,5) = \frac{5}{8} - \frac{5}{16}t - \frac{5}{2^6}t^2 - \frac{5^2}{2^{10}}t^3 - \frac{5}{2^9}t^4 - \frac{5\cdot 61}{2^{16}}t^5 - \frac{5\cdot 23^2}{2^{20}}t^6 - \frac{5\cdot 10099}{2^{25}}t^7 - \frac{5^2\cdot 71\cdot 73}{2^{27}}t^8 - \frac{5\cdot 281321}{2^{31}}t^9 + \cdots$$

$$\sigma_3(0,5) = -\frac{5}{8} + \frac{5}{16}t + \frac{5}{2^6}t^2 + \frac{5}{2^7}t^3 + \frac{3\cdot 5\cdot 13}{2^{13}}t^4 + \frac{5\cdot 53}{2^{14}}t^5 + \frac{5\cdot 11\cdot 449}{2^{21}}t^6 + \frac{5\cdot 19\cdot 397}{2^{22}}t^7 + \frac{3\cdot 5\cdot 15907}{2^{25}}t^8 + \frac{5\cdot 77527}{2^{26}}t^9 + \cdots$$

$$\sigma_4(0,5) = -\frac{5}{8} + \frac{5}{16}t + \frac{5}{2^6}t^2 + \frac{5}{2^7}t^3 + \frac{5\cdot 41}{2^{13}}t^4 + \frac{5\cdot 59}{2^{14}}t^5 + \frac{5\cdot 5813}{2^{21}}t^6 + \frac{5\cdot 47\cdot 199}{2^{22}}t^7 + \frac{5\cdot 13\cdot 97\cdot 197}{2^{27}}t^8 + \frac{5\cdot 13\cdot 97\cdot 197}{2^{28}}t^9 + \cdots$$

Nonlinear equation

For an appropriate choice of normalizations we used the program guessfunc and have found that all four σ_j satisfy the same equation of Cosgrove-Scoufis form

$$t^2(t-1)^2\sigma''^2 + 4\sigma'(t\sigma' - \sigma)\Big((t-1)\sigma' - \sigma\Big)$$
$$+\tfrac{1}{4}\Big((N^2+1)(t-1) - t^2\Big)\sigma'^2 - \tfrac{1}{2^6}\Big(16(N^2+1-2t)\sigma + N^2 t\Big)\sigma'$$
$$-\tfrac{1}{4}\sigma^2 + \tfrac{N^2}{2^6}\sigma - \tfrac{N^2(N^2-3)}{2^{10}} = 0.$$

Setting $\sigma_i(0,N;t) = h_i(t) + \frac{t}{16} + \frac{N^2-1}{32}$

the function h_i satisfies an Okamoto equation with parameters

$$n_1 = \tfrac{N+1}{4},\;\; n_2 = \tfrac{N-1}{4},\;\; n_3 = -\tfrac{1}{2},\;\; n_4 = 0$$

and

$$\sigma(0,N) = \sum_{i=1}^{4}\sigma_i(0,N) = \sum_{i=1}^{4} h_i + 4\left(\frac{t}{16} + \frac{N^2-1}{32}\right)$$

Boundary conditions

For class 4 boundary conditions

$n_1 + n_2 + n_3 + n_4 = \frac{N-1}{2}$

which makes the coefficient of $t^{(N+1)/2}$ arbitrary and for class 1

$n_1 + n_2 - n_3 - n_4 = \frac{N+1}{2}$

which makes the coefficient of $t^{(N+3)/2}$ arbitrary.

Thus we find for $j = 1, 2$ (case 4)

$\sigma_j(0, N; t; \lambda_j) = \frac{N}{8}\sqrt{1-t} + \sum_{n=1}^{\infty} \left(\lambda_j t^{(N+1)/2}\right)^n B_n^{(4)}(0, N; t)$,

and for $j = 3, 4$ (case 1)

$\sigma_j(0, N; t; \lambda_j) = -\frac{N}{8}\sqrt{1-t} + \sum_{n=1}^{\infty} \left(\lambda_j t^{(N+3)/2}\right)^n B_n^{(1)}(0, N; t)$,

where $B_n^{(4)}(0, N; t)$ and $B_n^{(1)}(0, N; t)$ are power series in t, normalized for both $i = 1$ and 4 and all N:

$B_1^{(i)}(0, N; 0) = 1 \quad i = 1, 4.$

Choice of λ_i

In order for factorization to hold we find that we need

$$\lambda_1 = -\lambda_2, \quad \lambda_3 = -\lambda_4$$

with

$$\lambda_3 = \frac{\lambda_1}{4(N+1)}$$

For this to reduce to a factorization of $C(M, N)$ we need in addition

$$\lambda_1 = -\frac{N!}{2^{2N+1}((\frac{N-1}{2})!)^2}$$

9. Toeplitz determinants

In 2004 Forrester and Witte arxiv:math-ph/0204008 gave a set of $\tilde{N} \times \tilde{N}$ Toeplitz determinants

$$D_{\tilde{N}}^{(p,p',\eta,\xi)}(t) =, \det\left[A_{j-k}^{(p,p',\eta,\xi)}(t)\right]_{j,k=0}^{\tilde{N}-1},$$

where $A_m^{(p,p',\eta,\xi)}(t) = A^{(1)}(t) + \xi A^{(2)}(t)$,

$$A_m^{(1)}(t) = \frac{\Gamma(1+p')\cdot t^{(\eta-m)/2}}{\Gamma(1+\eta-m)\,\Gamma(1-\eta+m+p')} \cdot {}_2F_1[-p,\ -p'+\eta-m],\ [1+\eta-m],\ t)$$

$$A_m^{(2)}(t) = \frac{\Gamma(1+p)\cdot t^{(m-\eta)/2}}{\Gamma(1-\eta+m)\,\Gamma(1+\eta-m+p)} \cdot {}_2F_1([-p',\ -p-\eta+m],\ [1-\eta+m],\ t),$$

where each $A_m^{(1)}$ and $A_m^{(2)}$ separately gives a Toeplitz matrix.

Okamoto parameters

It is sufficient to consider $\xi = 0$ and find that

$$A_m^{(p,p',\eta)}(t) = \begin{cases} A_m^{(1)}(t) & \text{for } m \le \eta \\ A_m^{(2)}(t) & \text{for } m \ge \eta \end{cases}$$

The sigma functions of these determinants satisfy Okamoto's equation with

$$n_1 = (\tilde{N} + \eta + p - p')/2, \qquad n_2 = (\tilde{N} - \eta - p + p')/2,$$
$$n_3 = (\eta - \tilde{N} - p - p')/2, \qquad n_4 = (\eta + \tilde{N} + p + p')/2.$$

and thus

$$\tilde{N} = n_1 + n_2, \;\; \eta = n_3 + n_4, \;\; p = -n_2 - n_3, \;\; p' = -n_1 + n_4.$$

Conclusion

We thus conclude that for $C(M,N)$ with $\nu = -k$ that $C(M,N)$ and for $M+N$ odd the factors of $C(M,N)$ all have a representation as Toeplitz determinants.

This very indirect method of finding a Toeplitz determinant representation is completely unsatisfactory.

Very open questions

1. Row versus diagonal

The row correlations $C(0,N)$ are described as an $N \times N$ determinant in the variables $\alpha_1,\ \alpha_2$.

The diagonal correlations $C(N,N)$ and $C(N-1,N)$ are described as an $N \times N$ determinant in the variables $k,\ \nu$.

In general $C(M,N)$ can be obtained as the Pfaffian obtained from a horizontal path of $N-M$ links with variables $\alpha_1,\ \alpha_2$ and diagonal path of M links with $k,\ \nu$ as variables.

The only correlation which is known in terms of both sets of variables is $C(0,1)$.

To find a generalization of the AuYang-Perk result to $C(N-n,N)$ we need to express $C(0,N)$ in terms of $k,\ \nu$.

This has not been done.

Nonlinear equations

2. Is there a non linear equation in $t = k^2$ (or k) with ν fixed as a parameter of fixed order and degree for $\sigma = t(t-1)\frac{\ln C(M,N)}{dt}$?

At present there is no evidence that such an equation needs to exist at all and there is no argument that if it does exist that it will have the Painlevé property.

A more complicated alternative is that $C(M,N)$ could satisfy a partial differential equation with both k and ν as variables.

ODE's for other systems?

In the 45 years since the Painlevé III representation was found for the scaling function of the Ising model no other model has been found for which the correlation functions satisfy nonlinear equations. Are there any other models?

The most promising candidate is the N state superintegrable chiral Potts spin chain

$$\mathcal{H} = A_0 + \lambda A_1$$
$$A_0 = -\sum_{j=1}^{\mathcal{N}} \sum_{n=1}^{N-1} \frac{e^{i\pi(2n-N)/2N}}{\sin \pi n/N} (Z_j Z_{j+1}^{\dagger})^n$$
$$A_1 = -\sum_{j=1}^{\mathcal{N}} \sum_{n=1}^{N-1} \frac{e^{i\pi(2n-N)/2N}}{\sin \pi n/N} (X_j)^n$$

X_j and Z_j are $N \times N$ matrices at site j with matrix elements

$$Z_{m,n} = \delta_{m,n} e^{2\pi i m/N} \quad X_{m,n} = \delta_{m+1,n}$$

When $N = 2$ this reduces to the transverse Ising chain.

Onsagers algebra

The matrices $A_0,\ A_1$ satisfy

$$[A_0,[A_0,[A_0,A_1]]] = \mathrm{const}[A_0,A_1]$$

$$[A_1,[A_1,[A_1,A_0]]] = \mathrm{const}[A_1,A_0]$$

and from this it follows that

$$[A_j,A_k] = 4G_{j-k},$$

$$[G_m,A_l] = 2A_{l+m} - 2A_{l-m}$$

$$[G_j,G_k] = 0$$

This is the algebra used by Onsager in his original computation of the free energy of the Ising model!

The significance of this algebra for correlation functions is unknown.

Conclusion

It is my hope that some of you will provide answers to these open questions.