Anisotropic spin generalisation of elliptic Ruijsenaars operators and R-matrix identities

(joint work with Andrei Zotov)

arXiv:2201.05944 2202.01177

Maria Matushko

Steklov Mathematical Institute of RAS

Recent advances
in quantum integrable systems

Lyon, France 29.08.2022

Congratulations to Nikita on his birthday!

Outline

- [Ruijenaars 87]

Ruijsenaars-Macdonald operators (scalar case)

commutativity $\Leftrightarrow$ functional identities

degenerations: elliptic $\to$ trigonometric $\to$ rational

- Construction of anisotropic version of Ruijsenaars-Macdonald operators in terms of elliptic R-matrix

commutativity $\Leftrightarrow$ R-matrix identities

- [Lamers 18], [Lamers, Pasquier, Serban 20], [Uglov 95] trigonometric case

„freezing" (applications to long-range spin chains

q-deformed Haldane-Shastry spin chain

Macdonald-Ruijsenaars operators

$$\mathcal{D}_k = \sum_{|I|=k} \prod_{\substack{i\in I\\ j\notin I}} \varphi(z_j - z_i) \prod_{i\in I} p_i \qquad k=1,\dots,N \qquad I\subset\{1,2,\dots,N\}$$

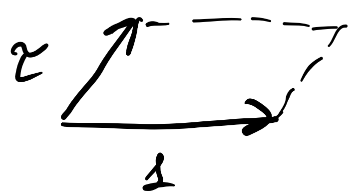
Here: $(p_i f)(z_1,\dots z_N) = e^{-\eta\partial_{z_i}} f(z_1,\dots z_N) = f(z_1,\dots, z_i-\eta,\dots z_N)$ - shift operator

Elliptic case

$$\varphi(z)=\varphi(z,h) = \frac{\theta'(0)\,\theta(z+h)}{\theta(z)\,\theta(h)}$$

$\theta(z)$ - odd theta function

$$\theta(z)=\theta(z|\tau) = \sum_{k\in\mathbb{Z}} \exp\left(\pi i\tau\left(k+\tfrac{1}{2}\right)^2 + 2\pi i\left(z+\tfrac{1}{2}\right)\left(k+\tfrac{1}{2}\right)\right)$$



$\mathrm{Im}(\tau)\to\infty$ $\rightsquigarrow$

trigonometric case

$$\varphi(z) = \pi\cot\pi z + \pi\cot\pi h$$

$$\theta(z) = 2\exp\left(\frac{\pi i\tau}{4}\right)\sin\pi z + O\left(\exp\left(\frac{9\pi i\tau}{4}\right)\right)$$

$t = \exp(-2\pi i h)$

$x_k = \exp(2\pi i z_k)$

$q = \exp(-2\pi i\eta)$

Macdonald operators:

$$\mathcal{D}_k^{Macd} = \sum_{|I|=k} \prod_{\substack{i\in I\\ j\notin I}} \frac{t x_i - x_j}{x_i - x_j} \prod_{i\in I} q^{x_i \frac{\partial}{\partial x_i}} \qquad k=1,\dots,N$$

[Ruijsenaars 87]:

Let $\mathcal{D}_k = \sum_{|I|=k} \prod_{\substack{i\in I \\ j\notin I}} \varphi(z_j - z_i) \prod_{i\in I} p_i$

Theorem 1 $[\mathcal{D}_k, \mathcal{D}_\ell] = 0 \quad \forall k, \ell = 1, \dots N$

$$\Updownarrow$$

(*) $$\sum_{|I|=k} \left(\prod_{\substack{i\in I \\ j\notin I}} \varphi(z_j - z_i)\varphi(z_i - z_j - \eta) - \prod_{\substack{i\in I \\ j\notin I}} \varphi(z_i - z_j)\varphi(z_j - z_i - \eta) \right) = 0$$

Theorem 2. The Kronecker elliptic function

$$\varphi(z,h) = \frac{\theta'(0)\theta(z+h)}{\theta(z)\theta(h)} \xrightarrow{\text{trig.}} \pi\cot\pi z + \pi\cot\pi h \xrightarrow{\text{rat}} \frac{1}{z} + \frac{1}{h}$$

satisfies (*).

Example $N=2$

$$\mathcal{D}_1 = \varphi(z_2 - z_1) p_1 + \varphi(z_1 - z_2) p_2$$

$$\mathcal{D}_2 = p_1 p_2$$

$$[\mathcal{D}_1, \mathcal{D}_2] = 0$$

In trigonometric case $\varphi(z) = \pi \cot \pi z + \pi \cot \pi \hbar$

Passing to the exponential variables

$t = \exp(-2\pi i \hbar)$

$x_k = \exp(2\pi i z_k)$

$q = \exp(-2\pi i \eta)$

$$\mathcal{D}_1^{Macd} = \frac{t x_1 - x_2}{x_1 - x_2} \cdot q^{x_1 \frac{\partial}{\partial x_1}} + \frac{t x_2 - x_1}{x_2 - x_1} \cdot q^{x_2 \frac{\partial}{\partial x_2}}$$

$$\mathcal{D}_2^{Macd} = q^{x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}}$$

Example $N=3$:

$$\mathcal{D}_1 = \varphi(z_2-z_1)\varphi(z_3-z_1)p_1 + \varphi(z_1-z_2)\varphi(z_3-z_2)p_2 + \varphi(z_1-z_3)\varphi(z_2-z_3)p_3$$

$$\mathcal{D}_2 = \varphi(z_3-z_1)\varphi(z_3-z_2)p_1p_2 + \varphi(z_2-z_1)\varphi(z_2-z_3)p_1p_3 + \varphi(z_1-z_2)\varphi(z_1-z_3)p_2p_3$$

$$\mathcal{D}_3 = p_1p_2p_3$$

$[\mathcal{D}_1, \mathcal{D}_2] = 0 \iff (*)$ with $N=3$ $k=1$

coefficient at $p_1p_2p_3$

Notations:

Let $I, J \subset \{1, \dots, N\}$; $I \cap J = \emptyset$

$$(I, J) = \prod_{\substack{i \in I \\ j \in J}} \varphi(z_i - z_j) \quad \text{and} \quad p_I = \prod_{i \in I} p_i$$

$$(I_-, J) = p_I (I, J) p_I^{-1} = \prod_{\substack{i \in I \\ j \in J}} \varphi(z_i - z_j - \eta)$$

I^c - complement set

$$D_k = \sum_{|I| = k} (I^c, I) p_I = \sum_{|I| = k} \prod_{\substack{i \in I \\ j \notin I}} \varphi(z_j - z_i) \prod_{i \in I} p_i$$

Theorem 1. $[D_k, D_\ell] = 0 \quad \forall k, \ell = 1, \dots N$

$$\Updownarrow$$

$$(*) \quad \sum_{|I| = k} \Big((I^c, I)(I_-, I^c) - (I, I^c)(I^c_-, I) \Big) = 0$$

$$\sum_{|I| = k} \Big(\prod_{\substack{i \in I \\ j \notin I}} \varphi(z_j - z_i) \varphi(z_i - z_j - \eta) - \prod_{\substack{i \in I \\ j \notin I}} \varphi(z_i - z_j) \varphi(z_j - z_i - \eta) \Big) = 0$$

Elliptic Baxter-Belavin R-matrix

GL_2 case 8-vertex R-matrix

$$R^{\hbar}_{12}(z) = \frac{1}{2}\left(\varphi_{00}\,\sigma_0\otimes\sigma_0 + \varphi_{01}\,\sigma_1\otimes\sigma_1 + \varphi_{11}\,\sigma_2\otimes\sigma_2 + \varphi_{10}\,\sigma_3\otimes\sigma_3\right) \in End(\mathbb{C}^2\otimes\mathbb{C}^2)$$

$$\varphi_{00} = \phi\left(z,\tfrac{\hbar}{2}\right) \quad \varphi_{10} = \phi\left(z,\tfrac{1}{2}+\tfrac{\hbar}{2}\right) \quad \varphi_{01} = e^{\pi i z}\phi\left(z,\tfrac{\tau}{2}+\tfrac{\hbar}{2}\right) \quad \varphi_{11} = e^{\pi i z}\phi\left(z,\tfrac{1+\tau}{2}+\tfrac{\hbar}{2}\right)$$

$$\phi(z,\hbar) = \frac{\vartheta'(0)\,\vartheta(z+\hbar)}{\vartheta(z)\,\vartheta(\hbar)}$$

Kronecker elliptic function

σ_a – Pauli matrices $a = 0,1,2,3$

$$\sigma_0 = \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad \sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$R^{h}_{12}(z) = \frac{1}{2}\begin{pmatrix}\varphi_{00}+\varphi_{10} & 0 & 0 & \varphi_{01}-\varphi_{11}\\ 0 & \varphi_{00}-\varphi_{10} & \varphi_{01}+\varphi_{11} & 0\\ 0 & \varphi_{01}+\varphi_{11} & \varphi_{00}-\varphi_{10} & 0\\ \varphi_{01}-\varphi_{11} & 0 & 0 & \varphi_{00}+\varphi_{10}\end{pmatrix}$$

$R^{\hbar}_{ij}(z)$ satisfies

Yang-Baxter equation

$$R^{\hbar}_{12}(u)R^{\hbar}_{13}(u+v)R^{\hbar}_{23}(v) = R^{\hbar}_{23}(v)R^{\hbar}_{13}(u+v)R^{\hbar}_{12}(u) \in \mathrm{End}(\mathbb{C}^{d}\otimes\mathbb{C}^{d}\otimes\mathbb{C}^{d})$$

Unitarity property

$$R^{\hbar}_{ij}(z)R^{\hbar}_{ji}(-z) = \mathrm{Id}\,\varphi(z,\hbar)\varphi(-z,\hbar) = \mathrm{Id}\left(\wp(\hbar)-\wp(z)\right)$$

We use $\bar{R}^{\hbar}_{ij}(z) = \frac{1}{\varphi(z,\hbar)}R^{\hbar}_{ij}(z)$, then $\bar{R}^{\hbar}_{ij}(z)\bar{R}^{\hbar}_{ji}(-z) = \mathrm{Id}$

Degenerations

$$R^{\hbar}_{12}(z) \longrightarrow \pi\begin{pmatrix} \cot\pi\hbar+\cot\pi z & 0 & 0 & 0 \\ 0 & \frac{\pi}{\sin\pi\hbar} & \frac{\pi}{\sin\pi z} & 0 \\ 0 & \frac{\pi}{\sin\pi z} & \frac{\pi}{\sin\pi\hbar} & 0 \\ 0 & 0 & 0 & \cot\pi\hbar+\cot\pi z \end{pmatrix} \xrightarrow{\text{rat}} \begin{pmatrix} \frac{1}{\hbar}+\frac{1}{z} & 0 & 0 & 0 \\ 0 & \frac{1}{\hbar} & \frac{1}{z} & 0 \\ 0 & \frac{1}{z} & \frac{1}{\hbar} & 0 \\ 0 & 0 & 0 & \frac{1}{\hbar}+\frac{1}{z} \end{pmatrix}$$

Anisotropic version of quantum spin Ruijsenaars operators

$$(\mathcal{I},\mathcal{J}) = \prod_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}} \varphi(z_i - z_j) \qquad R_{\mathcal{I}\mathcal{J}} = \prod_{\substack{i\in\mathcal{I},\, j\in\mathcal{J}\\ i<j}} R_{ij}(z_i - z_j) \qquad R'_{\mathcal{I}\mathcal{J}} = \prod_{\substack{i\in\mathcal{I},\, j\in\mathcal{J}\\ i>j}} R_{ij}(z_i - z_j)$$

$$P_{\mathcal{I}} = \prod_{i\in\mathcal{I}} P_i \qquad R_{ij}(z) = \varphi(z)\bar{R}_{ij}(z)$$

Spin operators:

$$\mathcal{D}_k^{spin} = \sum_{|\mathcal{I}|=k} (\mathcal{I}^c,\mathcal{I})\,\bar{R}_{\mathcal{I}^c\mathcal{I}}\, P_{\mathcal{I}}\, \bar{R}'_{\mathcal{I}\mathcal{I}^c} \qquad k=1,\dots,N$$

Theorem 1

$$[\mathcal{D}_k^{spin}, \mathcal{D}_\ell^{spin}] = 0 \qquad k,\ell = 1,\dots,N$$

$$\Updownarrow$$

$$(**)\quad \sum_{|\mathcal{I}|=k} \left(R_{\mathcal{I}^c\mathcal{I}} R'_{\mathcal{I}_-\mathcal{I}^c} R_{\mathcal{I}_-\mathcal{I}^c} R'_{\mathcal{I}^c\mathcal{I}} - R_{\mathcal{I}\mathcal{I}^c} R'_{\mathcal{I}^c_-\mathcal{I}} R_{\mathcal{I}^c_-\mathcal{I}} R'_{\mathcal{I}\mathcal{I}^c} \right) = 0$$

Theorem 2. Elliptic Baxter-Belavin R-matrix satisfies (**)

Example $N=3$

Operators:

$$\mathcal{D}_1^{spin} = \varphi(z_2-z_1)\varphi(z_3-z_1)p_1 + \varphi(z_1-z_2)\varphi(z_3-z_2)\bar{R}_{12}(z_1-z_2)p_2\bar{R}_{21}(z_2-z_1) +$$
$$+ \varphi(z_1-z_3)\varphi(z_2-z_3)\bar{R}_{23}(z_2-z_3)\bar{R}_{13}(z_1-z_3)p_3\bar{R}_{31}(z_3-z_1)\bar{R}_{32}(z_3-z_2)$$

$$\mathcal{D}_2^{spin} = \varphi(z_3-z_1)\varphi(z_3-z_2)p_1p_2 + \varphi(z_2-z_1)\varphi(z_2-z_3)\bar{R}_{23}(z_2-z_3)p_1p_3\bar{R}_{32}(z_3-z_2)$$
$$+\varphi(z_1-z_2)\varphi(z_1-z_3)\bar{R}_{12}(z_1-z_2)\bar{R}_{13}(z_1-z_3)p_2p_3\bar{R}_{31}(z_3-z_1)\bar{R}_{21}(z_2-z_1)$$

$$\mathcal{D}_3^{spin} = p_1p_2p_3$$

Identity:

Here $R_{ij}(z) = \varphi(z)\bar{R}_{ij}(z)$

$$R_{12}(z_1-z_2)R_{13}(z_1-z_3)R_{31}(z_3-z_1-\eta)R_{21}(z_2-z_1-\eta)$$
$$-R_{12}(z_1-z_2-\eta)R_{13}(z_1-z_3-\eta)R_{31}(z_3-z_1)R_{21}(z_2-z_1)$$
$$+R_{23}(z_2-z_3)R_{32}(z_3-z_2-\eta)R_{12}(z_1-z_2-\eta)R_{21}(z_2-z_1)$$
$$-R_{12}(z_1-z_2)R_{21}(z_2-z_1-\eta)R_{23}(z_2-z_3-\eta)R_{32}(z_3-z_2)$$
$$+R_{23}(z_2-z_3-\eta)R_{13}(z_1-z_3-\eta)R_{31}(z_3-z_1)R_{32}(z_3-z_2)$$
$$-R_{23}(z_2-z_3)R_{13}(z_1-z_3)R_{31}(z_3-z_1-\eta)R_{32}(z_3-z_2-\eta) = 0$$

Construction of spin operators

Notations

Fix $J = \{j_1, j_2, \dots, j_k\}$ $\quad j_1 < j_2 < \dots < j_k$

$$R_{IJ} = \prod_{\substack{i\in I\\ j\in J\\ i<j}} R_{ij}(z_i - z_j) = \overleftarrow{\prod_{\substack{i_1\in I\\ i_1<j_1}}} R_{i_1 j_1}(z_{i_1} - z_{j_1}) \overleftarrow{\prod_{\substack{i_2\in I\\ i_2<j_2}}} R_{i_2 j_2}(z_{i_2} - z_{j_2}) \cdots \overleftarrow{\prod_{\substack{i_k\in I\\ i_k<j_k}}} R_{i_k j_k}(z_{i_k} - z_{j_k})$$

$$R'_{IJ} = \prod_{\substack{i\in I\\ j\in J\\ i>j}} R_{ij}(z_i - z_j) = \overleftarrow{\prod_{\substack{i_1\in I\\ i_1>j_1}}} R_{i_1 j_1}(z_{i_1} - z_{j_1}) \overleftarrow{\prod_{\substack{i_2\in I\\ i_2>j_2}}} R_{i_2 j_2}(z_{i_2} - z_{j_2}) \cdots \overleftarrow{\prod_{\substack{i_k\in I\\ i_k>j_k}}} R_{i_k j_k}(z_{i_k} - z_{j_k})$$

Example

$N = 5 \quad J = \{2, 5\} \quad I = \{1, 3\}$

$$R_{IJ} = R_{12}(z_1 - z_2) R_{35}(z_3 - z_5) R_{15}(z_1 - z_5)$$

$$R'_{IJ} = R_{32}(z_3 - z_2)$$

In scalar case operators $\mathcal{D}_k = \sum_{|I|=k} (I^c, I) \, p_I$ are symmetric: $\sigma \mathcal{D}_k \sigma^{-1} = \mathcal{D}_k \quad \forall \sigma \in S_N$ (Acting as permutation of variables $z_1 z_2 \ldots z_N$; for example

$$\sigma_{i,i+1} f(z_1, \ldots z_i, z_{i+1}, \ldots z_N) = f(z_1, \ldots z_{i+1}, z_i \ldots z_N))$$

Thus $\mathcal{D}_k$ can be written as

$$\mathcal{D}_k = \frac{1}{k!} \sum_{\sigma \in S_N} \sigma (I_0^c, I_0) \, p_1 p_2 \cdots p_k \, \sigma^{-1},$$

where $I_0 = \{1, 2, \ldots k\}$

In spin case we use another representation of S_N:

$$\boxed{S_{i,i+1} = \bar{R}_{i,i+1}(z_i - z_{i+1}) P_{i,i+1} \sigma_{i,i+1}}$$

$$P_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Construction of spin operators

$$S_{i,i+1} = \bar{R}_{i,i+1}(z_i - z_{i+1}) P_{i,i+1} \sigma_{i,i+1}$$

$$S_{(12\ldots j)} = S_{12} S_{23} \cdots S_{j-1,j}$$

$$D_k^{spin} = \sum_{i_1 < i_2 < \ldots < i_k} s^{-1}_{\{i_1, i_2 \ldots i_k\}} \left(I_0^c, I_0\right) p_1 p_2 \ldots p_k \, S_{\{i_1, \ldots i_k\}}$$

$$S_{\{i_1, i_2, \ldots i_k\}} : \quad i_m \to m$$

$$S_{\{i_1, i_2 \ldots i_k\}} = S(k \ldots i_k) S(k-1, \ldots i_{k-1}) \ldots S(2 \ldots i_2) S(1 \ldots i_1)$$

Example N=3

$$\mathcal{D}_1^{spin} = \varphi(z_2-z_1)\varphi(z_3-z_1)\,p_1 + S_{12}\,\varphi(z_2-z_1)\varphi(z_3-z_1)\,p_1\,S_{12} + $$
$$+ S_{23}S_{12}\,\varphi(z_2-z_1)\varphi(z_3-z_1)\,p_1\,S_{12}S_{23}$$

$$S_{12} = \bar{R}_{12}P_{12}\sigma_{12} \; ; \; S_{23} = \bar{R}_{23}P_{23}\sigma_{23} \qquad \bar{R}_{ij} := R_{ij}(z_i - z_j)$$

$$\mathcal{D}_1^{spin} = \varphi(z_2-z_1)\varphi(z_3-z_1)\,p_1 + \bar{R}_{12}P_{12}\sigma_{12}\,\varphi(z_2-z_1)\varphi(z_3-z_1)\,p_1 P_{12}\sigma_{12}\cdot\bar{R}_{21} + $$
$$+ \bar{R}_{23}P_{23}\sigma_{23}\bar{R}_{12}P_{12}\sigma_{12}\cdot\varphi(z_2-z_1)\varphi(z_3-z_1)\,p_1 P_{12}\sigma_{12}\bar{R}_{21}P_{23}\sigma_{23}\bar{R}_{32} =$$

$$= \varphi(z_2-z_1)\varphi(z_3-z_1)\,p_1 + \varphi(z_1-z_2)\varphi(z_3-z_2)\,\bar{R}_{12}(z_1-z_2)\,p_2\,\bar{R}_{21}(z_2-z_1) +$$
$$+ \varphi(z_1-z_3)\varphi(z_2-z_3)\,\bar{R}_{23}(z_2-z_3)\bar{R}_{13}(z_1-z_3)\,p_3\,\bar{R}_{31}(z_3-z_1)\bar{R}_{32}(z_3-z_2)$$

Anisotropic version of quantum spin Ruijsenaars operators

$$(\mathcal{I},\mathcal{J}) = \prod_{\substack{i\in\mathcal{I}\\ j\in\mathcal{J}}} \varphi(z_i - z_j) \qquad R_{\mathcal{I}\mathcal{J}} = \prod_{\substack{i\in\mathcal{I},\, j\in\mathcal{J}\\ i<j}} R_{ij}(z_i - z_j) \qquad R'_{\mathcal{I}\mathcal{J}} = \prod_{\substack{i\in\mathcal{I},\, j\in\mathcal{J}\\ i>j}} R_{ij}(z_i - z_j)$$

$$P_{\mathcal{I}} = \prod_{i\in\mathcal{I}} P_i \qquad R_{ij}(z) = \varphi(z)\bar{R}_{ij}(z)$$

Spin operators:

$$\mathcal{D}_k^{\text{spin}} = \sum_{|\mathcal{I}|=k} (\mathcal{I}^c,\mathcal{I})\, \bar{R}_{\mathcal{I}^c\mathcal{I}}\, P_{\mathcal{I}}\, \bar{R}'_{\mathcal{I}\mathcal{I}^c} \qquad k=1,\dots,N$$

Theorem 1

$$[\mathcal{D}_k,\mathcal{D}_\ell]=0 \qquad k,\ell=1,\dots,N$$

$$\Updownarrow$$

(**) $$\sum_{|\mathcal{I}|=k}\left(R_{\mathcal{I}^c\mathcal{I}}\, R'_{\mathcal{I}_-\mathcal{I}^c}\, R_{\mathcal{I}_-\mathcal{I}^c}\, R'_{\mathcal{I}^c\mathcal{I}} - R_{\mathcal{I}\mathcal{I}^c}\, R'_{\mathcal{I}^c_-\mathcal{I}}\, R_{\mathcal{I}^c_-\mathcal{I}}\, R'_{\mathcal{I}\mathcal{I}^c}\right)=0$$

Theorem 2. Elliptic Baxter-Belavin R-matrix satisfies (**)

Application: Long Range Spin Chains

Polychronakos freezing trick

$z_k = \frac{k}{N}$ – equilibrium position of classical model

$\mathcal{D}_k^{spin} \rightsquigarrow$ Hamiltonians of spin chain

$$H_s = \sum_{k<i}^{N} \overline{R}_{i-1,i} \cdots \overline{R}_{k+1,i} \overline{R}_{k,i} \left(\frac{\partial}{\partial z_i} \overline{R}_{i,k}\right) \overline{R}_{i,k+1} \cdots \overline{R}_{i,i-1} \Big|_{z_k = \frac{k}{N}}$$

trigonometric case of $\mathcal{D}_k^{spin}$ $\rightsquigarrow$ q-deformed Haldane-Shastry spin chain

Lamers-Pasquier-Serban; Uglov

$\hbar \to 0$

Id

Id

spin Calogero-Sutherland model $\rightsquigarrow$ Haldane-Shastry spin chain

$$H^{HS} = \frac{1}{4} \sum_{j \neq k}^{N} \frac{1 - P_{jk}}{\sin^2\left(\frac{\pi}{N}(j-k)\right)}$$

Thank you for your attention!