SEPARATION OF VARIABLES AND CORRELATION FUNCTIONS

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based on

2103.15800 [Cavaglia, Gromov, FLM]
2011.08229 [Gromov, FLM, Ryan]
1910.13442 [Gromov, FLM, Ryan, Volin]
1907.03788 [Cavaglia, Gromov, FLM]
1805.03927 [Gromov, FLM]
1610.08032 [Gromov, FLM, Sizov]

+ work in progress

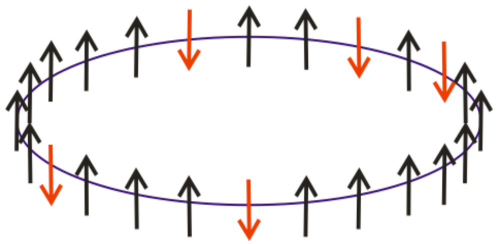
Motivation:

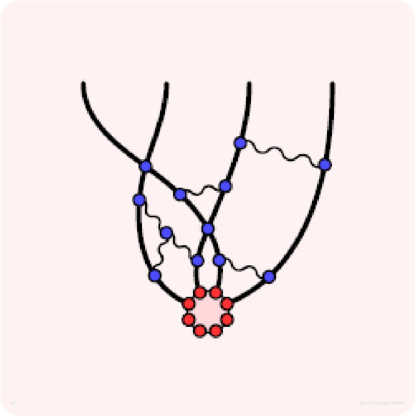
find new methods to compute correlators in integrable models
from spin chains to AdS/CFT

Should exist a basis where wavefunctions factorize $\langle x|\Psi\rangle \sim Q(x_1)Q(x_2)\dots Q(x_N)$

Separation of Variables (SoV)

Expected to be very powerful
But for a long time almost undeveloped beyond GL(2)





Would shed light on many open problems:
correlators, form factors,
3pt functions in N=4 super Yang-Mills, ...

Need to understand and develop SoV

For scalar products we need measure

In GL(2)-type models:

$$\langle \Psi_B | \Psi_A \rangle = \int d^L \mathbf{x} \left(\underbrace{\prod_{i=1}^{L} Q^{(A)}(x_i)}_{\text{state } A} \right) \underbrace{M(\mathbf{x})}_{\text{measure}} \left(\underbrace{\prod_{i=1}^{L} Q^{(B)}(x_i)}_{\text{state } B} \right)$$

e.g. for SL(2)

$$M(\mathbf{x}) = \frac{\prod_{j<k} (e^{2\pi x_j} - e^{2\pi x_k})(x_j - x_k)}{\prod_{j,k} (1 + e^{2\pi (x_j - \theta_k)})}$$

[Sklyanin 90-92]
[Derkachov Korchemsky Manashov 02]

GL(N) models are much harder
Only recently understood how to factorise wavefunctions

[Sklyanin 92] [Smirnov 2000]
[Gromov FLM Sizov 16]
[Maillet Niccoli 18]
[Ryan Volin 18]

Measure was not known at all

[cf Smirnov Zeitlin 02]

Focus of this talk – the measure and correlators

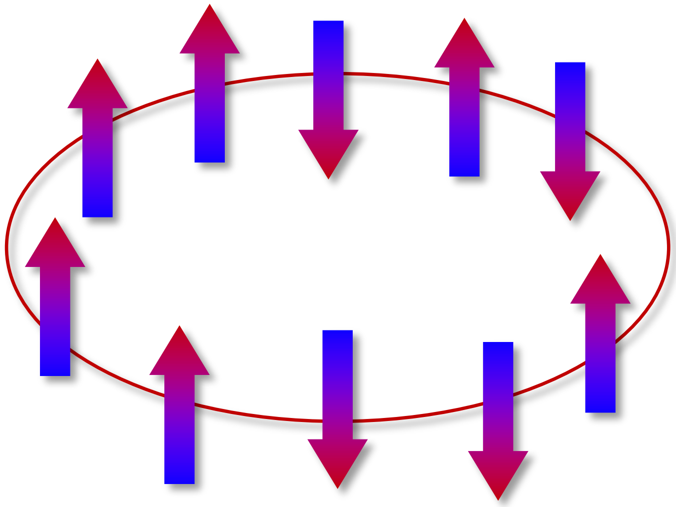
Plan

- Compact SU(N) spin chains [Gromov, FLM, Ryan, Volin 19]
- Noncompact case, [Cavaglia, Gromov, FLM 19 Gromov, FLM, Ryan 20]
  explicit result for measure, correlators
- Extensions to field theory [Cavaglia, Gromov, FLM 21 + in progress]

COMPACT SPIN CHAINS

SU(N) spin chains

Full Hilbert space for L sites is $\underbrace{\mathbb{C}^N \otimes \mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{L \text{ times}}$



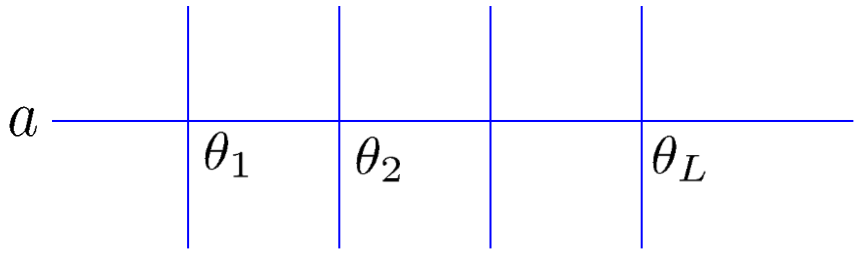
$$H = \sum_{n=1}^{L} (1 - P_{n,n+1})$$

(+ boundary terms, i.e. twist)

Monodromy matrix:

$$T(u) = R_{a1}(u - \theta_1) \ldots R_{aL}(u - \theta_L) g$$

$$R_{12}(u) = (u - \tfrac{i}{2}) + iP_{12}$$



We take generic inhomogeneities θ_n and diagonal twist $g = \mathrm{diag}(\lambda_1, \ldots, \lambda_{\mathrm{N}})$

Transfer matrix $\mathrm{Tr}_a\, T(u) = \sum_{n=0}^{L} T_n u^n$ gives commuting integrals of motion

Wavefunctions for spin chains

$$\langle x|\Psi\rangle = \prod_k Q_1(x_k)$$

$$Q_1 = e^{\phi_1 u}\prod_{j=1}^{N_u}(u-u_j)$$

$\langle x|$ = eigenstates of operator $B(u) = \prod(u-x_k)$ $[B(u),B(v)]=0$

SU(2): $T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$ $x_k = \theta_k \pm i/2, \quad k=1,\ldots L$ [Sklyanin 90-92]

Gives 2^L states, basis of the space

SU(N): B is a polynomial in elements of T [Sklyanin 92 for SU(3)]
[Smirnov 2000] [Gromov, FLM, Sizov 16]

Brief summary of results

SU(N) – results summary (1)

$$\langle x|\Psi\rangle = \prod_k Q_1(x_k)$$

[Gromov, FLM, Sizov 16]

For SU(N) we need to slightly modify Sklyanin's proposal

$$T \to T^{\text{good}} = KTK^{-1} \quad B \to B^{\text{good}}$$

1) Found spectrum of x
2) Found that we can build states nicely

$$|\Psi\rangle = B(u_1)B(u_2)\dots B(u_M)|0\rangle$$

Any SU(N) !

No need for nested BA,
use roots of 1 Baxter polynomial

We proved various special cases
Then part 2 proved for SU(3) [Lyashik, Slavnov 18]
Then full proof for SU(N) [Ryan, Volin 18] , who also showed equivalence with another way to build x

[Maillet, Niccoli 18,19,20]

$$\langle x| \sim \langle 0|\hat{T}(\theta_1 + i/2)^{n_1} \dots \hat{T}(\theta_L + i/2)^{n_L}$$

Analog of part 2 found for super SU(1|2) [Gromov, FLM 18]

To compute correlators
one inserts the complete basis

$$\mathbf{1} = \sum_x M_x \, |x\rangle\langle x|$$

measure $M_x = (\langle x|x\rangle)^{-1}$

Overlaps between these states look complicated

Can we find a way around this?

SU(N) – results summary (2)

[Cavaglia, Gromov, FLM 19; Gromov, FLM, Ryan, Volin 19]

- Constructed 'dual' C-operator for SU(N), gives SoV basis $|y\rangle$ for bra states $\langle\Psi|$
  B and C states have simple overlaps $\langle x|y\rangle$, are natural to pair!

- Found alternative way to compute overlaps (= SoV measure)
  Bypasses operator construction, gives measure from simple det of integrals

  Yet another way found later: recursion relations of [Maillet, Niccoli, Vignoli 20]

  More recently we found completely explicit result for measure [Gromov, FLM, Ryan 20]

- Get simple det expressions for form factors/scalar products
  for large class of operators (likely complete)

- Similar statements for SL(N) (infinite-dim rep)

Detailed example: SU(N) measure

SU(2) spin chain

Idea: orthogonality of states must imply same for Qs

$$Q_1 = e^{u\phi} \prod_{k=1}^{M} (u - u_k) \quad Q_\theta = \prod_{n=1}^{L} (u - \theta_n)$$

$$\tau_1 = 2\cos\phi \, u^L + \sum_{n=0}^{L-1} I_n u^n$$

$$f^\pm = f(u \pm i/2), \quad f^{[a]} = f(u + ia/2)$$

$$Df(u) = f(u + i/2)$$

Baxter equation can be written as

$$\hat{O} \circ Q_1 = 0 \qquad \hat{O} = \frac{1}{Q_\theta^+} D^2 + \frac{1}{Q_\theta^-} D^{-2} - \frac{\tau_1}{Q_\theta^+ Q_\theta^-}$$

Key property: self-adjointness

$$\langle f \hat{O} g \rangle = \langle g \hat{O} f \rangle$$

$$\langle f \rangle = \oint du \, f(u)$$

$$\langle f \hat{O} g \rangle = \oint du \, f \left[\frac{g^{++}}{Q_\theta^+} + \frac{g^{--}}{Q_\theta^-} - \frac{\tau_1}{Q_\theta^+ Q_\theta^-} g \right]$$

$u \to u - i$

$$= \oint du \left[\frac{f^{--}}{Q_\theta^-} + \frac{f^{++}}{Q_\theta^+} - \frac{\tau_1}{Q_\theta^+ Q_\theta^-} f \right] g$$

We can introduce L such brackets $\langle f\rangle_j = \oint du\, \mu_j\, f$

$$\langle f\hat{O}g\rangle_j = \langle g\hat{O}f\rangle_j \qquad \mu_j = e^{2\pi(j-1)u} \qquad j=1,\dots,L \qquad \tau_1 = 2\cos\phi\, u^L + \sum_{k=1}^{L} I_k u^{k-1}$$

uniquely identify the state

This gives orthogonality!

$$\langle Q^B(\hat{O}^A - \hat{O}^B)Q^A\rangle_j = 0 \qquad \hat{O} = \frac{1}{Q_\theta^+}D^2 + \frac{1}{Q_\theta^-}D^{-2} - \frac{\tau_1}{Q_\theta^+ Q_\theta^-}$$

$$\sum_{k=1}^{L}(I_k^A - I_k^B)\left\langle \frac{u^{k-1}Q^A Q^B}{Q_\theta^+ Q_\theta^-}\right\rangle_j = 0$$

Nontrivial solution means det=0

Sum of residues at $u=\theta_n \pm i/2$
i.e. at x eigenvalues as expected

$$\det_{1\le j,k\le L}\left\langle \frac{u^{k-1}Q^A Q^B}{Q_\theta^+ Q_\theta^-}\right\rangle_j \propto \delta_{AB}$$

Scalar product in SoV

Matches known results [Sklyanin; Kitanine, Maillet, Niccoli, ...]
[Kazama, Komatsu, Nishimura, Serban, Jiang, ...]

SU(3) spin chain

For SU(3) we have 2 types of Bethe roots

$$\prod_{n=1}^{L} \frac{u_j - \theta_n + i/2}{u_j - \theta_n - i/2} = e^{i(\phi_1 - \phi_2)} \prod_{k \neq j}^{N_u} \frac{u_j - u_k + i}{u_j - u_k - i} \prod_{l=1}^{N_v} \frac{u_j - v_l - i/2}{u_j - v_l + i/2}$$

$$1 = e^{i(\phi_2 - \phi_3)} \prod_{k \neq j}^{N_v} \frac{v_j - v_k + i}{v_j - v_k - i} \prod_{l=1}^{N_u} \frac{v_j - u_l - i/2}{v_j - u_l + i/2}$$

momentum-carrying $\{u_j\}_{j=1}^{N_u}$

auxiliary $\{v_j\}_{j=1}^{N_v}$

$$Q_1 = e^{\phi_1 u} \prod_{j=1}^{N_u} (u - u_j) \qquad\qquad Q_{12} = e^{(\phi_1 + \phi_2) u} \prod_{j=1}^{N_v} (u - v_j)$$

Main new feature: should use Q^i in addition to Q_i to get simple measure

Other Qs give dual roots $\quad Q^1 \equiv Q_{23}, \text{ etc}$

Baxter equations:

$$\tau_a(u) = u^L \chi_a(G) + \sum_{j=1}^{L} u^{j-1}\, I_{a,j-1},$$

$$\bar{O} = \frac{1}{Q_\theta^-} D^{-3} - \frac{\tau_2}{Q_\theta^+ Q_\theta^-} D^{-1} + \frac{\tau_1}{Q_\theta^+ Q_\theta^-} D - \frac{1}{Q_\theta^+} D^{+3}$$

$$O = \frac{1}{Q_\theta^{++}} D^{+3} - \frac{\tau_2^+}{Q_\theta^{++} Q_\theta} D + \frac{\tau_1^-}{Q_\theta Q_\theta^{--}} D^{-1} - \frac{1}{Q_\theta^{--}} D^{-3}$$

$$\bar{O} \circ Q^a = 0 \qquad O \circ Q_a = 0$$

$$\langle f \rangle_j = \oint du\, \mu_j\, f$$

These two operators are conjugate! $\langle f O \circ g \rangle_j = \langle g \bar{O} \circ f \rangle_j$

$$\mu_j = e^{2\pi (j-1) u}$$

$$j = 1, \ldots, L$$

$$\langle Q_b^B (\bar{O}^A - \bar{O}^B) Q^{a,A} \rangle_j = 0$$

$$\tau_a(u) = u^L \chi_a(G) + \sum_{j=1}^{L} u^{j-1} I_{a,j-1},$$

$$\bar{O} = \frac{1}{Q_\theta^-} D^{-3} - \frac{\tau_2}{Q_\theta^+ Q_\theta^-} D^{-1} + \frac{\tau_1}{Q_\theta^+ Q_\theta^-} D - \frac{1}{Q_\theta^+} D^{+3}$$

$$\langle Q_b^B (\bar{O}^A - \bar{O}^B) Q^{a,A} \rangle_j = 0$$

We have freedom which Qs to choose

Linear system:

$$\sum_{\alpha=\{1,2\},\ k=1,\ldots,L} (I^A_{\alpha,k} - I^B_{\alpha,k})(-1)^\alpha \left\langle \frac{u^k Q_1^B Q^{a,A[-3+2\alpha]}}{Q_\theta^+ Q_\theta^-} \right\rangle_j = 0$$

We have 2L variables, and two choices of a give 2L equations

[Gromov, FLM, Ryan, Volin 19]

$$\langle\Psi_B|\Psi_A\rangle \propto \begin{vmatrix} \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2-} \right\rangle_j \\ \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3-} \right\rangle_j \end{vmatrix}$$

$$1 \leq j, k \leq L$$

Each bracket is a sum of residues at $u = \theta_n \pm i/2$

$$N_A^2 \delta_{AB} = \sum_{x,y} M_{x,y} \prod_{k=1}^{L} Q_1^A(X_{k,1}) Q_1^A(X_{k,2}) \prod_{k=1}^{L} \left[Q_B^2(Y_{k,1}) Q_B^3(Y_{k,2}) - Q_B^2(Y_{k,2}) Q_B^3(Y_{k,1}) \right]$$

matches spectrum of $B(u)$!

Can we build the basis where these are the wavefunctions?

Operator realization for SU(3)

[Gromov, FLM, Ryan, Volin 19]

$$\langle \Psi_B | \Psi_A \rangle = \int \left(\prod_{a=1}^{N-1} \prod_{i=1}^{L} dx_{i,a} \right) \left(\underbrace{\prod_{a=1}^{N-1} \prod_{i=1}^{L} Q_1^{(A)}(x_{i,a})}_{\text{state A}} \right) \hat{M}(\mathbf{x}) \left(\underbrace{\prod_{a=1}^{N-1} \prod_{i=1}^{L} Q^{(B)^a}(x_{i,a})}_{\text{state B}} \right)$$

$$\langle x | \Psi_A \rangle \qquad \langle \Psi_B | y \rangle$$

Instead of integrals we have sums

$$\langle \Psi_B | \Psi_A \rangle = \sum_{x,y} M_{x,y} \langle \Psi_B | y \rangle \langle x | \Psi_A \rangle$$

Get scalar product from construction of two SoV bases $|y\rangle$ and $\langle x|$

[Sklyanin 92] [Gromov FLM Sizov 16]

$\langle x|$ are eigenstates of familiar operator $\hat{\mathbb{B}}(u) = \hat{T}^2{}_3(u)\hat{U}_3{}^1(u-i) - \hat{T}^1{}_3(u)\hat{U}_3{}^2(u-i)$

$|y\rangle$ are eigenstates of new “dual” operator $\hat{\mathbb{C}}(u) = \hat{T}^2{}_3(u-\frac{i}{2})\hat{U}_3{}^1(u-\frac{i}{2}) - \hat{T}^1{}_3(u-\frac{i}{2})\hat{U}_3{}^2(u-\frac{i}{2})$

$$M_{x,y} = (\langle x|y\rangle)^{-1}$$

Measure matches what we got from Baxter!

To build SoV basis we act on reference state with transfer matrices

B(u) is diagonalized by [Maillet, Niccoli 18] [Ryan, Volin 18]

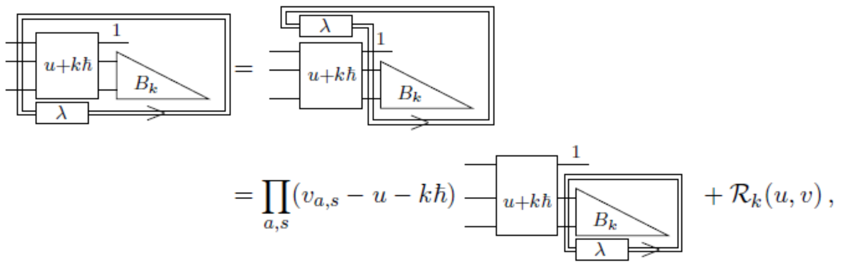
$$\langle x| \propto \langle 0| \prod_{k=1}^{L} [\hat{\tau}_2(\theta_k - i/2)]^{m_{k,1}+m_{k,2}} \qquad 0 \leq m_{k,1} \leq m_{k,2} \leq 1$$

C(u) is diagonalized by [Ryan, Volin 18] [Gromov FLM, Ryan, Volin 19]

$$|y\rangle \propto \prod_{k=1}^{L} \hat{\tau}_1(\theta_k - i/2)^{n_{k,2}-n_{k,1}} \; \hat{\tau}_2(\theta_k - i/2)^{n_{k,1}} |0\rangle \qquad 0 \leq n_{k,1} \leq n_{k,2} \leq 1$$

Proof is direct generalization of
highly nontrivial methods from [Ryan, Volin 18]

Based on commutation relations +
identifying Gelfand-Tsetlin patterns



$$M_{x,y} = (\langle x|y\rangle)^{-1}$$

$$\langle \Psi_B|\Psi_A\rangle = \sum_{x,y} M_{x,y} \langle \Psi_B|y\rangle\langle x|\Psi_A\rangle$$

Notice for SU(2) the overlaps matrix is diagonal

For SU(3) it is not, but the elements are still simple!

$$\langle \Psi_B|\Psi_A\rangle \propto \begin{vmatrix} \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2-} \right\rangle_j \\ \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3-} \right\rangle_j \end{vmatrix}$$

[Cavaglia, Gromov, FLM 19]

[Gromov, FLM, Ryan, Volin 19]

Alternative approach: [Maillet, Niccoli, Vignoli 20]
fix measure indirectly by deriving recursion relations for it
(+ another measure found in different basis)

Result should be same, would be interesting to prove

Diagonal form factors of type $\dfrac{\langle\Psi|\frac{\partial \hat{I}_n}{\partial p}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \dfrac{\partial I_n}{\partial p}$ are computable, give ratios of determinants.

From self-adjoint property:

$$0 = \langle Q(\hat{O} + \delta O) \circ (Q + \delta Q)\rangle = \underbrace{\langle QO \circ \delta Q\rangle}_{=0} + \langle Q\delta O \circ Q\rangle$$

$$\tau_1 = 2\cos\phi\, u^L + \sum_{k=0}^{L-1} I_k u^k$$

Link δI_n with $\delta\phi$

So $$\partial_\phi I_k = \frac{1}{2\sin\phi} \frac{\det\limits_{i,j=1,\ldots,L} m_{ij}^{(k)}}{\det\limits_{i,j=1,\ldots,L} m_{ij}}$$

norm

All this generalizes to SU(N)

NON-COMPACT SPIN CHAINS

[Cavaglia, Gromov, FLM 19]

General structure in SL(N):

$$\langle\Psi_A|\Psi_B\rangle = \int\left(\prod_{a=1}^{N-1}\prod_{i=1}^{L} dx_{i,a}\right)\left(\underbrace{\prod_{a=1}^{N-1}\prod_{i=1}^{L} Q_1^{(A)}(x_{i,a})}_{\text{state A}}\right)\hat{M}(\mathbf{x})\left(\underbrace{\prod_{a=1}^{N-1}\prod_{i=1}^{L} Q^{(B)^a}(x_{i,a})}_{\text{state B}}\right)$$

state-independent operator, contains shifts

$$\hat{M}(x) = \det\left|\underbrace{\left(\frac{\hat{x}^{j-1}}{1+e^{2\pi(\hat{x}-\theta_i)}}\right)}_{1\leqslant i,j\leqslant L}\otimes\underbrace{\begin{pmatrix}\mathcal{D}_x^{N-2} & \mathcal{D}_x^{N-4} & \dots & \mathcal{D}_x^{2-N}\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{D}_x^{N-2} & \mathcal{D}_x^{N-4} & \dots & \mathcal{D}_x^{2-N}\end{pmatrix}}_{(N-1)\times(N-1)}\right|$$

similar to conjecture of [Smirnov Zeitlin]
based on semi-classics
and quantization of alg curve

Representation with weight [s, 0, … 0], including infinite-dim case

Integral = sum over infinite set of poles in lower half-plane

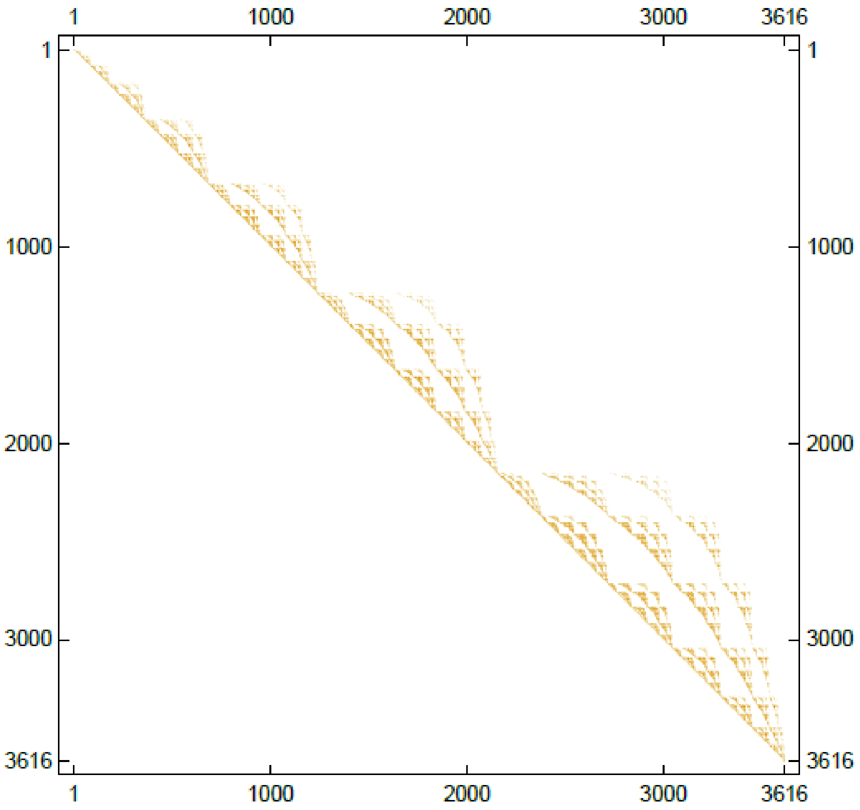
$$\mu_n = \frac{\Gamma(s - i(u-\theta_n))\Gamma(s + i(u-\theta_n))}{e^{\pi(u-\theta_n)}}$$

Everything works like before!

Recently we managed to compute measure for any GL(N) explicitly and for any spin [Gromov, FLM, Ryan 20]

$$M_{\mathsf{y},\mathsf{x}} = \sum_{k=\mathrm{perm}_\alpha n} \mathrm{sign}(\sigma) \left(\prod_{a=1}^{N-1} \frac{\Delta(\mathsf{x}_{\sigma^{-1}(a)})}{\Delta(\{\theta_a\})} \right) \prod_{a=1}^{N-1} \frac{r_{\alpha,n_{\alpha,a}}}{r_{\alpha,0}} \Big|_{\sigma_{\alpha,a}=k_{\alpha,a}-m_{\alpha,a}+a}$$

$$r_{\alpha,n} = -\frac{1}{2\pi} \prod_{\beta=1}^{L} (n+1-i\theta_\alpha + i\theta_\beta)_{2s-1}$$



$$\langle \Psi_B | \Psi_A \rangle \propto \begin{vmatrix} \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{2-} \right\rangle_j \\ \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3+} \right\rangle_j & \left\langle \frac{1}{Q_\theta^+ Q_\theta^-} u^{k-1} Q_1^A Q_B^{3-} \right\rangle_j \end{vmatrix}$$

Can also compute many other correlators in det form

E.g. overlaps with different twists $\langle \Psi^{\tilde{\lambda}_a} | \Psi^{\lambda_a} \rangle = \left[\!\left[\tilde{Q}_{12}, \tilde{Q}_{13} \middle| Q_1 \right]\!\right]$ [Gromov, **FLM**, Ryan 20]

Use that SoV basis is twist-independent [Ryan, Volin]

Also on-shell and off-shell overlaps involving B and C operators

$$|\Psi\rangle_{\text{off shell}} \equiv \mathrm{b}(v_1)\ldots\mathrm{b}(v_k)|\Omega\rangle$$

$$\frac{\langle \Phi | \mathbf{c}_{\gamma_1}(v_1)\ldots\mathbf{c}_{\gamma_K}(v_K)\mathbf{b}_{\beta_1}(w_1)\ldots\mathbf{b}_{\beta_J}(w_J)|\Theta\rangle}{\langle \Phi | \Psi \rangle}$$

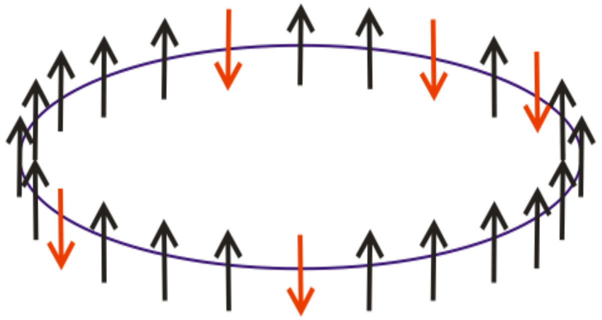
Likely this gives a complete set of operators

Further powerful generalization and simplification: see N. Primi's talk tomorrow [Gromov, Primi, Ryan 22]

EXTENSIONS TO FIELD THEORY

Integrability in N=4 super Yang-Mills

integrable spin chains



single trace operators

$$\mathrm{Tr}(\Phi_1(x)\Phi_2(x)\Phi_2(x)\Phi_1(x)\dots) \quad \Longleftrightarrow$$

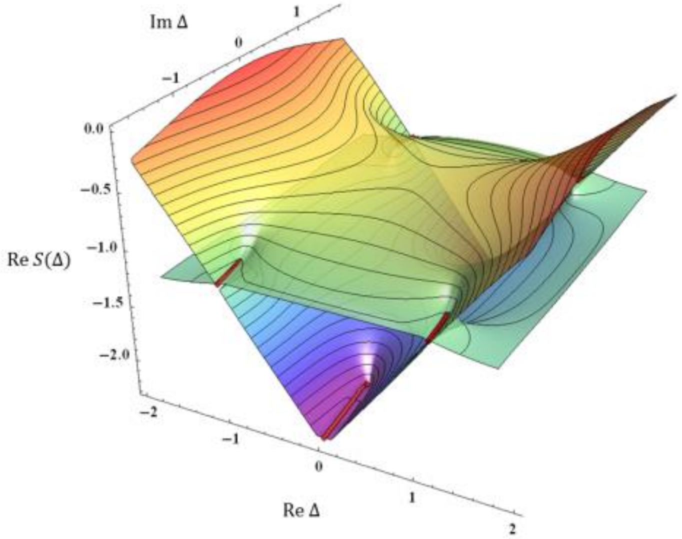
$$\Psi \sim Q(x_1)Q(x_2)\dots Q(x_n)$$

Q-functions are known at any coupling from Quantum Spectral Curve

[Gromov, Kazakov, Leurent, Volin 13]

Gives exact spectrum very efficiently !
All-loop, numerical, perturbative, …

[Marboe, Volin 14,16,17]
[Gromov, FLM, Sizov 13,14]
[Gromov, FLM, Sizov 15 x2]
[Gromov, FLM 15, 16]
[Alfimov, Gromov, Kazakov 14]
[FLM, Preti 20] …



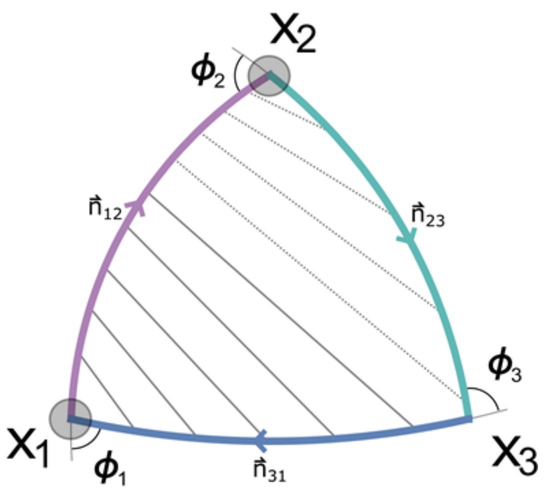
Hope to link with exact 3-pt functions which are much less understood

Goal: write correlators in terms of Q's

First all-loop example:
3 Wilson lines + scalars
in ladders limit

$$C_{123}^{\bullet\bullet\circ} = \frac{\langle q_1\, q_2\, e^{-\phi_3 u}\rangle}{\sqrt{\langle q_1^2\rangle\langle q_2^2\rangle}}$$

[Cavaglia, Gromov, FLM 18]



Similar structures seen in very different regime via localization
[Komatsu, Giombi 18,19]

Extension to local operators

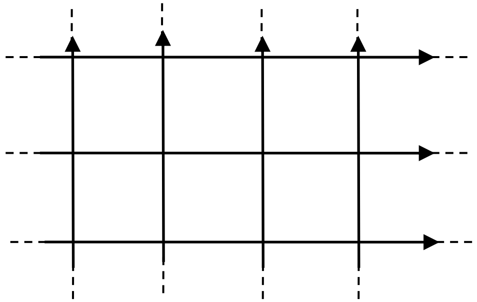
Gurdogan, Kazakov 2015

"fishnet CFT"

$$S = \frac{N}{2} \int d^4x \; \mathrm{tr} \; \left(\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + 2\xi^2 \, \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 \right)$$

Baby version of N=4 SYM, no susy but inherits integrability

Integrability visible
directly from Feynman graphs



We find very similar structures

$$C_{\mathcal{OOL}} \propto \frac{d\Delta}{d\xi^2} = \frac{\int_{|} \frac{q\bar{q}}{u} \frac{du}{2\pi i}}{\int_{|} i \left(q^+ \bar{q}^- - q^- \bar{q}^+ \right) \frac{du}{2\pi i}}$$

[Cavaglia, Gromov, **FLM** 21
+ with A. Sever]

Spin chain picture

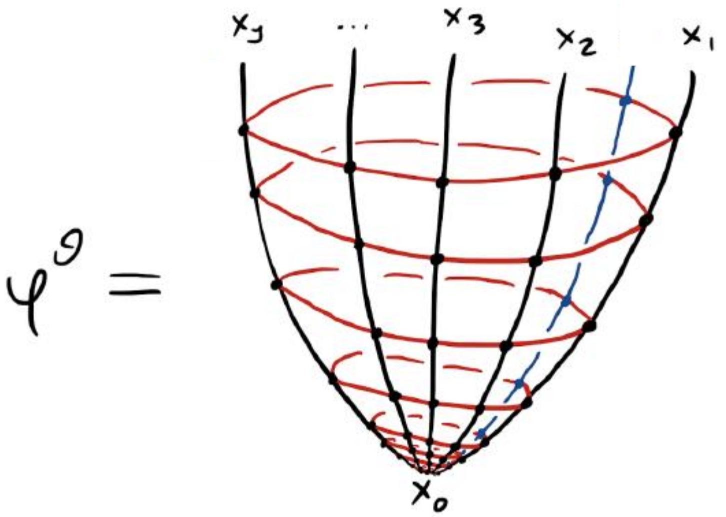
Get SO(4,2) spin chain in principal series rep

Wavefunction of spin chain = correlator in CFT

$$\varphi_{\mathcal{O}}(x_1,\dots,x_J) = \langle \mathcal{O}(x_0)\,\mathrm{tr}\,[\phi_1^\dagger(x_1)\dots\phi_J^\dagger(x_J)]\rangle\ .$$

[Gromov, Sever 19]

Tr $(\phi(x_0))^{\,J}$

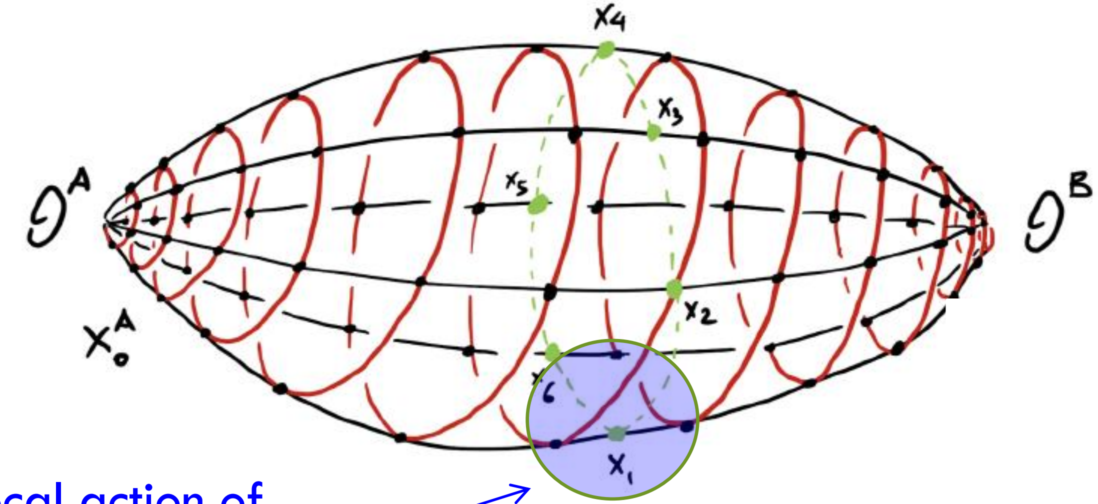


Spin chain form factors = more involved correlators

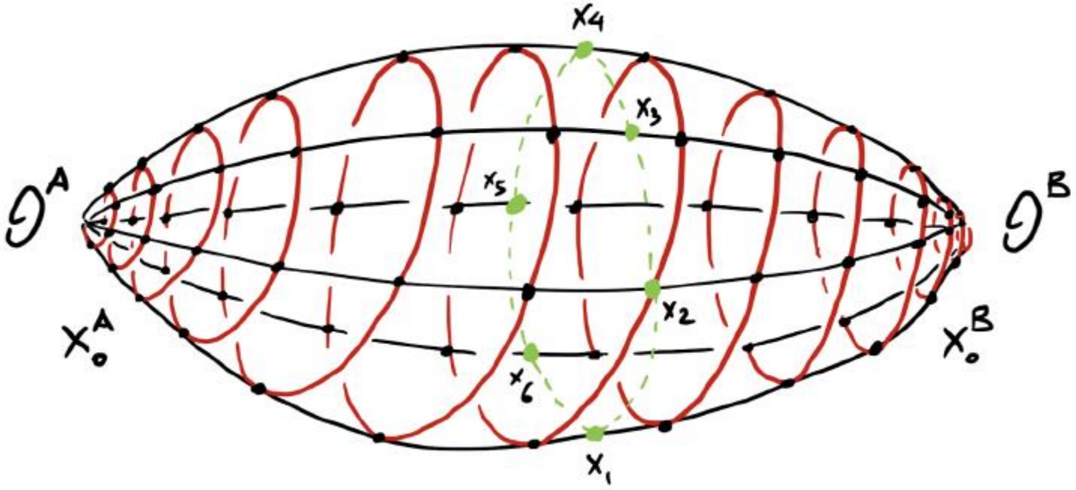
Can compute them via SoV! [Cavaglia, Gromov, **FLM** 21]

E.g. from $\partial I/\partial p$ compute 2pt function with local insertions to all loop orders

$$\frac{\partial \hat{H}}{\partial h_\alpha}\hat{H}^{-1} = -8\left[-\frac{x^2_{\alpha,\alpha-1}+x^2_{\alpha,\alpha+1}}{2}\left(1+x^\mu_\alpha\frac{\partial}{\partial x^\mu_\alpha}\right)+(x^2_{\alpha,\alpha-1}x^\mu_{\alpha+1}+x^2_{\alpha,\alpha+1}x^\mu_{\alpha-1})\frac{\partial}{\partial x^\mu_\alpha}\right]$$
$$\times\,\Box_\alpha^{-1}\frac{1}{x^2_{\alpha,\alpha-1}}\frac{1}{x^2_{\alpha,\alpha+1}}\ .$$



local action of differential operator



Hope to get experience for simpler 1d/2d fishnets [in progress],
then extend to 4d

[recent work on diagrams in 1d: Loebbert et al]

Proposal for g-function

[Cavaglia, Gromov, **FLM** 21]

Typical structure for g-function:

$$g \equiv \sqrt{\frac{\langle B|\Psi\rangle \langle \Psi|B\rangle}{\langle \Psi|\Psi\rangle}} = \underbrace{\exp\left(\int_0^\infty \Theta(u) \log(1+Y(u))du\right)}_{\text{boundary-dependent, simple}} \times \underbrace{\sqrt{\frac{\det\left[1-\hat{G}_-\right]}{\det\left[1-\hat{G}_+\right]}}}_{\text{universal factor, hard}}$$

we will guess it from norm

Like for GL(N) spin chains we conjecture the scalar product in SoV

$$\langle \Psi_A|\Psi_B\rangle \propto \det M_{AB}$$

built from integrals of Q-functions

For parity-symmetric states $M_{AA} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} \Rightarrow \det M = \det M_+ \det M_-$

We propose universal part of g-function $(g_{\text{universal}})^2 \propto \frac{|M_-|}{|M_+|_*}$ nontrivially satisfies selection rules!

inspired by spin chain/sin-Gordon results
[Gombor, Pozsgay 20, 21] [Caetano, Komatsu 20]

N=4 SYM

Still have the key starting point! [Cavaglia, Gromov, **FLM** 21]

$$\langle \bar{Q}_B (\mathcal{O}_A - \mathcal{O}_B) Q_A \rangle_\alpha = 0$$

Main difference with spin chains/fishnets:
infinitely many degrees of freedom

Implies infinitely many integrals of motion

Determinants of infinite size – should reduce to fixed size at each order in perturbation theory

Hope to uncover new structures

FUTURE

- Finally we know SoV measure for higher-rank spin chains
- Extensions: super case [Gromov, FLM 18; Maillet, Niccoli, Vignoli 20],
  SO(N) [Ferrando, Frassek, Kazakov; Ekhamar, Shu, Volin 20],
  principal series rep for fishnet, Slavnov scalar products, …
- Applications for generalized hydrodynamics? [Poszgay et al] Long range/Calogero?
  [in progress with Ferrando, Lamers, Serban]
- Algebraic meaning of $\int Q_1 Q_2 Q_3$?
- AdS/CFT: more general correlators, beyond ladders/fishnets, 1d/2d fishnet [in progress]
  Many hints of hidden SoV structures!

Happy Birthday, Nikita!

С Днём рождения!

Algebraic picture

Generating functional for transfer matrices in antisymmetric reps

$$W = (1 - \Lambda_1(u)D^2)(1 - \Lambda_2(u)D^2)\dots(1 - \Lambda_N(u)D^2) = \sum_{k=1}^{N}(-1)^k \tau_k(u) D^k$$

Define left and right action $\overrightarrow{D} f(u) = f(u + i/2), \quad f \overleftarrow{D} = f(u - i/2)$

Then $Q_a \overleftarrow{W} = 0$ and $\overrightarrow{W} Q^a = 0$

Using that for any operator $\oint g \overrightarrow{O} f = \oint f \overleftarrow{O} g$ we get $\oint Q_a^A (\overrightarrow{W}_A - \overrightarrow{W}_B) Q_B^b = 0$

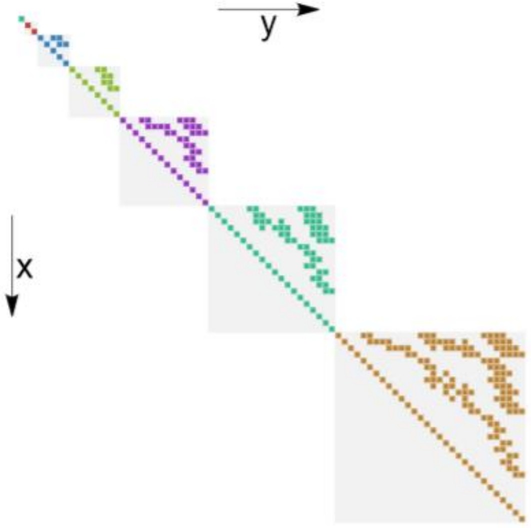
We also generalized to any spin s of the representation [Gromov FLM, Ryan 20]

$$\langle f\rangle_n = \int_{-\infty}^{\infty} du\; \mu_n\, f \qquad \mu_n = \frac{1}{1+e^{2\pi(u-\theta_n)}} \;\Rightarrow\; \mu_n = \frac{\Gamma(s-i(u-\theta_n))\Gamma(s+i(u-\theta_n))}{e^{\pi(u-\theta_n)}}$$

For SL(2) we reproduce [Derkachov, Manashov, Korchemsky]

To build SoV basis we need more involved T's in non-rectangular reps see [Ryan, Volin 20]

$$|y\rangle \propto \hat{T}_{\{m_1,m_2\}}\left(\theta_n + is + i\frac{m_1-\mu_1'}{2}\right)|0\rangle$$



Integral = sum over infinite set of poles in lower half-plane

The measure we get from Baxters again matches
the one from building the basis!

[Cavaglia, Gromov, FLM 19]

Infinite-dim highest weight representation of SL(N) on each site

Now we have integrals instead of sums $\langle f \rangle_j = \int_{-\infty}^{\infty} du\, \mu_j\, f$ $\mu_j = \frac{1}{1+e^{2\pi(u-\theta_j)}}$

$\bar{O} \circ Q^a = 0$ $O \circ Q_a = 0$

$$\bar{O} = Q_\theta^- D^{-3} - \tau_2 D^{-1} + \tau_1 D - Q_\theta^+ D^{+3}$$

$$O = Q_\theta^{++} D^{+3} - \tau_2^+ D + \tau_1{}^- D - Q_\theta^{--} D^{-3}$$

We would like $\langle g\bar{O} \circ f \rangle = \langle f O \circ g \rangle$

Now when we shift the contour we cross poles of the measure

$$\langle g\bar{O}\circ f \rangle = \int \mu\, g \left[Q_\theta^- f^{[-3]} - \tau_2 f^- + \tau_1 f^+ - Q_\theta^+ f^{[+3]} \right] = \langle f O \circ g \rangle + \text{pole contributions}$$

$Q_1(\theta_j + \frac{i}{2})\tau_1(\theta_j + \frac{i}{2}) - Q_1(\theta_j + \frac{3i}{2})Q_\theta(\theta_j + \frac{i}{2}) = 0$

Poles cancel when $g = Q_1$! Then everything works as before

The two Baxter equations are 'conjugate' to each other! [Cavaglia, Gromov, FLM 19]

$$\hat{O}\circ Q_1 \equiv Q_\theta^{++}Q_1^{[+3]} - \tau_1^+ Q_1^+ + \tau_2^- Q_1^- - Q_\theta^{--}Q_1^{[-3]} = 0$$
$$\hat{\hat{O}}\circ Q_{\bar{a}} \equiv Q_\theta^- Q_{\bar{a}}^{[-3]} - \tau_1 Q_{\bar{a}}^- + \tau_2 Q_{\bar{a}}^+ - Q_\theta^+ Q_{\bar{a}}^{[+3]} = 0$$

$$\langle g\, f\rangle_j \equiv \int_{-\infty}^{\infty} \mu_j(x) g(x)\, f(x)$$

Analog of self-adjointness property: $\langle Q_1\, \hat{\hat{O}}\circ f\rangle_j = 0$

$$\mu_j(u) = \frac{1}{1+e^{2\pi(u-\theta_j)}}$$

$$\begin{aligned}\langle\ g\ \hat{\hat{O}}\circ f\rangle_j &= \int_{-\infty}^{+\infty} \mu_j(u) g(u) \left[Q_\theta^- f^{[-3]} - \tau_1 f^- + \tau_2 f^+ - Q_\theta^+ f^{[+3]}\right] du \\ &= \int_{-\infty+i0}^{+\infty+i0} \mu_j(u+\tfrac{i}{2}) \left[\underbrace{Q_\theta^{++} g^{[+3]} - \tau_1^+ g^+ + \tau_2^- g^- - Q_\theta^{--} g^{[-3]}}_{\hat{O}\circ g}\right] f(u)\, du \\ &\quad + \text{residues from poles}\ ,\end{aligned}$$

Poles cancel if $g \equiv Q_1$! Use nontrivial relations between T's and Q's

Comment on chronology:

Such tricks with Baxters were used in [Cavaglia, Gromov, FLM 18] for cusp

Then in [Cavaglia, Gromov, FLM 19] for SL(N) spin chain

And then in [Gromov, FLM, Ryan, Volin 19] for SU(N) spin chain