Form factors and complex Bethe roots

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А. Г. Изергин, Н. А. Китанин, Н. А. Славнов, О корреляционных функциях XY модели, *Зап. научн. сем. ПОМИ*, 1995, том 224, 178–191

Izergin A. G., Kitanin N. A., and Slavnov N. A. On correlation functions of the XY model.

New determinant representation are obtained for the simplest correlation functions of the Heisenberg XY spin chain.

Slavnov determinant

Nikita Slavnov, 1989: $\{\lambda_1, \dots \lambda_N\}$ - solution of Bethe equations, $\{\mu_1, \dots \mu_N\}$ - generic set of parameters.

$$\langle \Psi(\{\mu\})\,|\Psi(\{\lambda\})\rangle = \frac{\prod_{k=1}^{N} q(\mu_k - i)}{\prod_{j>k} \sinh(\lambda_j - \lambda_k)\sinh(\mu_k - \mu_j)} \det_N \mathcal{M}(\{\lambda\}|\{\mu\}),$$

$$\mathcal{M}_{j,k}(\{\lambda\}|\{\mu\}) = \mathfrak{a}(\mu_k)t(\mu_k - \lambda_j) - t(\lambda_j - \mu_k), \quad t(\lambda) = \frac{\sin\zeta}{\sinh\lambda\sinh(\lambda + i\zeta)}.$$

Thermodynamic limit $N \to \infty$. **Asymptotic behaviour** of the scalar product?

Claim: If there is no external magnetic field

$$\text{Slavnov} = \text{Gaudin} \times \text{Cauchy}$$

XXZ spin chain

Defined on a one-dimensional lattice with M sites, with Hamiltonian, $H = H^{(0)} - hS_z$,

$$H^{(0)} = \sum_{m=1}^{M} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \right\},$$

$$S_z = \frac{1}{2} \sum_{m=1}^{M} \sigma_m^z, \qquad [H^{(0)}, S_z] = 0.$$

$\sigma_m^{x,y,z}$ are the local spin operators (in the spin-$\frac{1}{2}$ representation) associated with each site m of the chain and $\Delta = \cos(\zeta)$, ζ real or imaginary, is the anisotropy parameter, h - external magnetic field; $h \geqslant 0$. We impose the **periodic** boundary conditions.

If $\Delta = 1$ XXX Heisenberg chain (1928), solved by H. Bethe (1931).

For $h = 0$ if $\Delta > 1$ - massive antiferromagnetic regime, $|\Delta| < 1$ - massless regime.

Form Factors

Form factors: matrix elements of **local fields**, local spin operators σ_m^a, $a = x, y, z$

$|\Psi_g\rangle$ the ground state of the model $|\Psi_e\rangle$ - an excited state

The most basic form factors

$$|\mathcal{F}_a(\Psi_e)|^2 = \frac{\langle \Psi_g | \, \sigma_m^a \, | \Psi_e \rangle \, \langle \Psi_e | \, \sigma_m^a \, | \Psi_g \rangle}{\langle \Psi_g \, | \Psi_g \rangle \, \langle \Psi_e \, | \Psi_e \rangle}$$

Then more advanced questions can be studied like matrix elements of **currents**

- Integrable QFT - F. Smirnov 1992 **bootstrap approach**
- Massive XXZ, M. Jimbo and T. Miwa 1995 q-**vertex operator** approach
- General XXZ, N.K, J.M. Maillet, V. Terras, 1999 **Algebraic Bethe ansatz** approach

XXX chain: Algebraic Bethe ansatz

L.D. Faddeev, E.K. Sklyanin, L.A. Takhtajan (1979). Main object: quantum monodromy matrix:

$$T_a(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_a .$$

- Diagonal elements $\longrightarrow$ commuting conserved charges: transfer matrix

$$\mathcal{T}(\lambda) = \mathrm{tr}_a\, T_a(\lambda) = A(\lambda) + D(\lambda), \qquad [\mathcal{T}(\lambda), \mathcal{T}(\mu)] = 0$$

- Hamiltonian:

$$H = 2i \frac{\partial}{\partial \lambda} \log \mathcal{T}(\lambda)\Big|_{\lambda=\frac{i}{2}}, \qquad [H, \mathcal{T}(\lambda)] = 0$$

- Non-diagonal elements $\longrightarrow$ creation/annihilation operators.

Bethe states

Off-shell Bethe states:

$$|\Psi(\{\lambda_1, \ldots, \lambda_N\})\rangle = B(\lambda_1)\ldots B(\lambda_N)\,|0\rangle\,, \qquad |0\rangle = |\uparrow\uparrow\ldots\uparrow\rangle$$

For any Bethe state we define **Baxter polynomial** and **exponential counting function**

$$q(\lambda) = \prod_{j=1}^{N}(\lambda - \lambda_j), \qquad \mathfrak{a}(\lambda) = \left(\frac{\lambda - \frac{i}{2}}{\lambda + \frac{i}{2}}\right)^M \frac{q(\lambda + i)}{q(\lambda - i)}.$$

if the **Bethe equations** are satisfied (on-shell Bethe state)

$$\mathfrak{a}(\lambda_j) + 1 = 0, \qquad j = 1, \ldots N$$

then it is an eigenstate of the **transfer matrix** and the Hamiltonian

$$\mathcal{T}(\mu)\,|\Psi(\{\lambda\})\rangle = \tau(\mu)\,|\Psi(\{\lambda\})\rangle\,, \qquad \tau(\mu) = \big(\mathfrak{a}(\mu) + 1\big)\frac{q(\mu - i)}{q(\mu)}.$$

Multiplet structure

XXX chain: additional $\mathfrak{su}(2)$ symmetry:

$$[\mathcal{T}(\lambda), S_a] = 0, \qquad a = x, y, z.$$

On-shell Bethe vectors are $\mathfrak{su}(2)$ highest weight vectors

$$S_+ \left|\Psi(\{\lambda\})\right\rangle = 0, \qquad S_+ = \sum_{m=1}^{M} \sigma_m^+.$$

For XXX there are solutions of Bethe equations only if $N \leqslant \frac{M}{2}$. For $N = \frac{M}{2} - k$ they generate $2k + 1$ multiplets

$$\left|\Psi_\ell(\{\lambda\})\right\rangle = S_-^\ell \left|\Psi(\{\lambda\})\right\rangle, \quad \ell = 0, \ldots, 2k, \quad \mathcal{T}(\mu) \left|\Psi_\ell(\{\lambda\})\right\rangle = \tau(\mu) \left|\Psi_\ell(\{\lambda\})\right\rangle$$

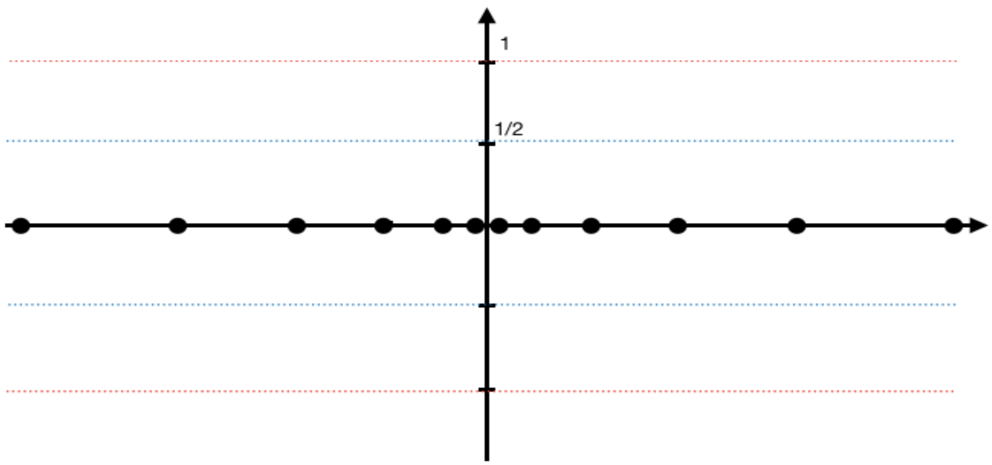
Multiplets can be seen as **Bethe states** with infinite rapidities $\lim_{\lambda\to\infty} \lambda B(\lambda) = S_-$.

The ground state

Ground state solution of the Bethe equations

$$\mathfrak{a}(\lambda_j) + 1 = 0, \qquad j = 1, \dots N$$

Yang and Yang 66: $N = \frac{M}{2}$ (singlet), all the roots are **real**. There is **no holes** *i.e.* all the real zeroes of $\mathfrak{a}_g(\lambda) + 1$ are Bethe roots.



The ground state density

The Bethe roots fill the real line with some density in the **thermodynamic limit**:

$$\frac{1}{M}\sum_{j=1}^{\frac{M}{2}} f(\lambda_j) = \int_{-\infty}^{\infty} f(\lambda)\rho(\lambda)d\lambda + o\left(\frac{1}{M}\right).$$

The ground state density solves the Lieb equation

$$\rho_g(\lambda) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} K(\lambda-\mu)\rho_g(\mu)d\mu = \frac{1}{2\pi i}t(\lambda - i/2),$$

Where $t(\lambda) = \frac{i}{\lambda(\lambda+i)}$ and $K(\lambda) = t(\lambda) + t(-\lambda)$.

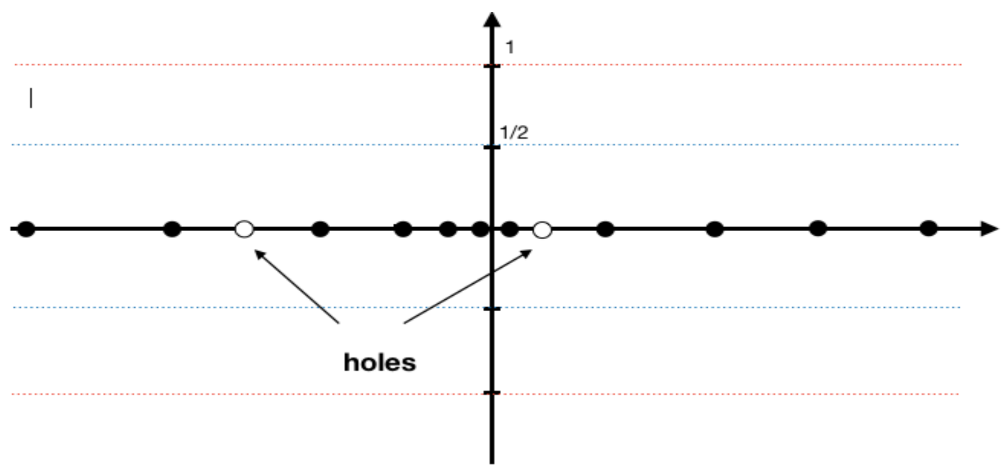
The ground state solution:

$$\rho(\lambda) = \frac{1}{2\cosh(\pi\lambda)}.$$

Excitations: spinons

Holes (spinons) μ_h is not a **Bethe root** but:

$$\mathfrak{a}_e(\mu_h) + 1 = 0$$



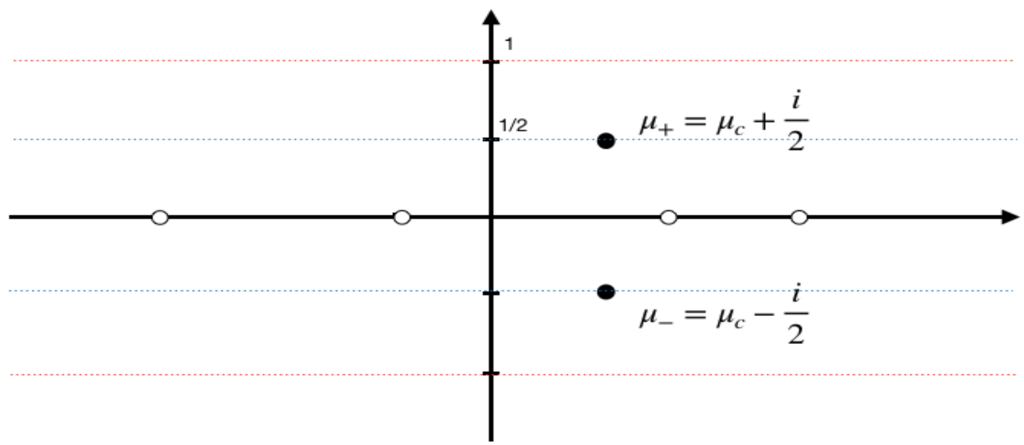
Number of holes n_h is always even, $N = \frac{M}{2} - \frac{n_h}{2}$ if all roots are real.

Complex roots: strings

Complex roots: if μ_+ is one of the Bethe roots then $\mu_- = \overline{\mu}_+$ is also a root (bound state). For a finite chain with large M the simplest configuration: **2-string**:

$$\mu_+ = \mu_c + \frac{i}{2} - i\delta, \quad \mu_- = \mu_c - \frac{i}{2} + i\delta,$$

Where $\mu_c \in \mathbb{R}$ - string center and $\delta = O(M^{-\infty})$ - string deviation.

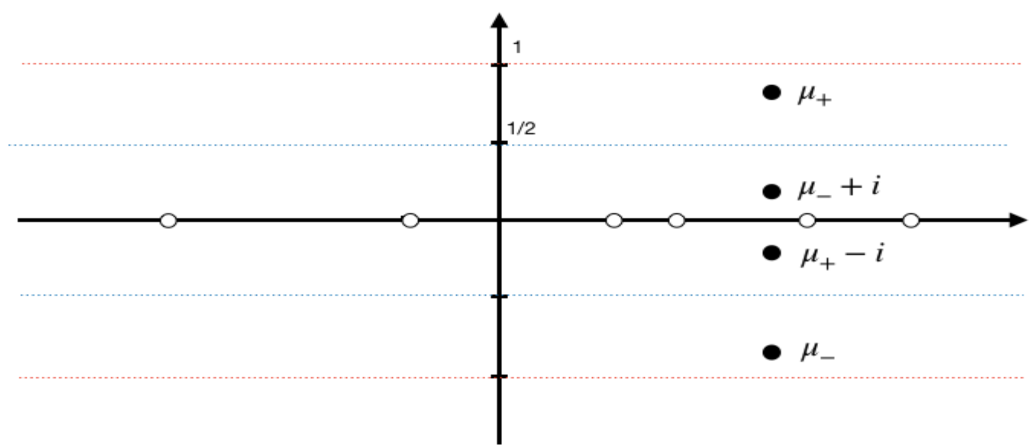


Complex roots: close pairs and quartets

Destri, Lowenstein 1982; Babelon, de Vega, Viallet 1983.

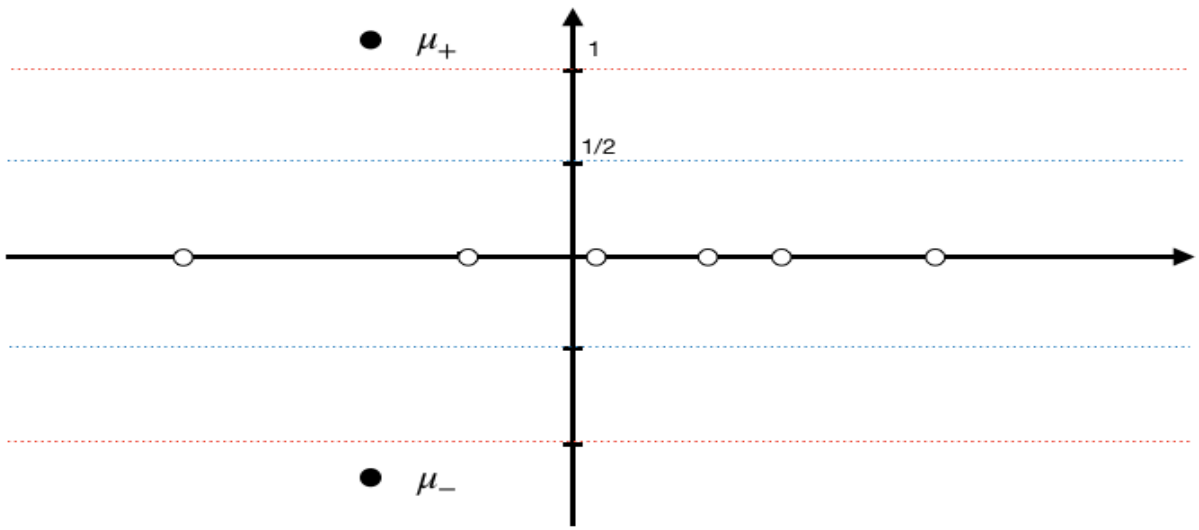
Close pair: $\mu_+, \mu_-, \quad 0 < \Im(\mu_+) < 1$. Note: **2-string** is a close pair. Otherwise close pairs form quartets:

$$\mu_+, \quad \mu_- + i - i\delta, \quad \mu_+ - i + i\delta, \quad \mu_-.$$



Complex roots: wide pairs

Wide pair: $\mu_+, \mu_-, \quad \Im(\mu_+) > 1$.



Excited states

We denote n_h - number of **holes** (even), n_s - number of **2-strings**, n_q - number of **quartets**, n_w - number of **wide pairs**. Total number of Bethe roots

$$N = \frac{M}{2} - \frac{n_h}{2} + n_s + 2n_q + 2n_w, \quad N \leqslant \frac{M}{2}$$

. Positions of holes $\mu_{h_a} \in \mathbb{R}$, $a = 1, \ldots n_h$ (arbitrary in the thermodynamic limit). The position of complex roots are defined from the position of holes from the **higher level Bethe equations** Destri, Lowenstein 1982; Babelon, de Vega, Viallet 1983.

$$\prod_{a=1}^{n_h} \frac{z_j - \mu_{h_a} - \frac{i}{2}}{z_j - \mu_{h_a} + \frac{i}{2}} \prod_{k=1}^{n_c} \frac{z_j - z_k + i}{z_j - z_k - i} + 1 = 0, \quad j = 1, \ldots, n_c.$$

here $n_c = n_s + 2n_q + 2n_w$,

$z_j = \mu_c$ for a **2 string**

$z_j = \mu_+ - \frac{i}{2}, \quad z_{j+1} = \mu_- + \frac{i}{2}$ for a **quartet** and a **wide pair**.

Energy and momentum

The complex roots don't influence **energy and momentum**, they depend only on hole positions (spinon rapidities).

$$\Delta E \equiv E_e - E_g = \sum_{a=1}^{n_h} \varepsilon(\mu_{h_a}), \qquad \varepsilon(\mu) = \frac{\pi}{2\cosh \pi\mu},$$

$$\Delta P \equiv P_e - P_g = \sum_{a=1}^{n_h} p(\mu_{h_a}), \qquad p(\mu) = \frac{\pi}{2} - \arctan(\sinh \pi\mu).$$

With fixed **spinon rapidities** 2^{n_h}-fold degeneracy.

Example: **two-spinon sector**, 4-fold degenerate:

- singlet with two holes μ_{h_1}, μ_{h_2} and one 2-string $\mu_c + \frac{i}{2}, \mu_c - \frac{i}{2}$, higher level Bethe equations have only one solution: $\mu_c = \frac{1}{2}(\mu_{h_1} + \mu_{h_2})$
- triplet with two holes μ_{h_1}, μ_{h_2} and no complex roots

Form factors and multiplet structure

We want to compute:

$$|\mathcal{F}_z|^2 = \frac{\langle \Psi_e \,|\, \sigma_m^z \,|\, \Psi_g \rangle \, \langle \Psi_g \,|\, \sigma_m^z \,|\, \Psi_e \rangle}{\langle \Psi_g \,|\, \Psi_g \rangle \, \langle \Psi_e \,|\, \Psi_e \rangle},$$

for XXX no difference between x, y or z form factors.

It is easy to see from $\sigma_m^z = [S^+, \sigma_m^-]$ that there are non-trivial form factors **only** for the **triplet** states ($N = \frac{M}{2} - 1$) and

$$\begin{aligned}
\left\langle \Psi_{e_1} \,\middle|\, \sigma_m^z \,\middle|\, \Psi_g \right\rangle &= -2 \left\langle \Psi_{e_0} \,\middle|\, \sigma_m^+ \,\middle|\, \Psi_g \right\rangle, \\
\left\langle \Psi_g \,\middle|\, \sigma_m^z \,\middle|\, \Psi_{e_1} \right\rangle &= \left\langle \Psi_g \,\middle|\, \sigma_m^+ \,\middle|\, \Psi_{e_2} \right\rangle, \\
\left\langle \Psi_{e_1} \,\middle|\, \Psi_{e_1} \right\rangle &= 2 \left\langle \Psi_{e_0} \,\middle|\, \Psi_{e_0} \right\rangle.
\end{aligned}$$

Form factors and inverse problem

Quantum inverse problem: local operators in terms of the monodromy matrix elements N.K., J.M. Maillet and V. Terras 1999:

$$\sigma_m^z = \mathcal{T}^{m-1}\left(\tfrac{i}{2}\right)\left\{A\left(\tfrac{i}{2}\right) - D\left(\tfrac{i}{2}\right)\right\}\mathcal{T}^{-m}\left(\tfrac{i}{2}\right),$$
$$\sigma_m^- = \mathcal{T}^{m-1}\left(\tfrac{i}{2}\right) B\left(\tfrac{i}{2}\right)\mathcal{T}^{-m}\left(\tfrac{i}{2}\right),$$
$$\sigma_m^+ = \mathcal{T}^{m-1}\left(\tfrac{i}{2}\right) C\left(\tfrac{i}{2}\right)\mathcal{T}^{-m}\left(\tfrac{i}{2}\right).$$

Due to the commutation relations of the monodromy matrix elements everything expressed in terms of **scalar products** of off-shell and on-shell **multiplet** Bethe states.

$$|\mathcal{F}_z|^2 = -\frac{\tau_e(\frac{i}{2})}{\tau_g(\frac{i}{2})}\frac{\left\langle \Psi_{e_0} \,\middle|\, C(\frac{i}{2}) \,\middle|\, \Psi_g \right\rangle \left\langle \Psi_g \,\middle|\, C(\frac{i}{2}) \,\middle|\, \Psi_{e_2} \right\rangle}{\left\langle \Psi_g \,\middle|\, \Psi_g \right\rangle \left\langle \Psi_{e_0} \,\middle|\, \Psi_{e_0} \right\rangle},$$

Scalar products and norms

N. Slavnov, 1989: $\{\lambda_1, \ldots \lambda_N\}$ - solution of Bethe equations, $\{\mu_1, \ldots \mu_N\}$ - generic set of parameters.

$$\langle \Psi(\{\mu\}) \,|\Psi(\{\lambda\})\rangle = \frac{\prod_{k=1}^{N} q(\mu_k - i)}{\prod_{j>k} (\lambda_j - \lambda_k)(\mu_k - \mu_j)} \det_N \mathcal{M}(\{\lambda\}|\{\mu\}),$$

$$\mathcal{M}_{j,k}(\{\lambda\}|\{\mu\}) = \mathfrak{a}(\mu_k) t(\mu_k - \lambda_j) - t(\lambda_j - \mu_k), \quad t(\lambda) = \frac{i}{\lambda(\lambda + i)}.$$

Norms of the on-shell Bethe states are given by the Gaudin formula

$$\langle \Psi(\{\lambda\}) \,|\Psi(\{\lambda\})\rangle = (-1)^N \frac{\prod_{j=1}^{N} q(\lambda_j - i)}{\prod_{j\neq k} (\lambda_j - \lambda_k)} \det \mathcal{N}(\{\lambda\}),$$

$$\mathcal{N}_{j,k}(\{\lambda\}) = \mathfrak{a}'(\lambda_j)\delta_{j,k} - K(\lambda_j - \lambda_k), \quad K(\lambda) = t(\lambda) + t(-\lambda).$$

Slavnov formula can be adapted for the **multiplet states** Foda, Wheeler 2012.

Finite chain determinant representation for the factors form factors corresponding to arbitrary triplet excited state:

$$|\mathcal{F}_z|^2 = -2 \prod_{j=1}^{\frac{M}{2}-1} \frac{q_g(\mu_j - i)}{q_e(\mu_j - i)} \prod_{k=1}^{\frac{M}{2}} \frac{q_e(\lambda_k - i)}{q_g(\lambda_k - i)}$$

$$\times \frac{\det_{\frac{M}{2}} \mathcal{M}(\{\lambda\}|\{\mu_1 \dots \mu_{\frac{M}{2}-1}, \frac{i}{2}\}) \det_{\frac{M}{2}+1} \mathcal{M}^{(2)}(\{\mu\}|\{\lambda_1, \dots \lambda_{\frac{M}{2}}, \frac{i}{2}\})}{\det_{\frac{M}{2}} \mathcal{N}(\{\lambda\}) \det_{\frac{M}{2}-1} \mathcal{N}(\{\mu\})}.$$

Here $\mathcal{M}^{(2)}$ - Foda Wheeler variant of the Slavnov matrix for triplets (two extra rows).

$$\mathcal{M}^{(\ell)}_{j,k}(\{\lambda\}|\{\mu\}) = \mathfrak{a}(\mu_k)(\mu_k + i)^{j-N-1} - \mu_k^{j-N-1}, \qquad \text{for } j > N.$$

Computation of determinants

The main idea is extremely simple: we compute the following matrices

$$F_g = \mathcal{N}^{-1}(\{\lambda\})\mathcal{M}\left(\{\lambda\}|\{\mu_1 \ldots \mu_{\frac{M}{2}-1}, \frac{i}{2}\}\right),$$
$$F_e = \mathcal{N}^{(2)^{-1}}(\{\mu\})\mathcal{M}^{(2)}\left(\{\mu\}|\{\lambda_1, \ldots \lambda_{\frac{M}{2}}, \frac{i}{2}\}\right),$$

For the first (ground state) matrix: system of linear equations

$$\mathfrak{a}_g'(\lambda_j)F_{g_{j,k}} - \sum_{a=1}^{\frac{M}{2}} K(\lambda_j - \lambda_a)F_{g_{a,k}} = \mathfrak{a}_g(\mu_k)t(\mu_k - \lambda_j) - t(\lambda_j - \mu_k).$$

We set

$$\mathfrak{a}_g'(\lambda_j)F_{g_{j,k}} = G_g(\lambda_j; \mu_k)$$

Linear equations $\longrightarrow$ Contour integral equation for $G_g(\lambda;\mu)$

$$G_g(\lambda;\mu_k) - \frac{1}{2\pi i}\oint\limits_{\Gamma} d\nu\, K(\lambda-\nu)\frac{G_g(\nu;\mu_k)}{1+\mathfrak{a}_g(\nu)} = (\mathfrak{a}_g(\mu_k)+1)t(\mu_k-\lambda),$$

We set

$$G_g(\lambda;\mu) = \big(1+\mathfrak{a}_g(\mu)\big)\rho_g(\lambda;\mu)$$

.

Thermodynamic limit $\longrightarrow$ Integral equation

$$\rho_g(\lambda;\mu) - \frac{1}{2\pi i}\int\limits_{\mathbb{R}+i\epsilon} d\nu\, K(\lambda-\nu)\rho_g(\nu;\mu) = t(\mu-\lambda).$$

Lieb equation for the density of Bethe roots!

Solution:

$$F_{g_{j,k}} = \frac{\mathfrak{a}_g(\mu_k)+1}{\mathfrak{a}'_g(\lambda_j)}\frac{\pi}{\sinh\pi(\mu_k-\lambda_j)}$$

F_e is slightly more complicated (holes contributions, Foda-Wheeler rows) but also has this basic Cauchy structure.

Example without complex roots:

$$F_{e_{j,k}} = \frac{\mathfrak{a}_e(\lambda_k)+1}{\mathfrak{a}'_e(\mu_j)} \left(\frac{\pi}{\sinh \pi(\lambda_k - \mu_j)} - 2\pi i \sum_{a=1}^{n_h} \frac{\rho_h(\mu_j - \mu_{h_a})}{\mathfrak{a}'_e(\mu_{h_a})} \frac{\pi}{\sinh \pi(\lambda_k - \mu_{h_a})} \right), \qquad j \leqslant \frac{M}{2} - 1$$

$$F_{e_{\frac{M}{2},k}} = \mathfrak{a}_e(\lambda_k) - 1, \qquad F_{e_{\frac{M}{2}+1,k}} = \mathfrak{a}_e(\lambda_k)(\lambda_k + i) - \lambda_k.$$

Composed only of **Cauchy columns**! Very similar structure with the complex roots.

Structure of results

After this step typically the form factor is written as a product of two **Cauchy** determinants and determinants of a $n_h \times n_h$ matrix and a $n_c \times n_c$ matrix.

$$|\mathcal{F}_z|^2 = -2 \prod_{j=1}^{M/2} \frac{\tau_e(\lambda_j)}{\tau_g(\lambda_j)} \prod_{k=1}^{M/2-1} \frac{\tau_g(\mu_k)}{\tau_e(\mu_k)} \det_{\frac{M}{2}} \mathcal{C}_g \det_{\frac{M}{2}+1} \mathcal{C}_e$$

$$\times \frac{\prod_{j=1}^{M/2} \prod_{k=1}^{M/2-1} (\lambda_j - \mu_k)^2}{\prod_{j \neq k}^{M/2} (\lambda_j - \lambda_k) \prod_{j \neq k}^{M/2-1} (\mu_j - \mu_k)} \det_{n_c} \mathcal{Q}_g \det_{n_h} \mathcal{Q}_e$$

,

$\mathcal{C}_{jk} = \frac{\pi}{\sinh \pi(\mu_k - \lambda_j)}$ - **Cauchy determinant** (can be computed)

Thermodynamic limit : 2-spinon case

Two-spinon form factor: no complex roots, the only “extra” matrix

$$\mathcal{Q}_e = \begin{pmatrix} \frac{1}{\mathfrak{a}'_e(\mu_{h_1})} & \frac{1}{\mathfrak{a}'_e(\mu_{h_2})} \\ \frac{\mu_{h_1}+\frac{i}{2}}{\mathfrak{a}'_e(\mu_{h_1})} & \frac{\mu_{h_2}+\frac{i}{2}}{\mathfrak{a}'_e(\mu_{h_2})} \end{pmatrix}, \qquad \det \mathcal{Q}_e = \frac{\mu_{h_1}-\mu_{h_2}}{\pi^2 M^2} \prod_{a=1}^{2} \cosh \pi \mu_{h_a}$$

With expected scaling $\frac{1}{M^2}$. Final result for the scaled form factor:

$$|\mathcal{Y}(\mu_{h_1}-\mu_{h_2})|^2 = \lim_{M\to\infty} M^2 |\mathcal{F}_z|^2 = \frac{2}{G^4\left(\frac{1}{2}\right)} \left| \frac{G\left(\frac{\mu_{h_1}-\mu_{h_2}}{2i}\right) G\left(1+\frac{\mu_{h_1}-\mu_{h_2}}{2i}\right)}{G\left(\frac{1}{2}+\frac{\mu_{h_1}-\mu_{h_2}}{2i}\right) G\left(\frac{3}{2}+\frac{\mu_{h_1}-\mu_{h_2}}{2i}\right)} \right|^2 .$$

Where $G(z)$ iz the Barnes G-function (related to the double Γ-function).

$$G(z+1) = \Gamma(z)G(z), \qquad G(1) = 1.$$

Relation with q-vertex operator approach

Using integral representations for $\log G(z)$ we obtain

$$|\mathcal{Y}(\mu_{h_1} - \mu_{h_2})|^2 = 2e^{-I(\mu_{h_1}-\mu_{h_2})},$$
$$I(\mu_{h_1} - \mu_{h_2}) = \int_0^\infty \frac{dt}{t}\, e^t \, \frac{\cos\left(2(\mu_{h_1} - \mu_{h_2})t\right)\cosh(2t) - 1}{\cosh(t)\sinh(2t)}.$$

This reproduces the result for the two-spinon form factor obtained in the q-vertex operator framework from the M. Jimbo and T. Miwa multiple integral formulas by A. H. Bougourzi, M. Couture and M. Kacir

Starting from **four-spinon** case the things become more interesting as we use a different basis with respect to the Jimbo-Miwa approach.

General case

General excited state with n_h **spinons** and $n_c = n_s + 2n_q + 2n_w$ **complex roots**

General representation: pre-factor (Cauchy determinants) and finite size determinant

$$\lim_{M\to\infty} M^{n_h}|\mathcal{F}_z|^2 = \prod_{j>k} |\mathcal{Y}(\mu_{h_j} - \mu_{h_k})|^2 \, \mathcal{R}(\mu_h, \mu_c) \det_{n_c} \widehat{\mathcal{Q}_g} \det_{n_h} \widehat{\mathcal{Q}_e}.$$

Here $\mathcal{R}(\mu_h, \mu_c)$ - rational function of holes and complex roots
$\widehat{\mathcal{Q}_g}$ and $\widehat{\mathcal{Q}_e}$ - finite matrices also written in terms of positions of **holes** and **complex roots**. Includes also a **Higher level Gaudin matrix**

Simpler form for finite matrices: work in progress.

Interesting interplay with Jimbo-Miwa-Smirnov fermionic approach.

Conclusion and outlook

Advantages of the new approach:

- Explicit results, no Fredholm determinants.
- We know how to deal with bound states
- Possibility to apply in a systematic way for all the regimes of the XXZ chain

Open problems: out-of-equilibrium systems, overlaps instead of form factors

- Second densification (holes, complex roots distributed with some density)
- Can we apply this method far from the **ground state**?
- Macroscopic changes in the system (**quenches**).

Happy birthday, Nikita!

Никита, С Днем Рождения!