# Polynomial rings theory and integrable systems

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**Yang-Baxter integrability**:

$$R_{12}(u_1,u_2)R_{23}(u_2,u_3)R_{13}(u_1,u_3) = R_{13}(u_1,u_3)R_{23}(u_2,u_3)R_{12}(u_1,u_2)$$

**Rationality condition I** *Let us $R(u,v)$ will be a rational matrix*

$$R(u,v) = \mathbb{I} + \frac{\mathbb{P}}{u-v}, \qquad \mathbb{P} = \sum_{1\le i,j\le N} E_{ij}\otimes E_{ji},$$

or

$$R(u,v) = f(u,v)\sum_{1\le i\le N} E_{ii}\otimes E_{ii} + \sum_{1\le i<j\le N}\left(E_{ii}\otimes E_{jj} + E_{jj}\otimes E_{ii}\right) + \sum_{1\le i<j\le N} g(u,v)\left(uE_{ij}\otimes E_{ji} + vE_{ji}\otimes E_{ij}\right)$$

where $f$, $g$ are proper rational functions

# ''Rational'' system

The system is given by Lax operator $L$ satisfying

$$R_{12}(u,v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u,v)$$

We assume existance of $|0\rangle$. Then Bethe ansatz provides as with a knowledge of eigenvectors of the system $|\bar{u}\rangle$ that are numerated by the rapidites $\bar{u}$ that are given solutions of Bethe ansatz equations (BAE)

$$e^{iP(u_j)L} = \prod_{1\leq i\leq N; i\neq j} S(u_j, u_i), \qquad j = 1, \ldots, N$$

where $p(u)$ is a (quasi)momentum of the system (defined by the Lax operator), $S$ scattering amplitude (defined by $R$-matrix and is rational)
**Rationality condition II** *If $L_{ij}|0\rangle$ is rational then eigenvectors become rational objects*
If conditions I-II are satisfied then BAE system becomes rational

Let us consider a finite size system, then for arbitrary operators $Q$, $O(x)$, $O_1(x)$, etc. we can compute matrix elements

$$\langle \bar{u}|O(x)|\bar{u}\rangle, \qquad \langle \bar{u}|O(x)|\bar{v}\rangle, \qquad \langle \bar{u}|O_1(x)O_2(y)|\bar{u}\rangle, \qquad \dots \tag{1.1}$$

(The simplest case will be averaging of the transfer matrix of the system $t^K$ that provide as with a statistical sum of the system)
If conditions I–III are satisfied then for finite case of Bethe roots obviously (1.1) are rational functions.

BAE can be rewritten via *QQ-system*

$$vQ_{a+1,s}(v)Q_{a,s+1}(v) = Q^{+}_{a+1,s+1}(v)Q^{-}_{a,s}(v) - Q^{-}_{a+1,s+1}(v)Q^{+}_{a,s}(v)$$

where $f^{\pm}(v) = f\left(v \pm \frac{c}{2}\right)$ and $Q$-functions $Q_{a,s}(v)$ are even polynomials on $v$ numerated by nodes of the Young tableau (Volin, Marboe, 2016)

$$Q_{a,s}(u) = u^{2M_{a,s}} + \sum_{k=0}^{M_{a,s}-1} c^{(k)}_{a,s} u^{2k} \tag{1.2}$$

+certain boundary conditions for $Q_{0,0}$
**Requirement for all solutions $Q_{a,s}$ to be polynomials provides us with a set of algebraic equations we call** *zero reminder conditions* (ZRC)

# Rational functions summation

We consider a set of $N$-variable algebraic equations

$$F_1(z_1, \dots, z_N) = F_2(z_1, \dots, z_N) = \dots = F_N(z_1, \dots, z_N) = 0 \qquad (1.3)$$

For a given polynomial $P$ we want to compute the sum over solutions of system (1.3)

$$S[P] = \sum_{sol} P(z_1, \dots, z_N)$$

Numeric solution is rather cumbersome and suffers from the numeric errors
For instance: not too much is known about Bethe equation solution beyond the string hypothesis conjecture. Especially problematic case of quantum transfer matrix (QTM) BAE
**At this moment the fact that we are dealing with exclusively rational functions can be applied** (D. Volin, J. Jacobsen, Y. Jiang)

# Polynomial ring theory

**Polynomial ring** $Q[z]$: set of polynomials of form

$$q = q_0 + q_1 z + q_2 z^2 + \cdots + q_N z^N$$

with the additive and multiplicative operations defined in the usual way. Similarly polynomial ring on multiple variables can be defined $Q[z_1, \ldots, z_n]$

**Ideal** $I$ of $Q[z]$

1. $f_1 + f_2 \in I$ if $f \in I$ and $f_2 \in I$
2. $gf \in I$ for $f \in I$ and $g \in Q[z]$

Important: any ideal of polynomial ring $Q[z]$ is *finitely generated* i.e. there exist a finite number of $f_i \in I$, such that for any $F \in I$

$$F = \sum_{1 \le i \le N} f_i g_i, \qquad g_i \in Q[z_1, \ldots, z_N]$$

# Polynomial reduction

Important: The choice of generating set of basis is no unique. Even the dimension of set can be different. The convenient basis can be chosen

**Polynomial reduction** of $F$ over the set $\langle f_1, \ldots, f_n \rangle$. For any $F \in Q[z_1, \ldots, z_N]$

$$F = \sum_{1 \le i \le N} f_i g_i + r, \qquad g_i \in Q[z_1, \ldots, z_N]$$

where $r$ is called **remainder**

Important: for a given set $\langle f_1, \ldots, f_n \rangle$ reduction is not unique! For instance: $F(x, y) = x^2 y + x y^2 + y^2$, $f_1 = y^2 - 1$ and $f_2 = xy - 1$, then

1. $F(x, y) = (x + 1) f_1 + x\, f_2 + (2x + 1)$,
2. $F(x, y) = f_1 + (x + y) f_2 + (x + y + 1)$

Hereby the remainder (and a whole expansion) is not well defined in a generic basis. Even dimensions of different bases can be different. More convenient basis is needed

# Gröbner basis

*Monomial ordering*:

- $u^s \prec u^{s+1}$
- If $u \prec v$ then for any monomial $w$ we have $uw \prec vw$
- For any monomial $w$ holds $1 \prec w$

*Leading term* of function $f$ ( $\mathrm{LT}(f)$) is defined as a highest monomial of $f$ with respect to the monomial order $\prec$

*Gröbner basis* $G(I)$ of an ideal $I$ with respect to the monomial order $\prec$ is a basis of the ideal $\langle g_1, \dots, g_n \rangle$, such that for any $f \in I$ there exists a $g_i \in G(I)$ such that $\mathrm{LT}(f)$ is divisible by $\mathrm{LT}(g_i)$.

*Minimal reduced Gröbner basis*

1. Coefficient in front of LT of each $g \in G(I)$ is equal 1.
2. Neither monomial of any $g \in G(I)$ is not divisible by the LT of any other $f \in G(I)$

**Lemma:** For a given monomial order $\prec$ minimal reduced Gröbner basis exists and is unique

The reminder of arbitrary $F$ is unique for the minimal reduced Gröbner basis

# Properties of the polynomial ring

**Quotient ring** $Q_I = Q[z_1, z_2, \ldots, z_N]/I$ where $I = \langle f_1 \ldots f_K \rangle$ is the ideal generated by $f_1, \ldots, f_K$, i.e. for $a \in Q_I$, $b \in Q_I$ $a \sim b$ if $a - b \in I$
We consider system

$$f_1(z_1, \ldots, z_N) = f_2(z_1, \ldots, z_N) = \cdots = f_K(z_1, \ldots, z_N) = 0 \qquad (1.4)$$

then ideal $I$ can be defined as $I = \langle f_1, \ldots, f_K \rangle$ and the quotient ring is given by $Q_I$ as $Q_I = Q[z_1, \ldots, z_N]/\langle f_1, \ldots, f_K \rangle$.
**Important**:
The dimension of the quotient ring dim $Q_I$ is equal to the number of solution of system (1.4).
If $G(I)$ is a Gröbner basis of the ideal $I$ the linear space $Q_I$ is spanned by monomials which are **not divisible** by any element in $\mathrm{LT}[G(I)]$

# Companion matrix

Any monomial $P \in Q[z_1, \ldots, z_N]$ can be presented as a matrix in the quotient ring which is finite dimension linear space. Let $(m_1, \ldots, m_N)$ be the monomial basis of $Q_I = Q[z_1, \ldots, z_N]/I$ which can be constructed by the Gröbner basis. Then

1. Perform the expansion

$$[Pm_i]_{G(I)} = \sum_j c_{ij} m_j$$

where $[F]_{G(I)}$ means the reminder of the polynomial reduction of $F$ with respect to the Gröbner basis $G(I)$

2. Define *a companion matrix* of function $P$

$$(M_P)_{ij} = c_{ij}.$$

# Properties of the companion matrix

For arbitrary functions $f$ and $g$ with companion matrices $M_f = M_g$ if and only if $[f] = [g]$ (i .e. $f - g \in Q_I$)
1.$M_{f+g} = M_f + M_g$
2.$M_{fg} = M_f M_g = M_g M_f$
3.$M_{f/g} = M_f M_g^{-1}$
The last property allows to deal with rational not only polynomial functions
Lemma:

$$S[P] = \sum_{sol} P(z_1, \ldots, z_N) = \mathrm{Tr}[\mathrm{M_P}]$$

# General algorithm

1. Generate the set of ZRC from rational QQ-system
2. Compute the Gröbner basis $G(I)$ of ZRC
3. Construct the quotient ring of the ZRC (provide us with a monomial basis $(m_1, \dots, m_N)$)
4. Compute companion matrices $M_P$
5. Compute traces of companion matrices

# Physical observables computation

6V (2020) and Potts (2022) models statistical sums J. Jacobsen *et. al.*
XXX model was studied by Y. Jiang *et al.*, (2021) with $L = 4, \dots, 20$ with a magnon numbers $N = L/2$
**Diagonal Rényi entropy** The diagonal ensemble is defined by the density matrix

$$\rho_d = \sum_{\bar{u}} O_{\bar{u}} |\bar{u}\rangle\langle\bar{u}|, \qquad O_{\bar{u}} = |\langle\psi_0|\bar{u}\rangle|^2,$$

then diagonal Rényi entropy is given by

$$S_d^{(\alpha)} = \frac{1}{1-\alpha} \log \mathrm{Tr} \rho_{\mathrm{d}}^{(\alpha)} = \frac{1}{1-\alpha} \log \sum_{\bar{\mathrm{u}}} \mathrm{O}_{\bar{\mathrm{u}}}^{\alpha}$$

## Loschmidt echo amplitude

$$M_L(it) = \langle\psi|e^{-itH}|\psi\rangle = \sum_{\bar{u}} |\langle\psi|\bar{u}\rangle|^2 e^{-itE(\bar{u})} \tag{1.5}$$

summation is taken over all solutions of Bethe equation, $|\psi\rangle$ is an initial state.
Integrable initial state $|\psi\rangle$ was considered

$$|\psi\rangle = \frac{1}{2}\left(|\uparrow\downarrow\rangle^{\otimes L/2} + |\downarrow\uparrow\rangle^{\otimes L/2}\right)$$

Remark: the similar approach can be done for XXZ
Remark: (1.5) contains factor $\exp(itE)$, that is **not** a rational function!

In case we are dealing with non-rational functions the following trick can be used: let us consider sum $\sum_{sol} p(u)F(q(u))$, where $p(u)$, $q(u)$ are rational functions and $F(z)$ any function that does not have singularities at $z = q$. Then

$$\sum_{sol} p(u)F(q(u)) = \oint_{\mathcal{C}} \frac{dz}{2\pi i} F(z) \sum_{sol} \frac{p(u)}{z - q(u)}$$

where $\mathcal{C}$ encircles all possible values of $q$
Hereby we proceed to the rational function by a cost of adding a **single** contour integral (could be computed straightforwardly)

# Conclusions

- Polynomial ring theory can be applied for any rational model
- Loschmidt echo, Rényi entropy could be calculated for relatively long chain

Possible future ideas

- Correlation functions?
- Summation over solutions of the QTM?
- Nested Bethe ansatz?