Effective form factors for free fermionic models at finite temperature

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Motivation: Integrable 1D models

- Heisenberg Spin-Chain

$$H_{XXZ} = -J \sum_{j=1}^{N} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z)$$

- Interacting Bose-Gas (Lieb-Liniger model)

$$H = \sum_{j=1}^{N} \frac{p_i^2}{2m} + c \sum_{i<j} \delta(x_i - x_j)$$

Observables

$$\langle \mathbf{q} | S_j^z(t) S_{j'}^z(0) | \mathbf{q} \rangle = ? \qquad \langle \mathbf{q} | \rho(x,t) \rho(0,0) | \mathbf{q} \rangle = ?$$

- inelastic neutron scattering
- Bragg spectroscopy
- Interference experiments

Motivation: Bethe Ansatz Solution

The eigenstate of N-quasiparticle (magnons, atoms etc) can be written as

$$|\mathbf{q}\rangle = |q_1, q_2, \dots q_N\rangle = \sum_{\sigma} (-1)^{[\sigma]} A_{\sigma}(\{\mathbf{q}\}) e^{i \sum_{j=1}^{N} q_{\sigma[j]} x_j}$$

Periodic boundary conditions provide quantization

$$\theta_{\mathrm{kin}}(q_j) + \frac{1}{L} \sum_{j \neq i} \theta_{\mathrm{scat}}(q_i - q_j) = \frac{2\pi}{L} n_j$$

Hilbert space is given by the ordered sets of integers

$$\langle \mathbf{q} | \mathcal{O}(x,t) \mathcal{O}(0,0) | \mathbf{q} \rangle = \sum_{\mathbf{k}} |\langle \mathbf{q} | \mathcal{O} | \mathbf{k} \rangle|^2 e^{-itE_{\mathbf{k}} + ixP_{\mathbf{k}}}$$

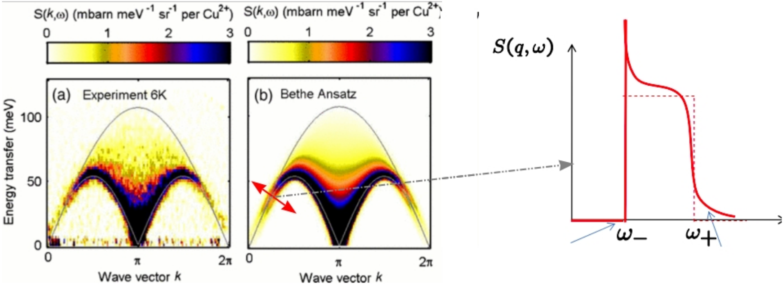
$E_{\mathbf{k}}$, $P_{\mathbf{k}}$ and $\langle \mathbf{q} | \mathcal{O} | \mathbf{k} \rangle$ are known by means of Algebraic Bethe Ansatz

$$|\mathbf{q}\rangle = B(q_1) B(q_2) \dots B(q_N) |0\rangle$$

Motivation: Summation over intermediate states

$$\langle \mathbf{q}|\mathcal{O}(x,t)\mathcal{O}(0,0)|\mathbf{q}\rangle = \sum_{\mathbf{k}} |\langle \mathbf{q}|\mathcal{O}|\mathbf{k}\rangle|^2 e^{-itE_{\mathbf{k}}+ixP_{\mathbf{k}}}$$

- Numerics (ABACUS)
- Field theory ($k_F x \gg 1,\ k_F^2 t \gg 1$)



Left: Comparison between ABACUS and inelastic neutron scattering for $KCuF_3$. [PRL 111 137205].

Right: The threshold singularities in the Non-Linear Luttinger Liquid.

Both approaches are problematic at finite temperature !!!

Motivation: Hardcore Lieb-Liniger

$$H = \sum_{j=1}^{N} \frac{p_i^2}{2m} + c \sum_{i<j} \delta(x_i - x_j) \equiv \int dx \left[\frac{\partial_x \psi^+ \partial_x \psi}{2m} + c \left[\psi^+ \psi\right]^2 \right]$$

$$\rho(x,t) = \langle \psi^+(x,t)\psi(0,0)\rangle = \sum_{\mathbf{k}} |\langle \mathbf{k}|\psi|\mathbf{q}\rangle|^2 e^{-it(E_{\mathbf{k}} - E_{\mathbf{q}}) - ix(P_{\mathbf{k}} - P_{\mathbf{q}})}$$

$$e^{ik_j L} = \prod_{i=1}^{N} \frac{k_j - k_i + ic}{k_j - k_i - ic} \qquad e^{iq_j L} = \prod_{i=1}^{N+1} \frac{q_j - q_i + ic}{q_j - q_i - ic}$$

$$c \to \infty$$

$$k_j = \frac{2\pi}{L}(n_j - 1/2) \qquad q_j = \frac{2\pi}{L} n_j$$

$$\rho(x,0) \sim \det_{[-k_F,k_F]} \left(1 - \frac{2}{\pi}\frac{\sin(x(p-q))}{p-q} + e^{-ix(p+q)}\right) - \det_{[-k_F,k_F]} \left(1 - \frac{2}{\pi}\frac{\sin(x(p-q))}{p-q}\right)$$

$$k_F = \pi N / L$$

Generalized Sine Kernels

$$\tau(x) = \det\left(1 + \frac{e^{2\pi i\nu} - 1}{\pi} n_F(q) \frac{\sin x(q-p)/2}{q-p}\right)$$

$$\tau_{XX} = \det_{[-\pi,\pi]}\left(1 - \frac{\omega_F(q)}{\pi}\frac{\sin\frac{x(p-q)}{2}}{\sin\frac{p-q}{2}} - \frac{\omega_F(q)}{\pi}e^{-\frac{ip(x-1)+iq(x+1)}{2}}\right) - \det\left(1 - \frac{\omega_F(q)}{\pi}\frac{\sin\frac{x(p-q)}{2}}{\sin\frac{p-q}{2}}\right)$$

- Random matrices
- Mobile impurity [SciPost Phys. 8, 053 (2020), New J. Phys. 18 (2016), 045005]
- Return probability from the domain wall initial state $|\mathrm{DW}\rangle = |\uparrow\uparrow\ldots\uparrow\downarrow\downarrow\ldots\downarrow\rangle$ [J.M. Stephan, J. Stat (2017)]

$$\langle\mathrm{DW}|e^{zH_{XXX}}|\mathrm{DW}\rangle = \det{}_{\mathbb{R}_+}\left(1 - e^{-p^2/4}\frac{\sin\sqrt{z}(p-q)}{\pi(p-q)}e^{-q^2/4}\right)$$

- Persistence of spin configurations [I. Dornic (2018)] $n_F(q) = 1/\cosh(q)$
- Classical integrable systems $n_F(q) = r(q)$

Effective numerical evaluation (F. Bornemann *"On the Numerical Evaluation of Fredholm Determinants"* [0804.2543])

$$\int_a^b f(q)dq = \lim_{N\to\infty}\sum_{k=1}^{N}\omega_k f(x_k), \qquad \det(1+\hat{V}) = \lim_{N\to\infty}\det(\delta_{ij}+\sqrt{\omega_i}V(x_i,x_k)\sqrt{\omega_k})\Big|_{1\le i,k\le N}$$

Check Nikita's works

Integral operators with the generalized sine kernel on the real axis

N. A. Slavnov

Theoretical and Mathematical Physics **165**, 1262–1274 (2010) | Cite this article

[K. K. Kozlowski, J. M. Maillet, N. A. Slavnov (J.Stat.Mech.1103:P03019,2011),(J.Stat.Mech.1103:P03018,2011)]

Riemann–Hilbert Approach to a Generalised Sine Kernel and Applications

N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov & V. Terras

Communications in Mathematical Physics **291**, 691–761 (2009) | Cite this article

Sine Kernel ($T = 0$)

$$\tau(x) = \tau(x, t=0) = \det_{[-k_F, k_F]} \left(1 + \frac{e^{2\pi i\nu} - 1}{\pi} \frac{\sin x(q-p)/2}{q-p} \right)$$

Form-factor presentation

$$\tau(x,t) = \langle \mathbf{q} | \mathcal{O}(x,t)\mathcal{O}(0,0) | \mathbf{q} \rangle = \sum_{k_1 < k_2 \cdots < k_N} |\langle \mathbf{q} | \mathcal{O} | \mathbf{k} \rangle|^2 e^{-itE_{\mathbf{k}} + iP_{\mathbf{k}} x}$$

$|\mathbf{q}\rangle$: free fermions: $q_i = \frac{2\pi n_i}{L}$; $|\mathbf{k}\rangle$: **shifted** free fermions: $k_i = \frac{2\pi(n_i - \nu)}{L}$

$$P_{\mathbf{k}} = \sum_i k_i, \qquad E_{\mathbf{k}} = \sum_i k_i^2/2 \sim L \int k^2 \rho(k) dk$$

The form-factor (overlap):

$$|\langle \mathbf{q} | \mathcal{O} | \mathbf{k} \rangle|^2 = \left(\frac{2}{L} \sin \pi\nu \right)^{2N} \left(\det_{N \times N} \frac{1}{k_i - q_j} \right)^2 \to |\langle \mathbf{q} | \mathbf{k} \rangle|^2 .$$

$$\tau = \sum_{\mathbf{k}} |\langle \mathbf{q} | \mathbf{k} \rangle|^2 e^{-itE_{\mathbf{k}} + iP_{\mathbf{k}} x} = \det \left(\frac{2}{L} \oint \frac{dk}{\cot \frac{kL}{2} + \cot \pi\nu} \frac{e^{-itk^2/2 + ikx}}{(k - q_i)(k - q_j)} \right) = \det(1 + \hat{V})$$

Field theory treatment = Microscopic bosonization ($T = 0$)

- Form-Factor summation

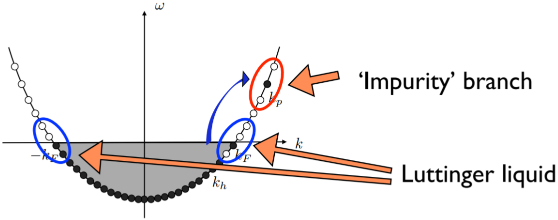
$$\tau(x,t) = \sum_{\mathbf{k}} |\langle \mathbf{q}|\mathcal{O}|\mathbf{k}\rangle|^2 e^{-ixP_{\mathbf{k}}+itE_{\mathbf{k}}} = \det(1+\hat{V})$$

- Orthogonality Catastrophe: $|\langle \mathbf{q}|\mathcal{O}|\mathbf{k}_{\mathrm{vac}}\rangle|^2 = \mathcal{A}/N^{2\alpha}$
- Soft-mode summation

$$\tau(x,t) \sim \sum_{\mathrm{IR}} |\langle \mathbf{q}|\mathcal{O}|\mathbf{k}\rangle|^2 e^{-ixP_{\mathbf{k}}+itE_{\mathbf{k}}} = \langle e^{\sqrt{\alpha}\varphi(x,t)} e^{-\sqrt{\alpha}\varphi(0,0)}\rangle = \frac{\mathcal{A}e^{-it\Delta E+ix\Delta P}}{(x-k_F t)^{\alpha}(x+k_F t)^{\alpha}}$$

- Nonlinear contributions

$$\tau(x,t) \sim \sum_{Q+\mathrm{IR}} |\langle \mathbf{q}|\mathcal{O}|\mathbf{k}\rangle|^2 e^{-ixP_{\mathbf{k}}+itE_{\mathbf{k}}+ix(Q-k_F)+it(Q^2-k_F^2)/2} = \frac{\mathcal{B}e^{-it\Delta E+ix\Delta P+ix^2/4t}}{\sqrt{t}(x-k_F t)^{\tilde{\alpha}}(x+k_F t)^{\alpha}}$$



Slavnov (1989); Slavnov and Korepin (1991); A. Shashi, L. I. Glazman, J.-S. Caux, and A. Imambekov (2011); N. Kitanine, K.K. Kozlowski, J.-M. Maillet, N.A. Slavnov, and V. Terras (2009-2012); K.K. Kozlowski, J.-M. Maillet (2015);

Combinatorics of orthogonality catastrophe

Generic overlap

$$|\langle \mathbf{k}_{\rm vac}|\mathbf{q}\rangle|^2 = \left(\frac{2}{L}\sin\pi\nu\right)^{2N}\left(\det_{N\times N}\frac{1}{k_i-q_j}\right)^2 = \left(\frac{2}{L}\sin\pi\nu\right)^{2N}\frac{\prod_{i>j}(k_i-k_j)^2\prod_{i>j}(q_i-q_j)^2}{\prod_{i,j}(k_i-q_j)^2}.$$

Fermi sea integers

$$k_j=\frac{2\pi}{L}(n_j-\nu),\qquad q_j=\frac{2\pi}{L}n_j,\qquad n_j=-\frac{N-1}{2}+j-1,\ \ j=1,2\ldots N$$

$$|\langle \mathbf{k}_{\rm vac}|\mathbf{q}\rangle|^2=\left(\frac{\sin\pi\nu}{\pi\nu}\right)^{2N}\prod_{i\neq j}\left(1-\frac{\nu}{i-j}\right)^{-2}=\frac{G^2(1-\nu)G^2(1+\nu)G^4(N+1)}{G^2(N-\nu+1)G^2(N+\nu+1)}.$$

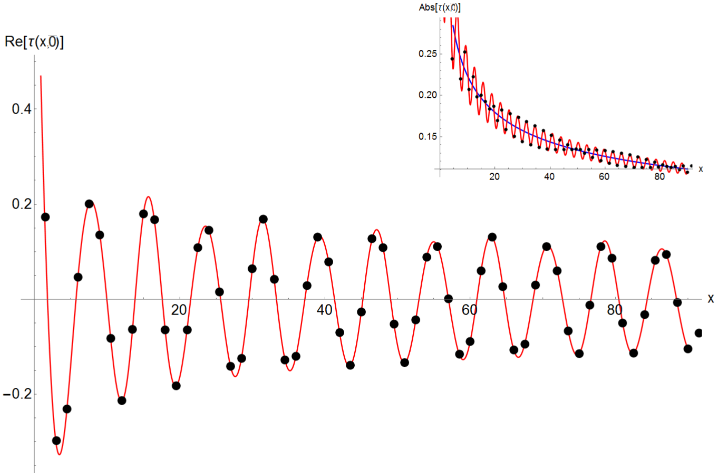
$$\boxed{|\langle \mathbf{k}_{\rm vac}|\mathbf{q}\rangle|^2=\frac{G^2(1-\nu)G^2(1+\nu)}{N^{2\nu^2}}}$$

For $\nu=\nu(k)$, $(\nu_\pm=\nu(\pm k_F)$, $\qquad k_F=\pi L/N)$

$$\boxed{|\langle \mathbf{k}_{\rm vac}|\mathbf{q}\rangle|^2=\frac{G^2(1-\nu_-)G^2(1+\nu_+)(2\pi)^{\nu_--\nu_+}}{N^{\nu_-^2+\nu_+^2}}\exp\left(\int_{[-k_F,k_F]^2}\left(\frac{\nu(\lambda)-\nu(\mu)}{\lambda-\mu}\right)^2 d\lambda d\mu\right)}$$

Static + zero temperature

$$\tau(x) = \frac{G^2(1-\nu)G^2(1+\nu)}{(-2ix)^{\nu^2}(2ix)^{\nu^2}} e^{-2i\nu x} + (\nu \to \nu + \mathbb{Z})$$

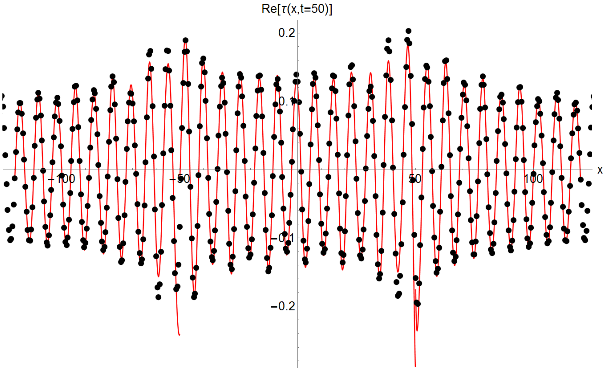


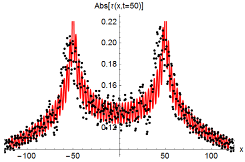
Dynamics $T = 0$

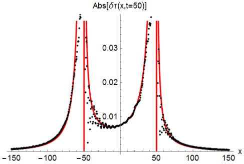
$$\tau(x,t) = \frac{G^2(1-\nu)G^2(1+\nu)}{(2i(t-x))^{\nu^2}(2i(x+t))^{\nu^2}} e^{-2i\nu x} +$$

$$\frac{G^2(1-\nu)G^2(\nu)}{\nu^2(2i(t-x))^{(1+\nu)^2}(2i(x+t))^{\nu^2}} \left(\frac{x-t}{x+t}\right)^{2\nu} \frac{e^{-i(t-x)^2/(2t)-2i\nu x}}{(x/t-1)^2} \sqrt{\frac{2\pi}{-it}}\theta(x^2 > t^2) +$$

$$+ \frac{G^2(-\nu)G^2(1+\nu)}{\nu^2(2i(t-x))^{(1-\nu)^2}(2i(x+t))^{\nu^2}} \left(\frac{x+t}{x-t}\right)^{2\nu} \frac{e^{i(t-x)^2/(2t)-2i\nu x}}{(x/t-1)^2} \sqrt{\frac{2\pi}{it}}\theta(x^2 < t^2) + (\nu \to \nu + \mathbb{Z})$$







Finite temperature (Static)

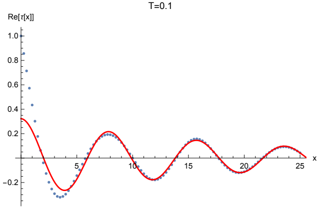
$$\tau(x) = \det\left(1 + \frac{n_F(q)}{\pi}(e^{2\pi i\nu} - 1)\frac{\sin(x(p-q))}{p-q}\right)$$

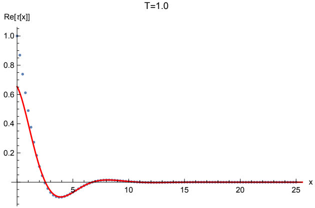
- It is challenging to do microscopic
  - Overlaps are to small $\sim e^{-cN}$
  - Too many soft modes $\sim e^{cN}$
- Heuristic approach instead!
  Dressing: inhomogeneous and complex valued!!!

$$\frac{n_F(q)}{\pi}(e^{2\pi i\nu}-1) = \frac{e^{2\pi i\nu_T(q)} - 1}{\pi}, \qquad \nu \to \nu_T(q) = \frac{1}{2\pi i}\log(1+(e^{2\pi i\nu}-1)n_F(q))$$

- Effectively **ONE** form factor

$$\tau(x) \approx \exp\left(-ix\int_{-\infty}^{\infty}\nu_T(q)dq - \frac{1}{2}\int_{\mathbb{R}^2}\left(\frac{\nu_T(q)-\nu_T(q')}{q'-q}\right)dqdq'\right)$$

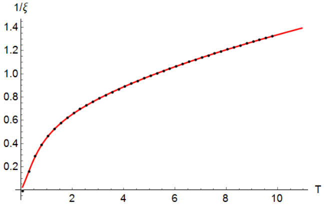




$$\mathrm{Tr}(e^{-\beta H}\mathcal{O}(x,t)\dots)/\mathrm{Tr}e^{-\beta \mathrm{H}} = \langle \mathcal{O}(z=x+it)\dots\rangle_{\mathbb{S}^1\times\mathbb{R}^1} \overset{z\to z'=e^{2\pi z/\beta}}{=} \sim \langle\mathcal{O}(z')\dots\rangle_{\mathbb{R}^2}$$

CFT prediction for correlation length:

$$\tau(x)\Big|_{T=0} = \frac{\mathcal{A}}{x^{\nu^2}} \Longrightarrow \qquad \tau(x) = \frac{\mathcal{A}}{(\sinh(xT)/T)^{\nu^2}} \sim e^{-x/\xi} \Longrightarrow 1/\xi \sim T???$$



XY spins chain

$$\mathbf{H}_{XY} = -\frac{1}{2}\sum_{j=1}^{L}\left[\frac{1+\gamma}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-\gamma}{2}\sigma_j^y\sigma_{j+1}^y + h\sigma_j^z\right]$$

Spectrum of fermionic (Majorana) excitations

$$E(q) = \sqrt{(h-\cos q)^2 + \gamma^2\sin^2 q}$$

Bogolyubov rotation angle

$$e^{i\theta(q)} = \frac{h - \cos q - i\gamma\sin q}{\sqrt{(h-\cos q)^2 + \gamma^2\sin^2 q}}$$

Finite temperature spin-spin correlation function

$$\tau(x) = \tau(x) \equiv \frac{\mathrm{Tr}\sigma_{x+1}^x\sigma_1^x e^{-\beta\mathbf{H}_{XY}}}{\mathrm{Tr}e^{-\beta\mathbf{H}_{XY}}} = \det_{[-\pi,\pi]}\left(1 + \hat{V} + \hat{W}\right) - \det_{[-\pi,\pi]}\left(1+\hat{V}\right)$$

XY model [A.G. Izergin, V.S. Kapitonov, N.A. Kitanine, solv-int/9710028]

$$V(p,q) = -\frac{\omega_F(q)}{\pi}\frac{\sin\frac{x(p-q)}{2}}{\sin\frac{p-q}{2}}, \qquad W(p,q) = -\frac{\omega_F(q)}{\pi}e^{-i(p+q)x/2}e^{-\frac{i(p-q)}{2}}$$

$$\omega_F(q) = \frac{1}{2}\left(1 - e^{i\theta(q)}\tanh\frac{\beta E(q)}{2}\right)$$

Form-factors

$$\tilde{\tau}(x) = \sum_{\mathbf{q}} |\langle \mathbf{k}|\mathbf{q}\rangle|^2 e^{-ix\left(\sum_{i=1}^{N+1} k_i - \sum_{i=1}^{N} q_i\right)}$$

with

$$e^{ikL} = e^{-2\pi i\nu(k)}, \qquad e^{iqL} = 1.$$

$$|\langle \mathbf{k}|\mathbf{q}\rangle|^2 = A\left(\prod_{i=1}^{N+1} \frac{\sin \pi\nu(k_i)}{L}\right)^2 \frac{\prod_{i>j}^{N+1} \sin^2 \frac{k_i - k_j}{2} \prod_{i>j}^{N} \sin^2 \frac{q_j - q_i}{2}}{\prod_{i=1}^{N+1}\prod_{j=1}^{N} \sin^2 \frac{k_i - q_j}{2}}$$

$$\boxed{\tilde{\tau}(x) = \det_{[-\pi,\pi]}\left(1 + \hat{V} + \hat{W}\right) - \det_{[-\pi,\pi]}\left(1 + \hat{V}\right)}$$

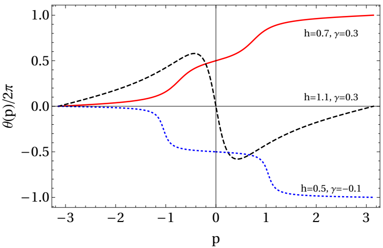
$$V(p,q) = -\frac{e^{2\pi i\nu(q)} - 1}{\pi} \frac{\sin \frac{x(p-q)}{2}}{\sin \frac{p-q}{2}} + O(e^{-\#x}), \qquad W(p,q) = -\frac{e^{2\pi i\nu(q)} - 1}{\pi} e^{\frac{-i(p+q)x}{2}} e^{\frac{-i(p-q)}{2}}$$

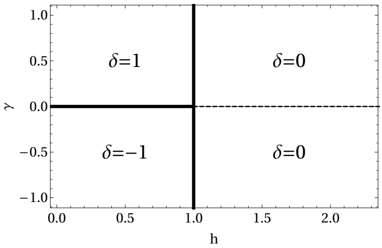
$$\boxed{e^{2\pi i\nu(k)} = 1 - 2\omega_F(k) = e^{i\theta(k)} \tanh \frac{\beta E(k)}{2}}$$

$$e^{2\pi i\nu(k)} = 1 - 2\omega_F(k) = e^{i\theta(k)} \tanh\frac{\beta E(k)}{2}$$

$$\nu(\pi) - \nu(-\pi) = \delta \in \mathbb{Z}$$

$$e^{ikL} = e^{-2\pi i\nu(k)}, \qquad e^{iqL} = 1.$$





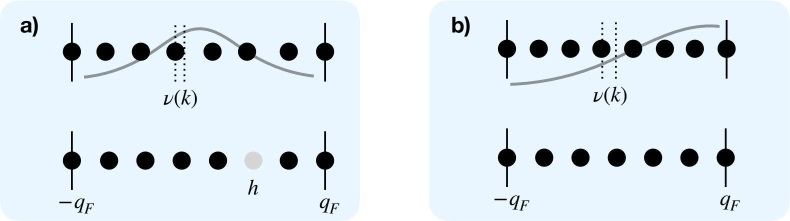
Winding of the effective phase

Total number of solutions

$$e^{iqL} = 1, \qquad q_j = \frac{2\pi}{L}\left(-\frac{L+1}{2} + j\right), \qquad j = 1, 2, \ldots L$$

$$e^{ikL} = e^{-2\pi i \nu(k)}, \qquad k_j \approx \frac{2\pi}{L}\left(-\frac{L+1}{2} + j - \nu_j\right), \qquad j = 1, 2, \ldots L{+}\delta$$

$$\nu(k) \to \nu_\delta(k) = \nu(k) - \delta\frac{k+\pi}{2\pi} \Longrightarrow e^{ik(L+\delta)} = (-1)^\delta e^{-2\pi i \nu_\delta(k)}$$



For $\delta = 1$ there is only one! form-factor

$\delta < 0$

$$\mathbf{q}^{a_1,\dots a_n} = \{q_1, \dots \hat{q}_{a_1}, \dots \hat{q}_{a_n}, \dots q_L\} \qquad \delta = 1 - n$$

$$\Delta P_{a_1,\dots a_n} = \sum_{i=1}^{L-n+1} k_i - \sum_{i=1}^{L} q_i + \sum_{i=1}^{n} q_{a_i} \approx \delta\pi - \int_{-\pi}^{\pi} \nu(q)dq + \sum_{i=1}^{n} q_{a_i}.$$

$$e^{-ix\Delta P_{a_1,\dots a_n}} |\langle \mathbf{k}|\mathbf{q}^{a_1,\dots a_n}\rangle|^2 = \mathcal{A}_\delta[\nu] \prod_{i>j}^{n} \left(2\sin\frac{q_{a_i} - q_{a_j}}{2}\right)^2 \prod_{i=1}^{n} \mathcal{Y}_{a_i},$$

$$\tau(x) = \det_{1\le j,k\le n} [Y_\delta(x+j-k)] \exp\left(ix\int_{-\pi}^{\pi}\nu_\delta(q)dq - \frac{1}{2}\int_{-\pi}^{\pi}dq\int_{-\pi}^{\pi}dk\left[\frac{\nu_\delta(q)-\nu_\delta(k)}{2\sin\frac{q-k}{2}}\right]^2\right),$$

where $\nu_\delta(q) \equiv \nu(q) - \delta(q+\pi)/(2\pi)$ has zero winding number and $Y_\delta(x)$ stands for

$$Y_\delta(x) = \int_{-\pi}^{\pi}\frac{dq}{2\pi}\left(e^{-2\pi i\nu(q)} - 1\right)\exp\left(-i(x-\delta)q + i\delta\pi - \int_{-\pi}^{\pi}dk\nu_\delta(k)\cot\frac{q-k+i0}{2}\right).$$

$$e^{2\pi i\nu(k)} = 1 - 2\omega_F(k) = e^{i\theta(k)}\tanh\frac{\beta E(k)}{2} \qquad \tau(x) = \mathcal{A}(T,h,\gamma)e^{-x/\xi(T,h,\gamma)}$$

Ferromagnetic $h \leq 1$ $(\delta = 1)$

$$\log\mathcal{A} = -\frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk \left[\frac{\nu(q)-\nu(k)-(q-k)/2\pi}{2\sin\frac{q-k}{2}}\right]^2$$

$$\xi^{-1} = i\pi - i\int_{-\pi}^{\pi}\nu(q)dq = -\frac{1}{2\pi}\int_{-\pi}^{\pi} dk \log\tanh\frac{\beta E(k)}{2}$$

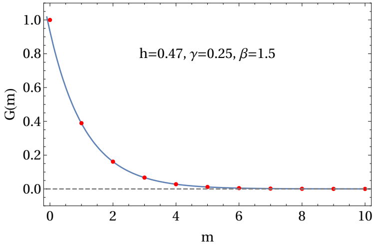
Paramagnetic $h > 1$ $(\delta = 0)$

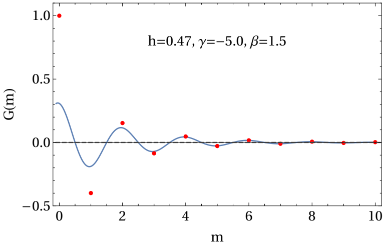
$$\xi^{-1} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} dk \log\tanh\frac{\beta E(k)}{2} + \log y_+, \qquad y_\pm = \frac{h+\sqrt{h^2+\gamma^2-1}}{1\pm\gamma}$$

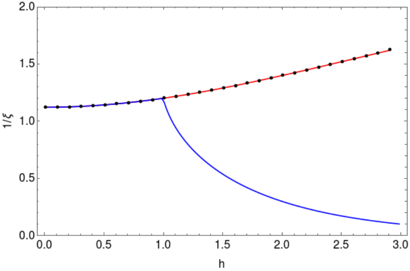
$$\log\mathcal{A} = \log\frac{2}{\beta\sqrt{h^2+\gamma^2-1}} - i\int_{-\pi}^{\pi} dq\,\nu(q)\frac{e^{iq}+y_+}{e^{iq}-y_+} - \frac{1}{2}\int_{-\pi}^{\pi} dq\int_{-\pi}^{\pi} dk\left(\frac{\nu(q)-\nu(k)}{2\sin\frac{q-k}{2}}\right)^2$$

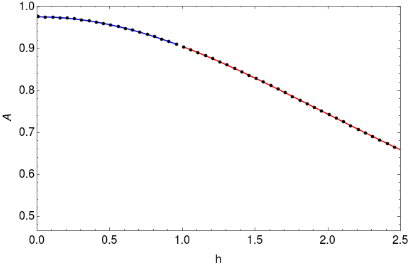
In agreement with E. Barouch and B. M. McCoy Phys. Rev. A **3**, 786 (1971)
Bonus: from Fredholm to Toeplitz and back

Correlation functions









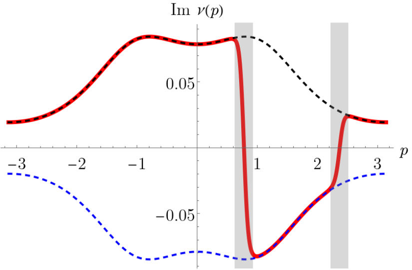
Dynamics

$$\tau(x,t) = \det\left(\mathbb{1} + \rho(q)\frac{e_+(p)e_-(q) - e_-(p)e_+(q)}{p-q}\right)$$

$$e_-(q) = e^{\frac{i}{2}xq - \frac{i}{4}tq^2}, \qquad e_+(q) = -\frac{\sin^2(\pi\nu)e^{-\frac{i}{2}xq + \frac{i}{4}tq^2}}{\pi}\left[\cot(\pi\nu) + i\mathrm{Erf}\left(\frac{(i+1)(x-qt)}{2\sqrt{t}}\right)\right]$$

$$\mathrm{Erf}\left(\frac{(x-qt)(1+i)}{2\sqrt{t}}\right) \to \mathrm{Sign}(x/t - q)$$

$$\nu_T(q) = \frac{1}{2\pi i}\log\left[1 + (e^{2\pi i\nu} - 1)\rho(q)\right]\theta_\epsilon(x/t - q) - \frac{1}{2\pi i}\log\left[1 + (e^{-2\pi i\nu} - 1)\rho(q)\right]\theta_\epsilon(q - x/t).$$



$$\theta_\epsilon(z) = \frac{1 + S(z/\epsilon)}{2}, \qquad \epsilon \sim \frac{1}{\sqrt{t}}$$

$$\log\tau \approx C_0 - \frac{1}{2}\Delta^2 \log t + i\int dk\, \nu_T(k)(x - tk)$$

$$\Delta = \frac{i}{2\pi}\log\left(1 + 2\Big(\cos(2\pi\nu) - 1\Big)\Big(\rho - \rho^2\Big)\right)$$

$$\rho = \rho(x/t)$$

Summary and outlook

- Riemann Hilbert Problem for Fredholm determinant
- Phase shift dressing
- Different types of soft mode contributions
- Universality
- Relation with QTM? Thermal form-factors?
- Asymptotic for classical integrable models?

Extra slides

Bonus: from Fredholm to Toeplitz and back

$$\det\left(1+\hat{S}_\nu\right) = \det_{0\le n,m\le x-1} c_{n-m}, \qquad c_k = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{2\pi i\nu(q)} e^{-ikq}$$

$$\hat{S}_\nu(p,q) = \frac{e^{2\pi i\nu(p)}-1}{2\pi} \frac{\sin\frac{x(p-q)}{2}}{\sin\frac{p-q}{2}} \sim \frac{e^{2\pi i\nu(p)}-1}{2\pi} \sum_{n=0}^{x-1} e^{in(q-p)} \sim \sum_n \mathcal{A}_{qn}\mathcal{B}_{np}$$

$$\det(1+\mathcal{A}\mathcal{B}) = \det(1+\mathcal{B}\mathcal{A})$$

$$\det\left(1+\hat{S}_\nu+\delta\hat{V}_\nu\right) - \det\left(1+\hat{S}_\nu\right) = \det_{0\le n,m\le x-1} \tilde{c}_{n-m}$$

$$\tilde{c}_k = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{2\pi i\nu_1(q)} e^{-ikq}, \qquad \nu_1(q) = \nu(q) - \frac{q+\pi}{2\pi}$$

$$\det\left(1+\hat{S}_\nu+\delta\hat{V}_\nu\right) - \det\left(1+\hat{S}_\nu\right) = \det\left(1+\hat{S}_{\nu_1}\right)$$

Szegő theorem for Toeplitz determinant [Szegő (1915), Fisher & Hartwig (1969)]

$$\log \det_{0\le i,j\le x-1} c_{i-j} = xk_0 + \sum_{n=1}^{\infty} nk_n k_{-n}$$

$$\nu_\delta(q) = \frac{-1}{2\pi i}\sum_{n=-\infty}^{\infty} k_n e^{iqn} \qquad -\frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp \left[\frac{\nu_\delta(q)-\nu_\delta(p)}{2\sin\frac{q-p}{2}}\right]^2 = \sum_{n=1}^{\infty} nk_n k_{-n},$$

Eytan Barouch and Barry M. McCoy Phys. Rev. A **3**, 786 (1971)

$$\mathcal{A} = XY,$$

where:

$$X = \prod_{l=1}^{\infty} \frac{\left(1-\lambda_1^{-1} f_{2l-1}\right)\left(1-\lambda_1^{-1} g_{2l-1}\right)\left(1-\lambda_2^{-1} f_{2l-1}\right)\left(1-\lambda_2^{-1} g_{2l-1}\right)}{\left(1-\lambda_1^{-1} f_{2l}\right)\left(1-\lambda_1^{-1} g_{2l}\right)\left(1-\lambda_2^{-1} f_{2l}\right)\left(1-\lambda_2^{-1} g_{2l}\right)}$$

$$Y = \prod_{i,j=1}^{\infty} \frac{(1-f_{2j}f_{2i-1})(1-f_{2i}f_{2j-1})(1-g_{2j}g_{2i-1})(1-g_{2i}g_{2j-1})}{(1-f_{2j}f_{2i})(1-f_{2j-1}f_{2i-1})(1-g_{2j}g_{2i})(1-g_{2j-1}g_{2i-1})} \times$$
$$\times \frac{(1-f_{2j}g_{2i-1})(1-f_{2i}g_{2j-1})(1-g_{2j}f_{2i-1})(1-g_{2i}f_{2j-1})}{(1-f_{2j}g_{2i})(1-g_{2j}f_{2i})(1-g_{2j-1}f_{2i-1})(1-f_{2j-1}g_{2i-1})}$$

and $\lambda_1, \lambda_2, f, g$ are defined as

$$\lambda_1 = \left\{h + \left[h^2 - \left(1-\gamma^2\right)\right]^{1/2}\right\}/(1-\gamma), \qquad \lambda_2 = \left\{h - \left[h^2 - \left(1-\gamma^2\right)\right]^{1/2}\right\}/(1-\gamma)$$

$$f_k = \frac{h+W_k}{1-\gamma^2} - \left[\left(\frac{h+W_k}{1-\gamma^2}\right)^2 - 1\right]^{1/2}, \qquad g_k = \frac{h-W_k}{1-\gamma^2} - \left[\left(\frac{h-W_k}{1-\gamma^2}\right)^2 - 1\right]^{1/2}$$

with

$$W_k = \left\{\gamma^2 h^2 - \left(1-\gamma^2\right)\left[\gamma^2 + (k\pi)^2\beta^{-2}\right]\right\}^{1/2}$$

Field theory treatment = Microscopic bosonization ($T = 0$)

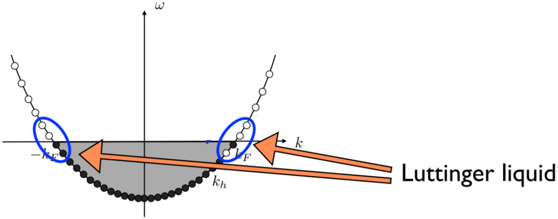
- Form-Factor summation

$$\tau(x,t) = \sum_{\mathbf{k}} |\langle \mathbf{q}|\mathcal{O}|\mathbf{k}\rangle|^2 e^{-ixP_{\mathbf{k}}} = \det(1+\hat{V})$$

- Orthogonality Catastrophe: $|\langle \mathbf{q}|\mathcal{O}|\mathbf{k}_{\mathrm{vac}}\rangle|^2 = \mathcal{A}/N^{2\alpha}$
- Soft-mode summation

$$\tau(x) \sim \sum_{\mathrm{IR}} |\langle \mathbf{q}|\mathcal{O}|\mathbf{k}\rangle|^2 e^{-ixP_{\mathbf{k}}} = \sum_{k=0}^{\infty} \sum_{\substack{p_1,\dots p_k \\ h_1,\dots h_k}} \langle \Omega | e^{\sqrt{\alpha}\varphi(x,t)} | \{p,h\}\rangle \langle \{p,h\} | e^{-\sqrt{\alpha}\varphi(0,0)} | \Omega \rangle$$

$$= \mathcal{A} e^{-i(P_\Omega - P_{\mathrm{vac}})x} \langle e^{\sqrt{\alpha}\varphi(x)} e^{-\sqrt{\alpha}\varphi(0)} \rangle = \frac{\mathcal{A}}{x^{2\alpha}} e^{-i(P_\Omega - P_{\mathrm{vac}})x}$$



Slavnov (1989); Slavnov and Korepin (1991); A. Shashi, L. I. Glazman, J.-S. Caux, and A. Imambekov (2011); N. Kitanine, K.K. Kozlowski, J.-M. Maillet, N.A. Slavnov, and V. Terras (2009-2012); K.K. Kozlowski, J.-M. Maillet (2015);

Soft mode summation

$$\frac{|\langle P_k, H_k|c_p|\mathbf{k}\rangle_Q|^2}{|\langle \mathrm{vac}|c_p|\mathbf{k}\rangle_Q|^2} = \left(\det_{1\le i,j\le k} \frac{1}{p_i + h_j - 1}\right)^2 \left(\frac{\sin \pi F_+}{\pi}\right)^{2k} \prod_{j=1}^{k} \frac{\Gamma(p_j - F_+)^2}{\Gamma(p_j)^2} \prod_{j=1}^{k} \frac{\Gamma(h_j + F_+)^2}{\Gamma(h_j)^2}$$

$$P_{\mathbf{q}} - Q = \int_{-1}^{1} dq F(q) + \frac{2\pi}{L} \sum_{j=1}^{k} (p_j + h_j - 1)$$

Auxiliary Fermions

$$\{\psi_n, \psi_m^+\} = \delta_{nm}, \qquad \{\psi_n, \psi_m\} = \{\psi_n^+, \psi_m^+\} = 0.$$

$$\psi_n|0\rangle = 0, \quad \text{if } \ n > 0, \qquad \psi_n^+|0\rangle = 0, \quad \text{if } \ n \le 0$$

$$|P_k, H_k\rangle = \psi_{p_1}^+ \dots \psi_{p_k}^+ \psi_{1-q_1} \dots \psi_{1-q_k}|0\rangle$$

Bosonization

$$\psi^+(z) = \sum_{n\in\mathbb{Z}} z^n \psi_n^+, \qquad \psi(z) = \sum_{n\in\mathbb{Z}} z^{-n}\psi_n.$$

$$J(z) =: \psi^+(z)\psi(z) := \sum_{k\in\mathbb{Z}} \frac{\sum_{j\in\mathbb{Z}} :\psi_j^+\psi_{j+k}:}{z^k} \equiv \sum_{k\in\mathbb{Z}} \frac{J_k}{z^k},$$

$$z\frac{\partial\varphi(z)}{\partial z} = J(z), \qquad \varphi(z) = \varphi_-(z) - \varphi_+(z) + J_0 \log z + N_0$$

$$\varphi_+(z) = \sum_{k>0} \frac{J_k}{kz^k}, \qquad \varphi_-(z) = \sum_{k>0} z^k \frac{J_{-k}}{k}$$

$$e^{iy(P_{\mathbf{q}}-Q)} \frac{|\langle P_k, H_k|c_p|\mathbf{k}\rangle_Q|^2}{|\langle \mathrm{vac}|c_p|\mathbf{k}\rangle_Q|^2} = e^{iy\Delta P}\langle 0|e^{F_+\varphi_+(1)}|P_k, H_k\rangle\langle P_k, H_k|e^{F_+\varphi_-(z)}|0\rangle$$

$$\Delta P = -\int_{-1}^{1} dqF(q), \qquad z = e^{2\pi iy/L}$$

Bosonization

$$\sum_{P_k,H_k} e^{iy(P_{\mathbf{q}}-Q)} \frac{|\langle P_k,H_k|c_p|\mathbf{k}\rangle_Q|^2}{|\langle \mathrm{vac}|c_p|\mathbf{k}\rangle_Q|^2} = e^{iy\Delta P} \sum_{P_k,H_k} \langle 0|e^{F_+\varphi_+(1)}|P_k,H_k\rangle\langle P_k,H_k|e^{F_+\varphi_-(z)}|0\rangle =$$

$$= e^{iy\Delta P}\langle 0|e^{F_+\varphi_+(1)}e^{F_+\varphi_-(z)}|0\rangle = e^{iy\Delta P}\langle 0|e^{F_+^2[\varphi_+(1),\varphi_-(z)]}|0\rangle = \frac{e^{iy\Delta P}}{(1-z)^{F_+^2}}$$

$$\sum_{P_k,H_k} e^{iy(P_{\mathbf{q}}-Q)}|\langle P_k,H_k|c_p|\mathbf{k}\rangle_Q|^2 = |\langle \mathrm{vac}|c_p|\mathbf{k}\rangle_Q|^2 \frac{e^{iy\Delta P}}{(1-z)^{F_+^2}} = \frac{\mathcal{A}}{N^{F_+^2}} \frac{e^{iy\Delta P}}{(1-e^{2\pi i/L})^{F_+^2}}$$

$$\boxed{\sum_{P_k,H_k} e^{iy(P_{\mathbf{q}}-Q)}|\langle P_k,H_k|c_p|\mathbf{k}\rangle_Q|^2 = \frac{\mathcal{A}e^{iy\Delta P}}{(-2k_F iy)^{F_+^2}}}$$

- Any soft modes excitations
- Non-linear bosonization

From Fredholm to Toeplitz1

$$\frac{\sin\frac{x(p-q)}{2}}{\sin\frac{p-q}{2}} \sim \sum_{n=0}^{x-1} e^{in(q-p)} \to \sum_{n=0}^{x-1} a_n(p) e^{in(q-p)}.$$

$$\hat{S}_\nu(p,q) = \frac{e^{2\pi i\nu(p)}-1}{2\pi}\frac{\sin\frac{x(p-q)}{2}}{\sin\frac{p-q}{2}} \sim \sum_n \mathcal{A}_{qn}\mathcal{B}_{np}$$

$$\mathcal{A}_{qn} = e^{iqn}, \qquad \mathcal{B}_{np} = \frac{e^{2\pi i\nu(p)}-1}{2\pi} a_n(p) e^{-inp}.$$

$$\det(1+\mathcal{AB}) = \det(1+\mathcal{BA})$$

$$\det\left(1+\widehat{S^a}\right) = \det_{0\le n,m\le x-1}\left(\delta_{nm}+T_{nm}\right), \qquad T_{nm} = \int_{-\pi}^{\pi}\frac{dq}{2\pi} a_n(q)(e^{2\pi i\nu(q)}-1)e^{-i(n-m)q}.$$

For $a_n = 1$ the matrix T_{nm} transforms into the Toeplitz one

$$\det\left(1+\widehat{S^a}\right) = \det_{0\le n,m\le x-1} c_{n-m}, \qquad c_k = \int_{-\pi}^{\pi}\frac{dq}{2\pi} e^{2\pi i\nu(q)} e^{-ikq}.$$

From Fredholm to Toeplitz2

$$\det\left(1+\hat{S}_\nu+\delta\hat{V}_\nu\right)-\det\left(1+\hat{S}_\nu\right)=\frac{\partial}{\partial\alpha}\det\left(1+\hat{S}_\nu+\alpha\delta\hat{V}_\nu\right)\Big|_{\alpha=0}.$$

$$\delta V_\nu(p,q)=-\frac{e^{2\pi i\nu(p)}-1}{2\pi}e^{-i(x+1)p/2}e^{-i(x-1)q/2}$$

We choose $a_0(q)=1-\alpha e^{-ixq}$ and $a_n(q)=1$ for $n\geq 1$

$$\det\left(1+\hat{S}_\nu+\alpha\delta\hat{V}_\nu\right)=\det\left(1+\widehat{S^a}\right)=\det\begin{pmatrix} c_0-\alpha c_x & c_{-1}-\alpha c_{x-1} & \dots & c_{-x+1}-\alpha c_1 \\ c_1 & c_0 & \dots & c_{-x+2} \\ \cdot & \cdot & & \cdot \\ & & \ddots & \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{x-1} & c_{x-2} & \dots & c_0 \end{pmatrix}.$$

$$\frac{\partial}{\partial\alpha}\det\left(1+\hat{S}_\nu+\alpha\delta\hat{V}_\nu\right)\Big|_{\alpha=0}=(-1)^x\det\begin{pmatrix} c_1 & c_0 & \dots & c_{-x+2} \\ c_2 & c_1 & \dots & c_{-x+3} \\ \cdot & \cdot & & \cdot \\ & & \ddots & \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_x & c_{x-1} & \dots & c_1 \end{pmatrix}=\det_{0\leq n,m\leq x-1}\tilde{c}_{n-m},$$

From Fredholm to Toeplitz3

$$\det\left(1+\hat{S}_\nu+\delta\hat{V}_\nu\right)-\det\left(1+\hat{S}_\nu\right)=\det_{0\le n,m\le x-1}\tilde{c}_{n-m}$$

$$\tilde{c}_k=-\int_{-\pi}^{\pi}\frac{dq}{2\pi}e^{2\pi i\nu(q)}e^{-i(k+1)q}=\int_{-\pi}^{\pi}\frac{dq}{2\pi}e^{2\pi i\nu_1(q)}e^{-ikq}.$$

$$\nu_1(q)=\nu(q)-\frac{q+\pi}{2\pi}$$

$$\boxed{\det\left(1+\hat{S}_\nu+\delta\hat{V}_\nu\right)-\det\left(1+\hat{S}_\nu\right)=\det\left(1+\hat{S}_{\nu_1}\right)}$$

Szegő theorem for Toeplitz determinant [Szegő (1915), Fisher & Hartwig (1969)]

$$\log\det_{0\le i,j\le x-1}c_{i-j}=xk_0+\sum_{n=1}^{\infty}nk_nk_{-n}$$

$$\nu_\delta(q)=\frac{-1}{2\pi i}\sum_{n=-\infty}^{\infty}k_ne^{iqn}$$

$$-\frac{1}{2}\int_{-\pi}^{\pi}dq\int_{-\pi}^{\pi}dp\left[\frac{\nu_\delta(q)-\nu_\delta(p)}{2\sin\frac{q-p}{2}}\right]^2=\sum_{n=1}^{\infty}nk_nk_{-n},$$

Fredholm determinants

$$\tau(x) = \det\left(1 + \hat{V}\right)$$

- $$\hat{V} f(q) = \int_\gamma dq' \, V(q, q') f(q')$$
- Infinite dimensional determinant
$$\det(1 + \hat{V}) = \lim_{N\to\infty} \det\left(\delta_{ij} + \frac{1}{N} V(x_i, x_k)\right)\Big|_{1\le i,k\le N}$$
- Effective numerical evaluation (F. Bornemann *"On the Numerical Evaluation of Fredholm Determinants"* [0804.2543])

$$\int_a^b f(q)dq = \lim_{N\to\infty} \sum_{k=1}^{N} \omega_k f(x_k), \qquad \det(1+\hat{V}) = \lim_{N\to\infty} \det(\delta_{ij} + \sqrt{\omega_i} V(x_i, x_k) \sqrt{\omega_k})\Big|_{1\le i,k\le N}$$