Absence of string excitations in the low-T spectrum of the quantum transfer matrix of the XXZ chain

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Recent **A**dvances in **Q**uantum **I**ntegrable **S**ystems
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Content of the talk

1. The Hamiltonian of the XXZ-chain
2. The dressed energy in the complex plane
3. Types of solutions for excited states of the quantum transfer matrix
4. Summary and Outlook

The Hamiltonian of the XXZ-chain

Hamiltonian of the spin 1/2 XXZ-chain with magnetic field

$$H_{XXZ} = J \sum_{j=1}^{L} (\sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta(\sigma_{j-1}^z \sigma_j^z - 1)) + \frac{h}{2} \sum_{j=1}^{L} \sigma_j^z$$

Eigenvalues of the quantum transfer matrix:

$$\Lambda_0 > |\Lambda_1| \geq |\Lambda_2| \geq \dots$$

Free energy

$$f(T,h) = -T \lim_{N\to\infty} \ln \Lambda_0$$

Static correlation functions:

$$\langle x_1 y_{m+1} \rangle_{T,h} \sim \sum_n A_n^{xy} \left(\frac{\Lambda_n}{\Lambda_0} \right)^m$$

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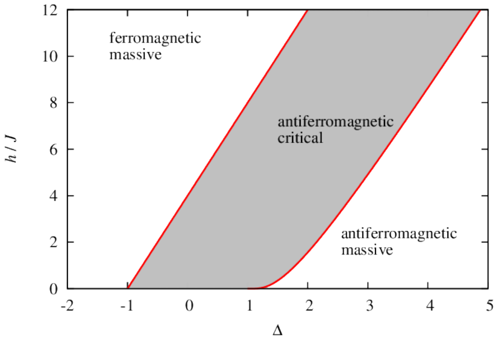
Upper critical magnetic field

$$h_c = 4J(1+\Delta)$$

In the following: Set $\Delta \in (0,1)$, $0 < h < h_c$



The auxiliary function for low temperatures

- Using the algebraic Bethe ansatz, we find the Bethe roots $\{\lambda_j^{(n)}\}$, which determine Λ_n
- Bethe ansatz equations $\Rightarrow$

$$\mathfrak{a}(\lambda_j|\{\lambda_j^{(n)}\}) = -1 \quad \forall j$$

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- For low temperatures, it is convenient to write

$$\mathfrak{a}(\lambda|\{\lambda_j^{(n)}\}) = \mathrm{e}^{-\frac{1}{T} u(\lambda|\mathbb{Y})}$$

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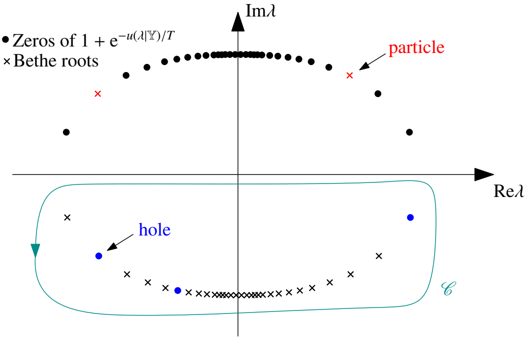
- $u(\lambda|\mathbb{Y})$ can be represented by a non-linear integral equation with integration contour $\mathscr{C}$
- $\mathbb{Y}$ denotes the finite set of Bethe roots outside $\mathscr{C}$ and zeros of $1 + \mathrm{e}^{-\frac{1}{T}u(\lambda|\mathbb{Y})}$ inside $\mathscr{C}$ that are not Bethe roots

- Each set $\mathbb{Y}$ relative to $\mathscr{C}$ corresponds to a state n of the quantum transfer matrix
- The parameters fulfill the "**higher level Bethe ansatz equations**"

$$\mathrm{e}^{-\frac{1}{T}u(y_i|\mathbb{Y})} = -1 \quad y_i \in \mathbb{Y}$$



The dressed energy as low-T-limit of the auxiliary function

Goal: Rigorous mathematical description of the auxiliary function for low temperatures.

Auxiliary function u for low-T

$$u(\lambda|\mathbb{Y}) = \varepsilon_c(\lambda) + \mathcal{O}(T)$$

The Bethe roots are determined by the solutions of

$$\mathrm{e}^{-\frac{u(\lambda|\mathbb{Y})}{T}} = -1 \Rightarrow u(\lambda|\mathbb{Y}) = \mathrm{i}\pi T(2n-1)$$

and are therefore located on the curve $\operatorname{Re}\varepsilon_c(\lambda) = 0$.

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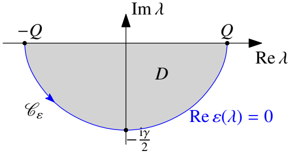


$D = \{x \in \mathbb{C} : \operatorname{Re}\varepsilon(\lambda) < 0, \operatorname{Im}(\lambda) < 0\}$
Continuation of $\varepsilon_c(\lambda)$:

$$\varepsilon_c(\lambda) = \varepsilon(\lambda) + \varepsilon(\lambda - i\gamma)\mathbf{1}_{\lambda - i\gamma \in D} - \varepsilon(\lambda + i\gamma)\mathbf{1}_{\lambda + i\gamma \in D}$$

Dressed energy

$$\varepsilon_c(\lambda) = \varepsilon_0(\lambda) - \int_{\mathscr{C}_\varepsilon} d\mu K(\lambda - \mu|\gamma)\varepsilon_c(\mu)$$

$$\varepsilon(\lambda) = \varepsilon_0(\lambda) - \int_{-Q}^{Q} d\mu K(\lambda - \mu|\gamma)\varepsilon(\mu)$$

with $\cos(\gamma) = \Delta$, $\gamma \in (0, \pi/2)$.
Kernel:

$$K(\lambda|\gamma) = \frac{1}{2\pi i}(\operatorname{cth}(\lambda - i\gamma) - \operatorname{cth}(\lambda + i\gamma))$$

Bare energy:

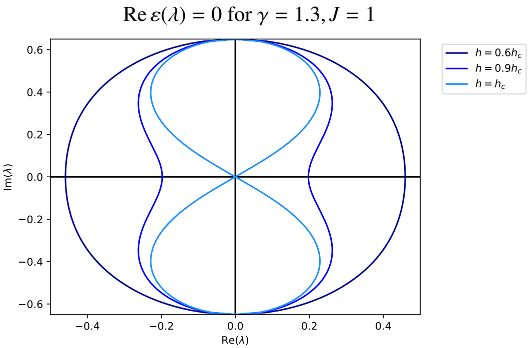
$$\varepsilon_0(\lambda) = h - 4\pi J \sin(\gamma) K(\lambda|\gamma/2)$$

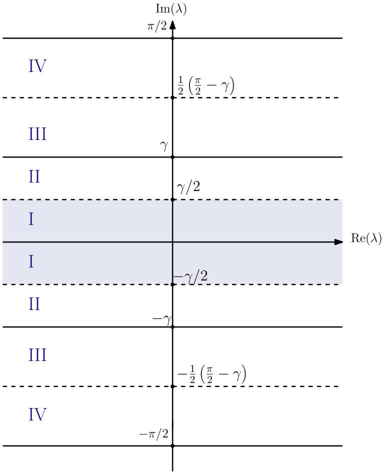
The "**Fermi-Point**" Q is the unique positive solution of

$$\varepsilon(Q) = 0$$

Dugave, Göhmann, Kozlowski 2014; SF, Göhmann, Kozlowski 2021

The dressed energy in the complex plane I





The dressed energy in the complex plane I

Theorem (SF, Göhmann, Kozlowski 2021)

$S_\gamma(Q_F) = \{z \in \mathbb{C} | -\pi/2 \leq \operatorname{Im}\lambda < \pi/2 \wedge z \notin [-Q, Q] \pm \mathrm{i}\gamma\}$

1. $\forall \lambda \in S_\gamma(Q_F)$ with $\operatorname{Re}\lambda = x$ and $\operatorname{Im}\lambda = y$ the function $\lambda \mapsto \operatorname{Re}\varepsilon(\lambda)$ is **even in x and y**.
2. Within the strip $0 \leq y < \gamma/2$ the function $x \mapsto \operatorname{Re}\varepsilon(x + \mathrm{i}y)$ is **monotonically increasing on $\mathbb{R}^+$** and, for every y, has a **single simple zero** $x(y)$.
3. Within the strip $|\operatorname{Im}\lambda| < \gamma/2$ the dressed energy is subject to the bounds

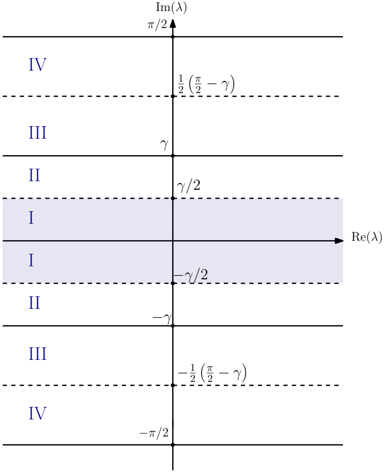
$$\operatorname{Re}\varepsilon_0(\lambda) < \operatorname{Re}\varepsilon(\lambda) < \operatorname{Re}\tilde{\varepsilon}(\lambda).$$

4. $\operatorname{Re}\varepsilon(\lambda) > 0$ for all $\lambda \in S_\gamma(Q_F)$ with $|\operatorname{Im}\lambda| > \gamma/2$, and we have the lower bounds

$$\operatorname{Re}\varepsilon(\lambda) > \min\left\{\frac{h}{2}, \frac{h\gamma}{\pi - \gamma}\right\} \quad \text{if} \quad \frac{\gamma}{2} < y < \gamma$$

$$\operatorname{Re}\varepsilon(\lambda) > \frac{h}{2} \quad \text{if} \quad \gamma < y < \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \gamma\right)$$

$$\operatorname{Re}\varepsilon(\lambda) > h \quad \text{if} \quad \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \gamma\right) < y < \frac{\pi}{2}$$



The dressed energy in the complex plane II

(2) Uniqueness of the zero

$\operatorname{Im}\lambda \in (0, \gamma/2)$, $\operatorname{Re}\lambda \in \mathbb{R}^+$

$$\lim_{\operatorname{Re}\lambda\to\infty} \varepsilon(\lambda) = h$$

Monotonicity:

$$\frac{\mathrm{d}\operatorname{Re}\varepsilon(\lambda)}{\mathrm{d}\operatorname{Re}(\lambda)} > 0$$

Lower and upper bound:

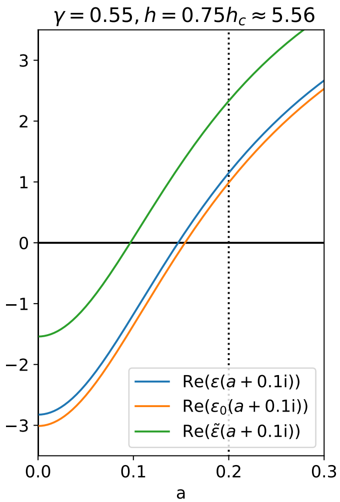
$$\operatorname{Re}\varepsilon_0(\lambda) < \operatorname{Re}\varepsilon(\lambda) < \operatorname{Re}\tilde{\varepsilon}(\lambda)$$

where

$$\tilde{\varepsilon}(\lambda) = h - \frac{2\pi J \sin(\gamma)}{\gamma \operatorname{ch}(\pi\lambda/\gamma)}$$

if $\tilde{\varepsilon}(\lambda)$ has a unique, positive zero.

→ Note that this is not the case for h large enough!



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The proof of the theorem is quite technical and requires different approaches in the respective strips, inter alia:

- Rewriting the equations using the resolvent kernel and Fourier transformations
- Deforming the integration contour of $\varepsilon(\lambda)$
- Direct estimations of integrals

Note that with the analytic continuation of $\varepsilon_c(\lambda)$,

$$\varepsilon_c(\lambda) = \varepsilon(\lambda) + \varepsilon(\lambda - \mathrm{i}\gamma)\mathbf{1}_{\lambda - \mathrm{i}\gamma \in D} - \varepsilon(\lambda + \mathrm{i}\gamma)\mathbf{1}_{\lambda + \mathrm{i}\gamma \in D}$$

we can use the Theorem for the analysis of $\mathrm{Re}\,\varepsilon_c(\lambda)$

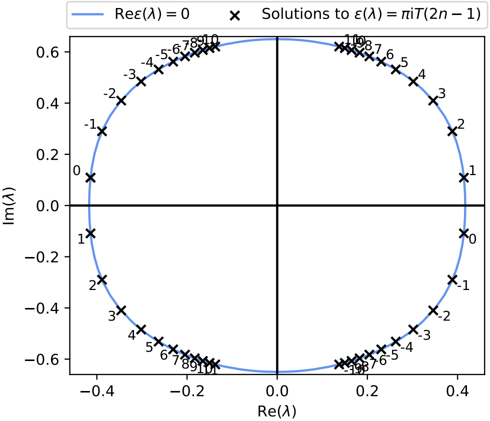
The dressed energy in the complex plane III

Theorem (SF, Göhmann, Kozlowski 2021)

5. Within the strip $-\gamma/2 < y < \gamma/2$, $\operatorname{Im}\varepsilon$ is **monotonically increasing** counterclockwise along the curve $x(y)$,

$$\frac{\mathrm{d}\operatorname{Im}\varepsilon(x(y)+\mathrm{i}y)}{\mathrm{d}y} > 0$$

$\Rightarrow$ This allows us to enumerate solutions to $\varepsilon(\lambda) = \pi\mathrm{i}T(2n-1)$ as well as the Bethe roots and other zeros of $1 + \mathrm{e}^{-u(\lambda|\mathbb{Y})/T}$



Types of solutions for excited states of the quantum transfer matrix

Particle-hole-type solutions

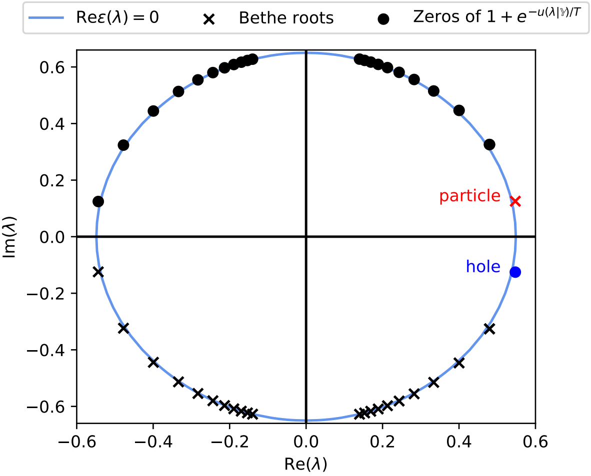
The particle and hole parameters $p_j^{(n)}$, $h_j^{(n)}$ solve the higher level Bethe ansatz equations:

$$1 + \mathrm{e}^{-u(h_j^{(n)}|\mathbb{Y})/T} = 0, \qquad \forall h_j^{(n)} \in \mathcal{H}_n$$

$$1 + \mathrm{e}^{-u(p_j^{(n)}|\mathbb{Y})/T} = 0, \qquad \forall p_j^{(n)} \in \mathcal{P}_n$$

→ Each particle-hole pattern corresponds to an excited state n of the quantum transfer matrix

→ Although in the Trotter limit there are infinitely many Bethe roots, the states are characterized by a finite number of parameters



$\gamma = 1.3,\ h = 0.5h_c \approx 2.53,\ J = 1.0,\ T = 0.2$

String-type solutions

For low T we can rewrite the equation for $u(\lambda|\mathbb{Y})$

$$e^{-\frac{1}{T}u(\lambda|\mathbb{Y})} = e^{-\frac{1}{T}\varepsilon_c(\lambda)} \prod_{y\in\mathbb{Y}_Q} \frac{\sinh(i\gamma + y - \lambda)}{\sinh(i\gamma - y + \lambda)} \cdot e^{-\Phi(\lambda|\mathbb{Y})} \cdot \frac{\left(1 + e^{-\frac{1}{T}u(\lambda - i\gamma|\mathbb{Y})}\right)^{\mathbf{1}_{\lambda - i\gamma \in \mathrm{Int}(\mathscr{C})}}}{\left(1 + e^{-\frac{1}{T}u(\lambda + i\gamma|\mathbb{Y})}\right)^{\mathbf{1}_{\lambda + i\gamma \in \mathrm{Int}(\mathscr{C})}}}$$

with $\Phi(\lambda|\mathbb{Y}) = u_{1;reg}(\lambda|\mathbb{Y}) + O(T)$

Let $\mathcal{Y} = \{y_0, y_1, ...\}$ be the finite set of Bethe roots outside $\mathscr{C}$, therefore fulfilling the higher level Bethe ansatz equations:

$$e^{-\frac{1}{T}u(y_i|\mathbb{Y})} = -1$$

Pick $y_0 \in \mathcal{Y}$ with $\mathrm{Im}(y_0) > \mathrm{Im}(y_i)$. Now, considering $e^{-\frac{1}{T}u(y_0|\mathbb{Y})}$ for $T \to 0^+$ there are two possibilities:

1: $\mathrm{Re}\,\varepsilon_c(y_0) = o(1) \Rightarrow$ we get a particle (1-string)

2: $\mathrm{Re}\,\varepsilon(y_0) < 0$ but one can find $y_1 \in \mathcal{Y}\backslash\{y_0\}$ s. th. $y_0 = y_1 + i\gamma + O(T^\infty)$, to compensate the exponential blowup of $e^{-\frac{1}{T}\varepsilon_c(\lambda)}$.

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In case 2, we consider the product

$$(-1)^2 = e^{-\frac{1}{T}u(y_0|\mathbb{Y})} e^{-\frac{1}{T}u(y_1|\mathbb{Y})}$$

and again, get:

2.1: $\mathrm{Re}\,\varepsilon(y_0) + \mathrm{Re}\,\varepsilon(y_1) = o(1) \Rightarrow$ we get a 2-string.

2.2: $\mathrm{Re}\,\varepsilon(y_0) + \mathrm{Re}\,\varepsilon(y_1) < 0$ but one can find $y_2 \in \mathcal{Y}\backslash\{y_1\}$ s. th. $y_1 = y_2 + i\gamma + \mathcal{O}(T^\infty)$

Repeat these steps for the remaining roots in $\mathcal{Y}$.

Non-existence of string-type solutions

String-type solutions

A point $y \in \mathbb{C}$ is called the top of a thermal r-string, $r \in \mathbb{N}$ if

$$\mathrm{Re}(\varepsilon_k^{(-)}(y)) < 0 \quad \text{for} \quad k = 1, \dots, r-1$$

and

$$\mathrm{Re}\,\varepsilon_r^{(-)}(y) = 0$$

with

$$\varepsilon_k^{(-)}(\lambda) = \sum_{s=0}^{k-1} \varepsilon_c(\lambda - \mathrm{i}s\gamma)$$

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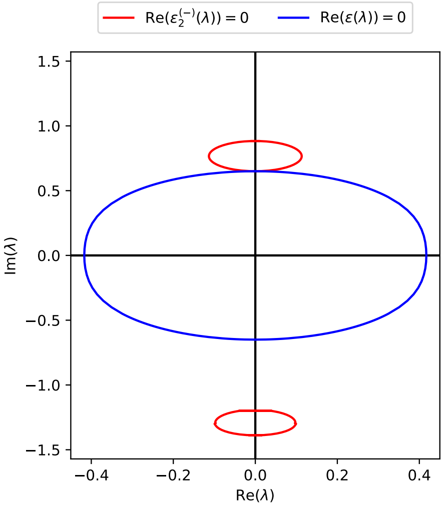
$$\varepsilon_k^{(-)}(\lambda) = \sum_{s=0}^{k-1} \varepsilon_c(\lambda - \mathrm{i}s\gamma)$$

But, using the properties of $\mathrm{Re}(\varepsilon(\lambda))$ we find that if

$$\mathrm{Re}\,\varepsilon_c(\lambda) < 0 \quad \Rightarrow \quad \mathrm{Re}\,\varepsilon_2^{(-)}(\lambda) > 0$$

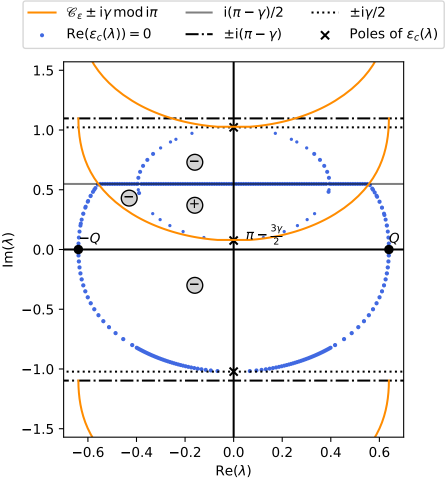
which shows, that **the condition for strings cannot be fulfilled for** $\Delta \in (0,1)$

SF, Göhmann, Kozlowski, SF in preparation



$\gamma = 1.3, h = 0.65 h_c \approx 3.3, J = 1.0$

Outlook for $\Delta \in (-1, 0)$



$\gamma = 2.04, h = 0.65 h_c \approx 1.41, J = 1.0$

The integral equation representations for $\varepsilon(\lambda)$, $\varepsilon_c(\lambda)$

$$\varepsilon(\lambda) = \varepsilon_0(\lambda) - \int_{-Q}^{Q} \mathrm{d}\mu K(\lambda-\mu|\gamma)\varepsilon(\mu|Q)$$

$$\varepsilon_c(\lambda) = \varepsilon_0(\lambda) - \int_{\mathscr{C}_\varepsilon} \mathrm{d}\mu K(\lambda-\mu|\gamma)\varepsilon_c(\mu)$$

have cuts at $\pm\mathrm{i}\gamma \bmod \mathrm{i}\pi$, $\mathscr{C}_\varepsilon \pm \mathrm{i}\gamma \bmod \mathrm{i}\pi$
$\Rightarrow$ If $\gamma > \pi/2$, $\mathscr{C}_\varepsilon \pm \mathrm{i}\gamma \bmod \mathrm{i}\pi$ intersects the curve $\mathrm{Re}\, \varepsilon_c(\lambda) = 0$

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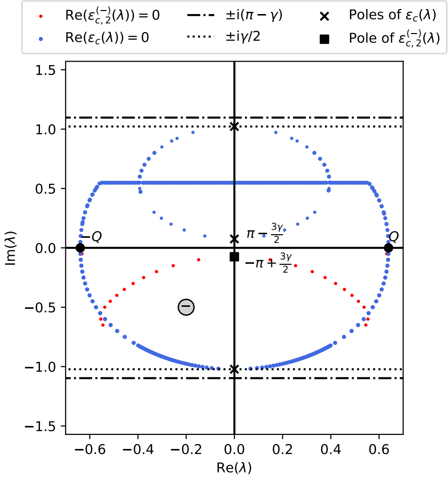

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have cuts at $\pm i\gamma \bmod i\pi$, $\mathscr{C}_\varepsilon \pm i\gamma \bmod i\pi$
$\Rightarrow$ If $\gamma > \pi/2$, $\mathscr{C}_\varepsilon \pm i\gamma \bmod i\pi$ intersects the curve $\mathrm{Re}\,\varepsilon_c(\lambda) = 0$

An analysis of the functions $\mathrm{Re}\,\varepsilon_c(\lambda)$ and $\mathrm{Re}\,\varepsilon_2^{(-)}(\lambda)$ shows, that the condition for the existence of 2-strings can be fulfilled for $\Delta \in (-1/2, 0)$

For $\Delta \in (-1, -1/2)$, $\mathrm{Re}\,\varepsilon_c(\lambda) = 0$ is intersected by more cuts at $\mathscr{C}_\varepsilon \pm i\gamma \bmod i\pi$

Summary and Outlook

Summary

- Each Eigenvalue Λ_n of the QTM is connected to a set $\mathbb{Y}$ and an auxiliary function $u(\lambda|\mathbb{Y})$.
- Static correlation functions:

$$\langle x_1 y_{m+1} \rangle_{T,h} \sim \sum_n A_n^{xy} \left(\frac{\Lambda_n}{\Lambda_0} \right)^m$$

- In the low-T limit the dressed energy $\varepsilon_c(\lambda)$ describes the behaviour of $u(\lambda|\mathbb{Y})$
- For $\Delta \in (0,1)$ the dressed energy is rigorously mathematically characterized in the complex plane

$\Rightarrow$ String-type solutions for Bethe roots can be excluded for low-T and $h > 0$, $\Delta \in (0,1)$

Outlook

- Massive regime: No strings $\rightarrow$ explicit expression for Form Factor series (Babenko, Göhmann, Kozlowski, Suzuki 2021)

$\rightarrow$ Is this also possible for the critical regime?

- Consider $\Delta \in (-1, 0)$ (Currently in work)
  - $\rightarrow$ Several technical difficulties, e.g the cuts intersecting with $\mathrm{Re}\, \varepsilon_c(\lambda)$