E_8 symmetry of the van Diejen model through gauge and integral transformations

Based on joint work with M. Noumi (Rikkyo University, Tokyo): arXiv:2203.00498

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Recent Advances in Quantum Integrable Systems
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The van Diejen model

Introduced by J. F. van Diejen as a model for relativistic particles with hyperoctahedral ($W_m = \{\pm 1\}^m \rtimes \mathfrak{S}_m$) symmetry (van Diejen '94, Hikami-Komori '97).

Defined by a family of mutually commuting analytic difference operators, of the form

$$D_x^{(r)} = \sum_{(I;\epsilon)\ :\ |I|\leq r} B_{I^c}^{(r)}(x;a|p,q,t) A_{\epsilon I, I^c}(x;a|p,q,t) \prod_{i\in I} T_{q,x_i}^{\epsilon_i}$$

where T_{q,x_i} are shift operators (acts by shifting $x_i \to qx_i$ while leaving remaining variables unaffected), acting on $x \in (\mathbb{C}^*)^m$ and dependent on 8 'external' couplings $a = (a_0, \ldots, a_7) \in (\mathbb{C}^*)^8$, one 'pair' coupling $t \in \mathbb{C}^*$, shift parameter $q \in \mathbb{C}^*$, and elliptic nomé p ($|p| < 1$).

The van Diejen model has known relations to

- Sakai's elliptic Painlevé equation,
- Affine R-matrices,
- Elliptic $6j$-symbols,
- Super-symmetric models,

$\vdots$

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The principal operator (Hamiltonian) given by

$$\mathcal{D}_x(a|p,q,t) = A^0(x;a|p,q,t) + \sum_{1\le i\le m;\ \varepsilon=\pm} A_i^{\varepsilon}(x;a|p,q,t)T_{q,x_i}^{\varepsilon}$$

with shift coefficients

$$A_i^{\varepsilon}(x;a|p,q,t) = \frac{\prod_{0\le s\le 7}\theta(a_s x_i^{\varepsilon};p)}{\theta(x_i^{\varepsilon 2};p)\theta(q x_i^{\varepsilon 2};p)} \prod_{j\ne i}\prod_{\delta=\pm} \frac{\theta(t x_i^{\varepsilon} x_j^{\delta};p)}{\theta(x_i^{\varepsilon} x_j^{\delta};p)} \quad (\varepsilon \in \{\pm\})$$

where $\theta(x;p) = \prod_{\ell=0}^{\infty}(1-p^{\ell}x)(1-p^{\ell+1}x^{-1})$ ($|p|<1$) and zeroth order coefficient $A^0(x;a|p,q,t) = \frac{1}{2}\sum_{0\le r\le 3} A_r^0(x;a|q,t)$, where

$$A_r^0(x;a|p,q,t) = L_r^{(m)}(a|p,q,t)\frac{\prod_{0\le s\le 7}\theta(c_r q^{-\frac{1}{2}}a_s;p)}{\theta(t;p)\theta(q^{-1}t;p)} \prod_{1\le j\le m}\prod_{\delta=\pm}\frac{\theta(c_r q^{-\frac{1}{2}} t x_j^{\delta};p)}{\theta(c_r q^{-\frac{1}{2}} x_j^{\delta};p)}$$

with $L_0^{(m)} = L_1^{(m)} = 1$, $L_2^{(m)} = p^2(L_3^{(m)})^{-1} = p^{-1}q^{-2}t^m(a_0\cdots a_7)^{\frac{1}{2}}$,
$c_0 = -c_1 = 1$, and $c_2 = -c_3^{-1} = p^{\frac{1}{2}}$.

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with shift coefficients $A_i^\varepsilon(x;a|p,q,t)$ and zeroth order coefficient $A^0(x;a|p,q,t)$.

The $\mathfrak{S}_8$ symmetry in the parameters a is straightforward from the coefficients, but the model is known to have

- $W(D_8)$ symmetry for arbitrary parameters ,
- $W(E_8)$ symmetry for particular values of t (depending on the other parameters).

The Weyl group actions can be realized explicitly through gauge and integral transformations.

Open problem: finding $\psi(x;a|p,q,t) \in \mathcal{O}(\mathbb{T}^m)^{W_m}$ that satisfies

$$\mathcal{D}_x(a|p,q,t)\psi(x;a|p,q,t) = \Lambda(a|p,q,t)\psi(x;a|p,q,t)$$

for some constant $\Lambda(a|p,q,t) \in \mathbb{C}^*$.

The D_8 Weyl group

Let $\mathcal{V} = \mathbb{C}^8 = \mathbb{C}\epsilon_0 \oplus \mathbb{C}\epsilon_1 \cdots \oplus \mathbb{C}\epsilon_7$ be a complex vector space with canonical basis $\{\epsilon_0, \ldots, \epsilon_7\}$ and $(\cdot|\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ a symmetric bilinear form s.t. $(\epsilon_i|\epsilon_j) = \delta_{i,j}$.

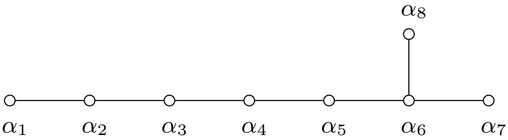

Figure: The Dynkin diagram corresponding to the D_8 root system.

Root system: $\Delta(D_8) = \{\delta\epsilon_i + \delta'\epsilon_j | 1 \leq i < j \leq 8,\ \delta, \delta' \in \{\pm\}\}$.

Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for all $i \in \{1, \ldots, 6\}$, $\alpha_7 = \epsilon_7 + \epsilon_0$ and $\alpha_8 = \epsilon_7 - \epsilon_0$.

Reflections: for each $\alpha \in \mathcal{V} \setminus \{\alpha \in \mathcal{V} | (\alpha|\alpha) = 0\}$ we define the reflection with respect to α by

$$\mathfrak{r}_\alpha : \mathcal{V} \to \mathcal{V}, \quad \mathfrak{r}_\alpha.u = u - 2\frac{(\alpha|u)}{(\alpha|\alpha)}\alpha \quad (u \in \mathcal{V}).$$

Simple reflections: $\mathfrak{r}_i = \mathfrak{r}_{\alpha_i}$ for all $i \in \{1, 2, \ldots, 8\}$.

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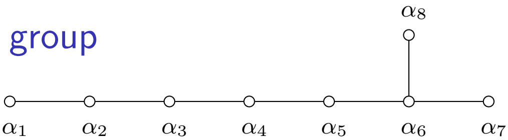

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Reflections: $\mathfrak{r}_\alpha.u = u - (\alpha^\vee|u)\alpha \quad (u \in \mathcal{V})$.
Simple reflections: $\mathfrak{r}_i = \mathfrak{r}_{\alpha_i}$ for all $i \in \{1, 2, \dots, 8\}$.

Action: $\mathfrak{r}_{\epsilon_i \underset{(-)}{+} \epsilon_j}.(u_0, \dots, u_7) = (v_0, \dots, v_7)$, where

$$v_k = \begin{cases} \underset{(+)}{-} u_j & \text{if } k = i \\ \underset{(+)}{-} u_i & \text{if } k = j \\ u_k & \text{if } k \neq i, j \end{cases}.$$

$$\langle \mathfrak{r}_1, \dots, \mathfrak{r}_6, \mathfrak{r}_8 \rangle = W(A_7) \subset W(D_8) = \langle \mathfrak{r}_1, \dots, \mathfrak{r}_7, \mathfrak{r}_8 \rangle$$

The group acts through permutations and even number of sign flips.

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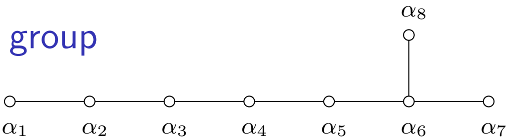

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We are working with multiplicative variables:
Let $\mu = (\mu_0, \dots, \mu_7) = (\mathrm{e}^{u_0}, \dots, \mathrm{e}^{u_7})$, then

$$r_{\epsilon_i \underset{(-)}{+} \epsilon_j}.(\mu_0, \dots, \mu_7) = (\nu_0, \dots, \nu_7), \quad \nu_k = \begin{cases} \mu_j^{\overset{-}{(+)}1} & \text{if } k = i \\ \mu_i^{\overset{-}{(+)}1} & \text{if } k = j \\ \mu_k & \text{if } k \neq i, j \end{cases}.$$

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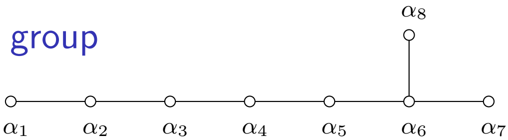

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We are working with multiplicative variables:
Let $a = (a_0, \ldots, a_7) = p^{\frac{1}{2}} q^{\frac{1}{2}} \mu = (p^{\frac{1}{2}} q^{\frac{1}{2}} \mu_0, \ldots, p^{\frac{1}{2}} q^{\frac{1}{2}} \mu_7)$, then

$$\mathfrak{r}_{\epsilon_i+\epsilon_j}.(a_0, \ldots, a_7) = (b_0, \ldots, b_7), \quad b_k = \begin{cases} pqa_j^{-1} & \text{if } k = i \\ pqa_i^{-1} & \text{if } k = j \\ a_k & \text{if } k \neq i, j \end{cases}.$$

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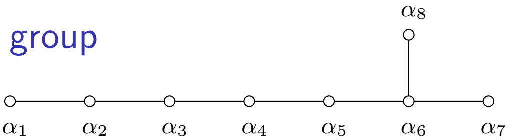
The D_8 Weyl group



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For any $K \subseteq \{0, 1, \dots, 7\}$ with even cardinality, we can write

$$K = \{i_1, \dots i_r, j_1, \dots, j_r\}$$

and define

$$w_K = \mathfrak{r}_{\epsilon_{i_1} + \epsilon_{j_1}} \mathfrak{r}_{\epsilon_{i_2} + \epsilon_{j_2}} \cdots \mathfrak{r}_{\epsilon_{i_r} + \epsilon_{j_r}} \in W(D_8).$$

These will be the elements of $W(D_8)$ that naturally correspond to our gauge transformations.

Gauge transformation

For any $K \subseteq \{0, 1, \ldots, 7\}$ with even cardinality, define the gauge function

$$U_K(x; a|p, q) = \prod_{1 \le i \le m} \prod_{s \in K} \frac{1}{\Gamma(a_s x_i; p, q)\Gamma(a_s x_i^{-1}; p, q)},$$

where

$$\Gamma(x; p, q) = \prod_{m,n \in \mathbb{Z}_{\ge 0}} \frac{1 - p^{m+1} q^{n+1} x^{-1}}{1 - p^m q^n x} \quad (|p|, |q| < 1).$$

is (essentially) Ruijsenaars' elliptic Gamma function.

Lemma

The van Diejen operator satisfies the relation

$$U_K(x; a|p, q)^{-1} \circ \mathcal{D}_x(a|p, q, t) \circ U_K(x; a|p, q) = \mathcal{D}_x(w_K.a|p, q, t).$$

for any $K \subseteq \{0, 1, \ldots, 7\}$ *with* $|K| \in 2\mathbb{Z}_{\ge 0}$.
If $\varphi(x; w_K.a|p, q, t)$ *is an eigenfunction of* $\mathcal{D}_x(w_K.a|p, q, t)$ *with eigenvalue* $\Lambda(w_K.a|p, q, t)$, *then* $\psi_K(x; a|p, q, t) = U_K(x; a|p, q)\varphi(x; w_K.a|p, q, t)$ *satisfies*

$$\mathcal{D}_x(a|p, q, t)\psi_K(x; a|p, q, t) = \Lambda(w_K.a|p, q, t)\ \psi_K(x; a|p, q, t).$$

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for any $K \subseteq \{0, 1, \ldots, 7\}$ *with* $|K| \in 2\mathbb{Z}_{\geq 0}$.
In particular, $U_K(x; a|p, q)$ *is an exact eigenfunction of the van Diejen model – with explicit eigenvalues – if the parameters satisfy*

$$(\prod_{j \notin K} a_j) p^{|K|} q^{|K|} t^{2m} = (\prod_{i \in K} a_i) p^2 q^2 t^2.$$

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Let $\mathcal{V} = \mathbb{C}^8 = \mathbb{C}\epsilon_0 \oplus \mathbb{C}\epsilon_1 \cdots \oplus \mathbb{C}\epsilon_7$ be a complex vector space with canonical basis $\{\epsilon_0, \ldots, \epsilon_7\}$ and $(\cdot|\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ a symmetric bilinear form s.t. $(\epsilon_i|\epsilon_j) = \delta_{i,j}$.

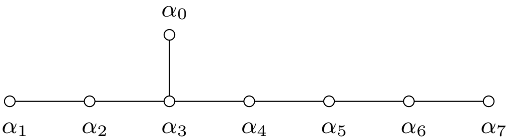

Figure: The Dynkin diagram corresponding to the E_8 root system.

The E_8 root system consists of two classes of roots

i) $\pm\epsilon_i \pm \epsilon_j, \quad 0 \leq i < j \leq 7$ (with arbitrary combination of signs)

ii) $\frac{1}{2}(\pm\epsilon_0 \pm \epsilon_1 \cdots \pm \epsilon_7),$ (with even number of minus signs)

The first class are the D_8 roots.

Let $\phi = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in \mathcal{V}$ (highest root). The second class of roots can then be expressed as

$$\alpha = \phi - \epsilon_K, \quad \epsilon_K = \sum_{i \in K} \epsilon_i$$

with any subset $K \subseteq \{0, 1, \ldots, 7\}$ with even cardinality.

The E_8 Weyl group

Simple roots:

$$\alpha_0 = \phi - \epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3, \quad \alpha_i = \epsilon_i - \epsilon_{i+1}, \ (i \in \{1, 2, \dots, 6\}), \quad \alpha_7 = \epsilon_7 + \epsilon_0.$$

Reflections: $\mathfrak{r}_\alpha.u = u - (\alpha^\vee|u)\alpha \quad (u \in \mathcal{V})$.
Simple reflections: $\mathfrak{r}_i = \mathfrak{r}_{\alpha_i}$ for all $i \in \{0, 1, \dots, 7\}$.

For $(u_0, \dots, u_7)$ the canonical coordinates of $\mathcal{V}$, we have that $\mathfrak{r}_{\phi-\epsilon_K}.(u_0, \dots, u_7) = (v_0, \dots, v_7)$ where

$$v_k = \begin{cases} u_k - \frac{1}{4}(\sum_{i\in K} u_i - \sum_{j\notin K} u_j) & \text{if } k \in K \\ u_k + \frac{1}{4}(\sum_{i\in K} u_i - \sum_{j\notin K} u_j) & \text{if } k \notin K \end{cases}.$$

$$W(A_7) \subset \langle \mathfrak{r}_1, \dots, \mathfrak{r}_8 \rangle = W(D_8) \subset W(E_8) = \langle \mathfrak{r}_0, \dots, \mathfrak{r}_7 \rangle$$

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Simple reflections: $\mathfrak{r}_i = \mathfrak{r}_{\alpha_i}$ for all $i \in \{0, 1, \ldots, 7\}$.

For $\mu_s = \exp(u_s)$ $(s \in \{0, 1, \ldots, 7\})$, then
$\mathfrak{r}_{\phi - \epsilon_K} . (\mu_0, \ldots, \mu_7) = (\nu_0, \ldots, \nu_7)$ where

$$\nu_k = \begin{cases} \mu_k \prod_{i \in K} \mu_i^{-\frac{1}{4}} \prod_{j \notin K} \mu_j^{\frac{1}{4}} & \text{if } k \in K \\ \mu_k \prod_{i \in K} \mu_i^{\frac{1}{4}} \prod_{j \notin K} \mu_j^{-\frac{1}{4}} & \text{if } k \notin K \end{cases}.$$

$$W(A_7) \subset \langle \mathfrak{r}_1, \ldots, \mathfrak{r}_8 \rangle = W(D_8) \subset W(E_8) = \langle \mathfrak{r}_0, \ldots, \mathfrak{r}_7 \rangle$$

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Simple reflections: $\mathfrak{r}_i = \mathfrak{r}_{\alpha_i}$ for all $i \in \{0, 1, \dots, 7\}$.

For $a_s = p^{\frac{1}{2}} q^{\frac{1}{2}} \mu_s$ $(s \in \{0, 1, \dots, 7\})$, then
$\mathfrak{r}_{\phi-\epsilon_K}.(a_0, \dots, a_7) = (b_0, \dots, b_7)$ where

$$b_k = \begin{cases} (pq)^{\frac{1}{4}(|K|-4)} a_k \prod_{i \in K} a_i^{-\frac{1}{4}} \prod_{j \notin K} a_j^{\frac{1}{4}} & \text{if } k \in K \\ (pq)^{-\frac{1}{4}(|K|-4)} a_k \prod_{i \in K} a_i^{\frac{1}{4}} \prod_{j \notin K} a_j^{-\frac{1}{4}} & \text{if } k \notin K \end{cases}.$$

In particular,

$$\mathfrak{r}_\phi.a = (pq(a_0 \cdots a_7)^{-\frac{1}{4}} a_0, \dots, pq(a_0 \cdots a_7)^{-\frac{1}{4}} a_7)$$

Integral transformation; Key ingredients 1

Lemma (Ruijsenaars '09, Komori-Noumi-Shiraishi '09)

Under the balancing condition

$$a_0 \cdots a_7 t^{2(m-n)} = p^2 q^2 t^2,$$

the function

$$\Phi(x, y|p, q, t) = \prod_{1 \le i \le m} \prod_{1 \le k \le n} \prod_{\delta, \delta' = \pm} \Gamma(p^{\frac{1}{2}} q^{\frac{1}{2}} t^{-\frac{1}{2}} x_i^{\delta} y_k^{\delta'}; p, q)$$

satisfies the functional identity

$$\mathcal{D}_x(a|p, q, t)\Phi(x, y|p, q, t) = \mathcal{D}_y(b|p, q, t)\Phi(x, y|p, q, t)$$

where $b = (p^{\frac{1}{2}} q^{\frac{1}{2}} t^{\frac{1}{2}} a_0^{-1}, \ldots, p^{\frac{1}{2}} q^{\frac{1}{2}} t^{\frac{1}{2}} a_7^{-1}) \in (\mathbb{C}^*)^8$.

If $t = p^{-1} q^{-1} a_0^{\frac{1}{2}} \cdots a_7^{\frac{1}{2}}$ then $(pqb^{-1})_s = (\mathfrak{r}_\phi . a)_s$ for all $s \in \{0, 1, \ldots, 7\}$.

Main idea: Use $\Phi(x, y|p, q, t)$ as the kernel of an integral transformation. (Ruijsenaars '15)

Integral transformation; Key ingredients 2

Define

$$\langle f, g\rangle = \int_{\mathcal{C}_m} d\omega_m(x)\; w_m(x;a|p,q,t)\; f(x)g(x), \quad d\omega_m(x) = \frac{1}{(2\pi \mathrm{i})^m}\frac{dx_1\cdots dx_m}{x_1\cdots x_m},$$

with *weight function* $w_m(x;a|p,q,t)$ given by

$$w_m(x;a|p,q,t) = \prod_{1\le i\le m}\prod_{\delta=\pm}\frac{\prod_{0\le s\le 7}\Gamma(a_s x_i^{\delta};p,q)}{\Gamma(x_i^{\delta 2};p,q)}\prod_{1\le i<j\le m}\prod_{\delta,\delta'=\pm}\frac{\Gamma(t x_i^{\delta}x_j^{\delta'};p,q)}{\Gamma(x_i^{\delta}x_j^{\delta'};p,q)}.$$

Lemma

The van Diejen operator is (formally) self-adjoint with respect to the symmetric $\mathbb{C}$*-bilinear form, i.e.*

$$\langle \mathcal{D}_x(a|p,q,t)f, g\rangle = \langle f, \mathcal{D}_x(a|p,q,t)g\rangle.$$

Integral transformations

Theorem

Let the parameters $p, q, t \in \mathbb{C}^*$ *and* $b = (b_0, \ldots, b_7) \in (\mathbb{C}^*)^8$ *satisfy* $b_0 \cdots b_7 t^{2(n-m)} = p^2 q^2 t^2$ *and*

$$\begin{cases} |p| < |qt|, & \text{if } K \neq \emptyset \\ |p| < \min(|qt|, |q^{-1} t^{n-m-1}|) & \text{if } K = \emptyset \end{cases}, \quad |q| < 1, \quad |t| < 1, \quad |b_s| < 1.$$

Let $\varphi(y; b|p,q,t) \in \mathcal{O}(\mathbb{T}^n)^{W_n}$ *be an eigenfunction of the van Diejen operator* $\mathcal{D}_y(b|p,q,t)$ *with eigenvalue* $\Lambda(b|p,q,t) \in \mathbb{C}$*, then the function*

$$\psi(x; a|p,q,t) = \int_{\mathbb{T}^n} d\omega_n(y) w_n(y; b|p,q,t) \Phi(x,y|p,q,t) \varphi(y; b|p,q,t)$$

is an eigenfunction of the van Diejen operator $\mathcal{D}_x(a|p,q,t)$ $(b_s = p^{\frac{1}{2}} q^{\frac{1}{2}} t^{\frac{1}{2}} a_s^{-1})$*, with the same eigenvalue* $\Lambda(b|p,q,t)$ *and* $\psi(x) \in \mathcal{O}(\mathbb{T}^m)^{W_m}$*.*

Setting $\varphi(y; b|p,q,t) = U_K(y; b|p,q)$ yields exact eigenfunctions in terms of BC_n elliptic hypergeometric integrals of Selberg type (in the broad sense) with two constraints on the model parameters.

Integral transformations

Setting $m = n = 1$ yields the elliptic hypergeometric integral:

Corollary (Komori '05, Spiridonov '04)

Let $K = \{6, 7\}$ and suppose the model parameters satisfy $a_0 \cdots a_5 = p^2 q^2 t^2$ and $a_6 a_7 = 1$, then the elliptic hypergeometric integral

$$I(p^{\frac{1}{2}} q^{\frac{1}{2}} t^{\frac{1}{2}} a_0^{-1}, \dots, p^{\frac{1}{2}} q^{\frac{1}{2}} t^{\frac{1}{2}} a_5^{-1}, p^{\frac{1}{2}} q^{\frac{1}{2}} t^{-\frac{1}{2}} x, p^{\frac{1}{2}} q^{\frac{1}{2}} t^{-\frac{1}{2}} x^{-1}; p, q),$$

where

$$I(b_0, \dots, b_5, b_6, b_7; p, q) = \int_{|y|=1} \frac{dy}{2\pi \mathrm{i} y} \frac{\prod_{0 \le s \le 7} \prod_{\delta = \pm} \Gamma(b_s y^{\delta}; p, q)}{\Gamma(y^{\delta 2}; p, q)},$$

is an exact eigenfunction of the van Diejen operator $\mathcal{D}_x(a|p, q, t)$ (with explicit eigenvalues).

Setting (say) $b_0 b_1 = p^M q^N$ $(M, N \in \mathbb{Z}_{\ge 0})$ in $I(b_0, \dots, b_5, p^{\frac{1}{2}} q^{\frac{1}{2}} t^{-\frac{1}{2}} x, p^{\frac{1}{2}} q^{\frac{1}{2}} t^{-\frac{1}{2}} x^{-1}; p, q)$ yields eigenfunction in terms of the ${}_{12}V_{11}$ elliptic hypergeometric series (Date, Jimbo, *et al.* '88, Frenkel & Turaev '97).

E_8 symmetry

Theorem

Let $K \subseteq \{0, 1, \ldots, 7\}$ with even cardinality.

(1) By the transformation $\psi_K(x; a|p, q, t) = U_K(x; a|p, q)\varphi(x; w_K.a|p, q, t)$, the model parameters of eigenfunctions of the van Diejen operator transform according to the action of the elements $w_K \in W(D_8)$.

(2) Let $a_0 \cdots a_7 = p^2 q^2 t^2$. By the transformation

$$\psi(x; a|p, q, t) = U_K(x; a|p, q) \int_{\mathcal{C}_m} d\omega_m(y)\ w_m(y; b|p, q, t)\Phi(x, y|p, q, t) \cdot U_{K^c}(y; b|p, q)\varphi(y; d|p, q, t),$$

where $b_s = p^{\frac{1}{2}} q^{\frac{1}{2}} t^{\frac{1}{2}} (w_K.a)_s^{-1}$ and $d_s = (w_{K^c}.b)_s$ for all $s \in \{0, 1, \ldots, 7\}$, the model parameters of eigenfunctions of the van Diejen operator transform according as the action of the reflection $\mathfrak{r}_{\phi-\epsilon_K}$ by the E_8 root $\phi - \epsilon_K$.

Conclusions

To summarize:

- Show that the $W(D_8)$ and $W(E_8)$ symmetries of the van Diejen model can be realized in terms of explicit transformations
- Found several classes of exact eigenfunctions of the van Diejen operator:
  1. eigenfunctions by BC_n elliptic hypergeometric integrals under two restrictions on the model parameters.
  2. gauge eigenfunctions under one restriction on the model parameters.
- Showed that BC_n elliptic hypergeometric integrals of Selberg type (in the broad sense) with even no. of parameters satisfy an eigenvalue equation for a van Diejen type operator
- Implicitly: Obtained joint eigenfunctions of so-called modular pairs $(\mathcal{D}_x(a|p,q,t), \mathcal{D}_x(a|q,p,t))$ due to $p \leftrightarrow q$ symmetry.

Higher symmetries?

In this talk, I presented the $W(D_8)$ and $W(E_8)$ symmetry of the van Diejen model, but we found several other symmetries in the parameter space $(a|q,t) \in (\mathbb{C}^*)^8 \times (\mathbb{C}^*)^2$.

Gauge transformations:

$$\begin{aligned}(a|q,t) &\leftrightarrow (w_K.a|q,t) \quad (K \subseteq \{0,1,\ldots,7\},\ |K| \in 2\mathbb{Z}_{\geq 0}\\ (a|q,t) &\leftrightarrow (p^P a|q,t)\\ (a|q,t) &\leftrightarrow (a|q,pqt^{-1})\end{aligned}$$

where $P = Q(E_8) = \{v \in \mathbb{Z}^8 \cup (\phi + \mathbb{Z}^8)|(\phi|v) \in \mathbb{Z}\}$ is the E_8 root lattice.

Integral transformations:

$$\begin{aligned}(a|q,t) \leftrightarrow ((p^{\frac{1}{2}}q^{\frac{1}{2}}t^{\frac{1}{2}}a_0^{-1},\ldots,p^{\frac{1}{2}}q^{\frac{1}{2}}t^{\frac{1}{2}}a_7^{-1})|q,t) \qquad &\text{if } a_0\cdots a_7 t^{2(m-n)} = p^2q^2t^2,\\ (a|q,t) \leftrightarrow (a|t,q) \qquad &\text{if } a_0\cdots a_7 q^{2n}t^{2m} = p^2q^2t^2.\end{aligned}$$

Conclusions and Outlook

To summarize:

- Show that the $W(D_8)$ and $W(E_8)$ symmetries of the van Diejen model can be realized in terms of explicit transformations
- Found several classes of exact eigenfunctions of the van Diejen operator:
  1. eigenfunctions by BC_n elliptic hypergeometric integrals under two restrictions on the model parameters.
  2. gauge eigenfunctions under one restriction on the model parameters.
- Showed that BC_n elliptic hypergeometric integrals of Selberg type (in the broad sense) with even no. of parameters satisfy an eigenvalue equation for a van Diejen type operator
- Implicitly: Obtained joint eigenfunctions of so-called modular pairs.

To do:

- Finding the totality of eigenfunctions obtained by these transformations.
- Finding transformation for a generalization of the van Diejen model (F.A. '20)
- Finding the equivalent transformation and their interpretations in the related models, *e.g.* transformations of e-P_{VI}, affine R-matrices, Yang-Baxter eq., ...

Thank you!