New bootstrap solutions in two-dimensional percolation models

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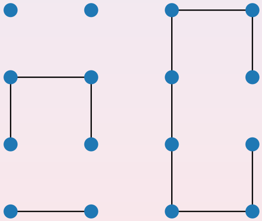
Q-Potts random cluster model

$$\mathrm{Probability}(\mathcal{G}) = p^{\#\text{bonds}}(1-p)^{\#\text{edges without bond}}Q^{\#\text{clusters}}$$

$\mathcal{G}$:



$\#$bonds $= 11$

$\#$edges without bond $= 5$

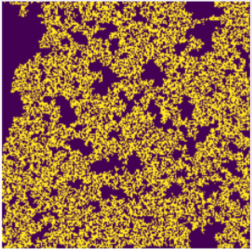
$\#$clusters $= 5$

$$\mathrm{Prob}(\mathcal{G}) = p^{11}(1-p)^5 Q^5$$

Study object : the connectivity properties of clusters

Ex: $\exists$ infinite cluster (connecting $0 \leftrightarrow \infty$)?

Percolation transition



$$p_c = \frac{\sqrt{Q}}{\sqrt{Q}+1}, \quad Q \in [0,4]$$

Clusters $\xrightarrow[\text{scaling}]{}$ conformal random fractals

- Fractal dimension of cluster, curves, pivotal bonds$\cdots$

  (Di Francesco, Saleur, Zuber '87, De Nijs, Duplantier, Nienhuis, Saleur...'89)
- Crossing probabilities (Cardy formula)

  (Cardy '92)
- New set-up in probability theory and complex analysis (ex: SLE, lattice parafermion etc..)

  (Werner, Smirnov, Bernard, Bauer'01)

Potts CFT ? A 30 y.o. open problem...

What was known:

- The CFT torus partition function: central charge, and the set of Virasoro representations (spectre $\mathcal{S}^{\text{Potts}}$)
- The representation properties of certain boundary fields

What was NOT known:

- The structure constants, necessary to compute all the field correlation function
- The fine structure of the Virasoro representations

Why is difficult and at the same time interesting:

- So far NO consistent CFT that is non-unitary, non-rational and logarithmic has been found

...towards a solution!

2D Bootstrap approach to four-point connectivities

(Picco, Ribault, Santachiara '15,'16, '19)

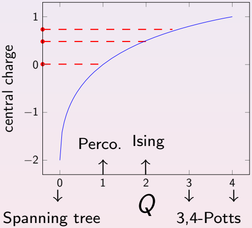
(Ninevisat, Ribault, Samuelsson, Liu, He, Jacobsen, Saleur '18 '19 '20)

key inputs

- Monte-Carlo simulations
- Transfer matrix simulations
- Representation theory of Affine Temperley-Algebra

Potts CFT central charge: non-unitarity

(Di Francesco, Saleur, Zuber '87, De Nijs, Duplantier, Nienhuis, Saleur...'89)



$c \in [-2, 1]$

Unitary CFT series:

$$c = 1 - \frac{6}{p(p+1)} \quad p = 2, 3 \cdots$$

- Local and positives Boltzmann weights $\xrightarrow[\text{scaling}]{}$ unitary CFT
- Potts model is not local (or local but with complex Boltzmann weights) $\xrightarrow[\text{scaling}]{}$ non-unitary CFT

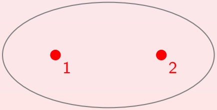
Potts CFT spectre $\mathcal{S}^{\text{Potts}}$: non-rational

$$\mathcal{S}^{\text{Potts}} = \{\underbrace{V^D_{1,1}, V^D_{1,2}, \dots}_{\text{Termal sector}}, \underbrace{V^N_{0,\frac{1}{2}}, V^N_{0,\frac{3}{2}}, \dots}_{\text{Magnetic Sector}}, \underbrace{V^N_{2,0}, V^N_{3,0}, V^N_{2,1}, V^N_{2,\frac{1}{2}}, V^N_{4,\frac{1}{2}}, \dots}_{\text{Other sectors}}\}$$

$$V^D_{r,s}(z,\bar{z}) \rightarrow (\Delta_{r,s}, \Delta_{r,s}),\ \ V^N_{r,s}(z,\bar{z}) \rightarrow (\Delta_{r,s}, \Delta_{r,-s})$$

Correlation lenght $\nu = (2 - 2\Delta_{1,2})^{-1} \rightarrow$ energy field $V^N_{1,2}$

Order parameter $\beta = \Delta_{0,\frac{1}{2}}/(2 - 2\Delta_{1,2}) \rightarrow$ connectivity field $V^N_{0,\frac{1}{2}}$



$p_{12} = |x|^{-4\Delta_{0,\frac{1}{2}}}$

Potts CFT representations: Logarithmic

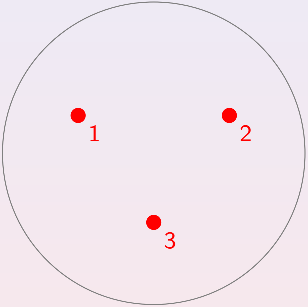
Log CFT: not semi-simple (indecomposable but not irreducible)
Virasoro representation

Rank 2 example

$$\begin{pmatrix} \langle V_1 | L_0 V_1 \rangle & \langle V_1 | L_0 V_2 \rangle \\ \langle V_2 | L_0 V_1 \rangle & \langle V_2 | L_0 V_2 \rangle \end{pmatrix} = \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix},$$

$$\begin{aligned} \langle V_1(x) V_1(0) \rangle &= 0 \\ \langle V_1(x) V_2(0) \rangle &\sim |x|^{-4\Delta} \\ \langle V_2(x) V_2(0) \rangle &\sim \ln |x|^2 |x|^{-4\Delta} \end{aligned}$$

Potts CFT 3-pt connectivity: the Delfino-Viti conjecture



$$p_{123} = \frac{\text{Constant}}{\left|x_{12}x_{13}x_{23}\right|^{\Delta_{0,\frac{1}{2}}}}, \quad \text{Constant} = \sqrt{2}C^{(0,\frac{1}{2})}_{(0,\frac{1}{2}),(0,\frac{1}{2})}$$

(Delfino-Viti '10)

$c \leq 1$ Liouville structure constant

- Shift relations:

$$\frac{C^{(r_3+2,s_3)}_{(r_1,s_1),(r_2,s_2)}}{C^{(r_3,s_3)}_{(r_1,s_1),(r_2,s_2)}} = \text{Product of } \Gamma, \quad \frac{C^{(r_3,s_3+2)}_{(r_1,s_1),(r_2,s_2)}}{C^{(r_3,s_3)}_{(r_1,s_1),(r_2,s_2)}} = \text{Product of } \Gamma$$

(Teschner '95)

- Admits an unique solution (product of double Γ_2):
  - for $c \geq 25$: C^{DOZZ}: Liouville theory for 2D quantum gravity
  - for $c \leq 1$: C, used in Delfino-Viti conjecture

(Schomerus '03, Kostov, Petkova, Zamolodchikov '05)

(Delfino,Picco, S., Viti 2012, Dotsenko 2013)

- Liouville $c \leq 1$ theory

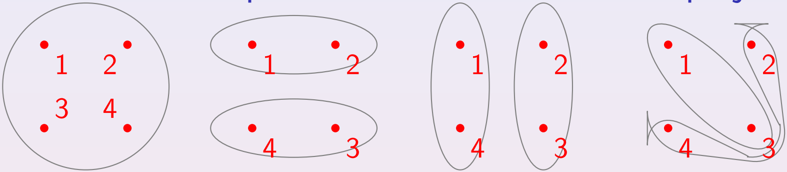
(Ribault, S.'15)

(Gavrilenko, S.'18)

- Generalized to other three-point observables

(Estienne, Ihklef, Jacobsen Saleur, '15)

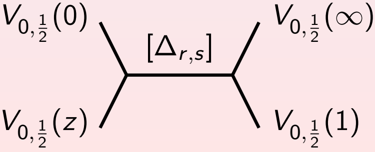
Potts CFT and 4−pts functions: an ambitious project



p_{1234} $p_{12;34}$ $p_{14;23}$ $p_{13;24}$

$$p_\sigma = \left|x_{13}x_{24}\right|^{-4\Delta_{0,\frac{1}{2}}} \sum_{(r,s)\in\mathcal{S}_\sigma} \left[D_\sigma^{(r,s)}\right]^2 \left|\mathcal{F}_{r,s}\left(z = \frac{x_{12}x_{34}}{x_{13}x_{24}}\right)\right|^2$$

$$\sigma = 1234, \quad 12;34, \quad 14;23, \quad 13;24$$



4-pts functions: crossing relations

	z	$1-z$	$1/z$	$z/(z-1)$	$1-1/z$	$1/(1-z)$
	id	(13)	(23)	(12)	(123)	(132)
	(13)(24)	(24)	(14)	(34)	(243)	(234)
	(12)(34)	(1234)	(1342)	(1324)	(134)	(143)
	(23)(14)	(1432)	(1243)	(1423)	(142)	(124)

$$p_{1234}(1 \leftrightarrow 3) = p_{1234}, \ p_{13;24}(1 \leftrightarrow 3) = p_{13;24}, \ p_{12;34}(1 \leftrightarrow 3) = p_{14;23}$$

$$\sum_{(r,s)\in\mathcal{S}_{1234}} \left[D_{1234}^{(r,s)}\right]^2 \left(\left|\mathcal{F}_{r,s}(z)\right|^2 - \left|\mathcal{F}_{r,s}(1-z)\right|^2\right) = 0$$

$$\sum_{(r,s)\in\mathcal{S}_{13;24}} \left[D_{13;24}^{(r,s)}\right]^2 \left(\left|\mathcal{F}_{r,s}(z)\right|^2 - \left|\mathcal{F}_{r,s}(1-z)\right|^2\right) = 0$$

$$\dots$$

A new (not-log) bootstrap solution

$\Big($Picco, Ribault, R.S '16,'19$\Big)$

$$R_1 = \sum_{(r,s)\in\left(2\mathbb{Z},\mathbb{Z}+\frac{1}{2}\right)} (-1)^{rs} C^{(r,s)} C^{(r,-s)} \mathcal{F}^s_{r,s}(1-z)\mathcal{F}^s_{r,-s}(1-\bar{z})$$

$$R_2 = \sum_{(r,s)\in\left(2\mathbb{Z},\mathbb{Z}+\frac{1}{2}\right)} C^{(r,s)} C^{(r,-s)} \mathcal{F}^s_{r,s}(z)\mathcal{F}^s_{r,-s}(\bar{z})$$

$$R_3 = \sum_{(r,s)\in\left(2\mathbb{Z},\mathbb{Z}+\frac{1}{2}\right)} (-1)^{rs} C^{(r,s)} C^{(r,-s)} \mathcal{F}^s_{r,s}(z)\mathcal{F}^s_{r,-s}(\bar{z})$$

$$R_1(z) = R_3(1-z) = |1-z|^{-4\Delta_{(0,\frac{1}{2})}} R_1\left(\frac{z}{z-1}\right)$$

$$R_2(z) = R_2(1-z) = |1-z|^{-4\Delta_{(0,\frac{1}{2})}} R_3\left(\frac{z}{z-1}\right)$$

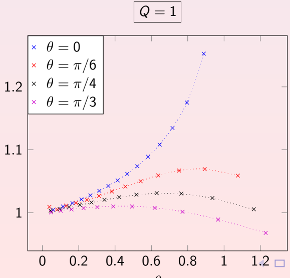
An educated guess...

$$R_1 = p_{1234} + \frac{2}{Q-2}\, p_{12;34}$$
$$R_2 = p_{1234} + \frac{2}{Q-2}\, p_{13;24}$$
$$R_3 = p_{1234} + \frac{2}{Q-2}\, p_{14;23}$$

- Exact for $Q = 0, 3, 4$.

$\big($$Q = 4$, Ashkin-Teller model, Zamolodchikov '86$\big)$

- Very good agreement with Monte-Carlo simultations

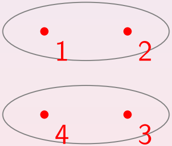


The correct statistical interpretation

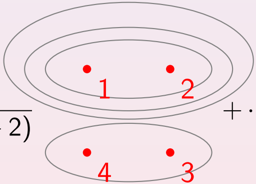
(Samuelsson, Liu, He, Jacobsen, Saleur '18,'19)

$$\tilde{p}_{12;34}$$

$$\frac{2}{Q-2}$$



$$-\frac{2(3Q-10)}{(Q-2)(Q^2-4Q+2)}$$



$$+\cdots$$

$$R_1 = p_{1234} + \tilde{p}_{12;34}$$

Bootstrap solution: $\lim_{p,q\to\infty}$ D(p,q) RSOS models

The Potts bootstrap log solutions

(Ninesvivat, Ribault '20, Samuelsson, Liu, He, Jacobsen, Saleur '20)

$$\begin{aligned}
p_{1234} + p_{12;34} = &\sum_{n \in 2\mathbb{N}+1} \left[C^{(0,\frac{n}{2})}\right]^2 \left|\mathcal{F}_{0,\frac{n}{2}}\right|^2 + \\
&+ \frac{2}{Q-2} \sum_{n \in 2\mathbb{N}+1} \left[C^{(2,\frac{n}{2})}\right]^2 \mathcal{F}_{2,\frac{n}{2}} \mathcal{F}_{2,-\frac{n}{2}} + \\
&- \frac{4}{(Q-1)(Q-2)(Q^2-4Q+2)} \sum_{n \in 2\mathbb{N}+1} \left[C^{(2,\frac{n}{2})}\right]^2 \mathcal{F}_{4,\frac{n}{2}} \mathcal{F}_{4,-\frac{n}{2}} + \ldots \\
&+ \sum_{n \in 4\mathbb{N}+1} \left[D^{(4,\frac{1}{4})}\right]^2 \mathcal{F}_{4,\frac{n}{4}} \mathcal{F}_{4,-\frac{n}{4}} + \sum_{n \in 3\mathbb{N}+1} \left[D^{(6,\frac{1}{3})}\right]^2 \mathcal{F}_{6,\frac{n}{3}} \mathcal{F}_{6,-\frac{n}{3}} + \cdots \\
&+ \left[D^{(2,0)}\right]^2 \mathcal{F}_{2,0} \mathcal{F}_{2,0} + \sum_{n \in \mathbb{N}^*} \left[D^{(2,n)}\right]^2 |\mathcal{G}^{\text{reg}}_{2,n}|^2 + \cdots
\end{aligned}$$

Origin of log structures (I)

- Null-vectors: If $(r, s) \in (\mathbb{N}^*, \mathbb{N}^*)$

$$\exists\ \eta_{rs} \in \mathcal{V}_{r,s} \text{ of zero norm } \langle \eta_{rs} | \eta_{rs} \rangle = 0$$

  - $\eta_{r,s} = 0$, always true for positive definite inner-product $\langle ... | ... \rangle$ (unitary CFT)

$$\mathcal{V}_{r,s} \to \frac{\mathcal{V}_{r,s}}{[\eta_{rs}]} \text{ implies fusion rules}$$

$$\text{Example: } \langle V_{1,2} V_{r,s} V_{r',s'} \rangle = 0 \text{ if } r \neq r' \ \&\ s' \neq s \pm 1$$

  - $\eta_{r,s} \neq 0$, possible if inner-product $\langle ... | ... \rangle$ is non definite (non-unitary CFT)

$$\text{Example: } \langle V_{1,1} V_{r,s} V_{r',s'} \rangle \neq 0 \text{ for } r \neq r' \ \&\ s' \neq s$$

Origin of log structures (II)

(R.S., Viti '13)

A fixed c, log-conformal block con be obtained by a limit of no-log ones

$$\left[C^{(r,s+\epsilon)}\right]^2 \left|\mathcal{F}^{s}_{r,s+\epsilon}\right|^2 + \left[C^{(r,-s-\epsilon)}\right]^2 \left|\mathcal{F}^{s}_{r,-s-\epsilon}\right|^2$$

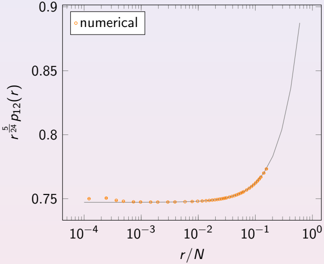
for $r,s \in \mathbb{N}^*$

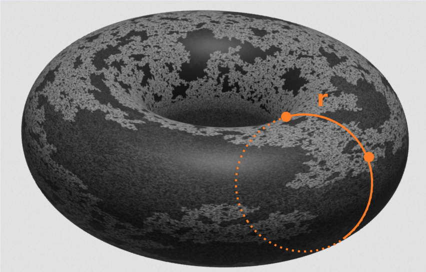
$$\rightarrow \frac{0}{\epsilon^2} + \frac{0}{\epsilon} + |\mathcal{G}^{\mathrm{reg}}_{r,s}(x)|^2$$

$V_{r,-s}V_{r,-s}$, and $\eta_{r,s}\bar{\eta}_{r,s}$

form the rank 2 log-partner

Potts CFT on a torus





(Javerzat, Picco, R.S. '18' 19)

$$p_{12} = \frac{1}{r^{\frac{5}{24}}} \left[1 + \left(\frac{r}{N}\right)^{\frac{5}{4}} \left((2\pi)^{\frac{5}{4}} \pi \sqrt{3} \left(\frac{4}{9} \frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{1}{4})} \right)^2 e^{-\frac{5\pi}{24}} + O\left(e^{-\frac{53}{24}\pi \frac{M}{N}} \right) \right) + \right.$$

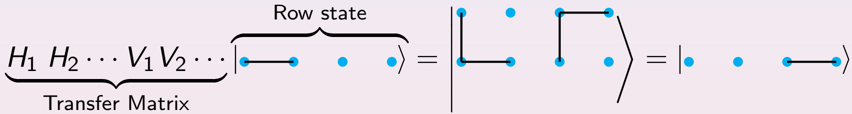
$$\left. + O\left(\left(\frac{r}{N} \right)^2 \right) \right].$$

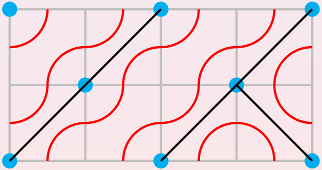
Conclusions

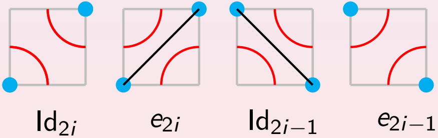
- The two boostrap solutions are probably the first of a hierarchy of solutions describing the scaling limit of Temperlie-Lieb algebra models
- Some of the structure constants have been obtained numerically using bootstrap. Analytic expressions will be probably soon derived
- Another class of 2D systems hints to new boostrap solutions: the disordered fixed points (i.e. Potts bond quenched disorder)
- We are exploring new 2D critical points in non-integrable deformation of percolation model. Results are quite promising..

Transfer matrix and Temperley-Lieb algebra

$$Z = \sum_{\mathcal{G}} \text{Probability}(\mathcal{G}) = \text{Tr}\left[\text{Transfer Matrix}^{M}\right]$$





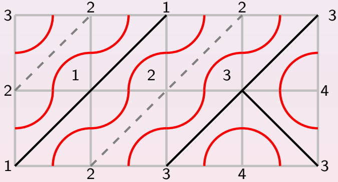


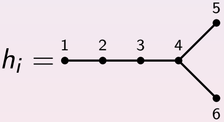
$$H_i = Q\ \text{Id}_{2i-1} + \ e_{2i-1}, \quad V_i = \text{Id}_{2i} + Q\ e_{2i}$$

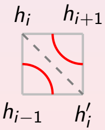
$$e_i^2 = Q\ e_i,\ \ e_{i-1}e_ie_{i+1} = e_i, e_ie_j = e_je_i \text{ for } |i-j| > 2$$

Other Temperley-Lieb models: D type RSOS model

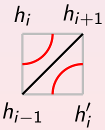
$$\sqrt{Q} = 2\cos\left(\frac{\pi(p-q)}{p}\right),\ p = 2 \bmod 4, p > q, p \wedge q = 1$$







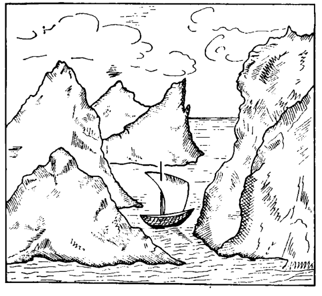
$\mathrm{Id}_i = \delta_{h_i,h'_i}$



$$e_i = \frac{(S_{h_{i-1}}S_{h_{i+1}})^{\frac{1}{2}}}{S_{h_i}}\delta_{h_{i-1},h_{i+1}}$$

Conformal symmetry in percolative Gaussian random fields

(Nina Javerzat, Sebastian Grijalva, Alberto Rosso, R.S.)



$$u(\mathbf{x}) = \sum_{\mathbf{k}} \lambda_{\mathbf{k}}^{-\frac{H+1}{2}} \; \hat{w}(\mathbf{k}) \; e^{i \, \mathbf{k} \, \mathbf{x}}$$

$$\hat{w}(\mathbf{k}) = \text{i.i.d } \mathcal{N}(0,1)$$

$$\lambda_{\mathbf{k}}, e^{i \, \mathbf{k} \, \mathbf{x}} = \text{Eigensyst of discrete torus } \nabla$$

- Long-range correlated site percolation model

$$\text{Prob}(\mathbf{x} \text{ and } \mathbf{y} \text{ activated}) \sim |\mathbf{x}-\mathbf{y}|^{2H} :$$

- Universality critical point:
  $H < 3/4$: pure percolation ($H = -1$). $3/4 < H < 0$: New ones (conformal?)