Riemann surfaces for KPZ fluctuations in finite volume

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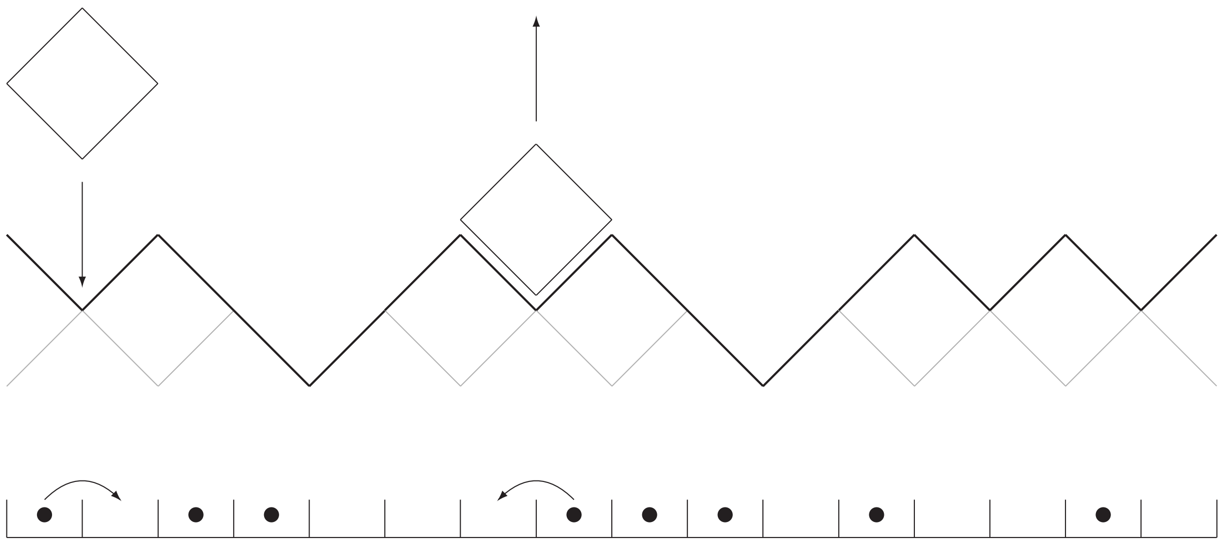
I ASEP and KPZ fluctuations



II Bethe ansatz and Riemann surfaces

III Height fluctuations of TASEP

IV $g \to \infty \quad \Rightarrow \quad$ KPZ fluctuations

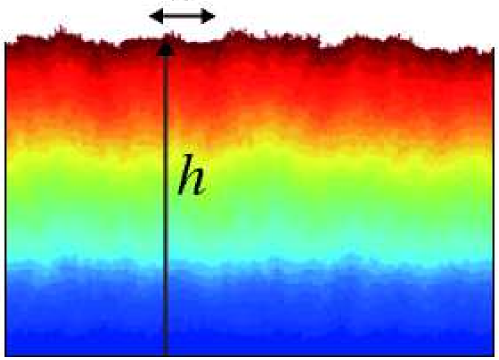
KPZ fluctuations in $1+1$ dimension

universal statistics random field $h(x,t)$

time evolution of probabilities integrable

Interface growth

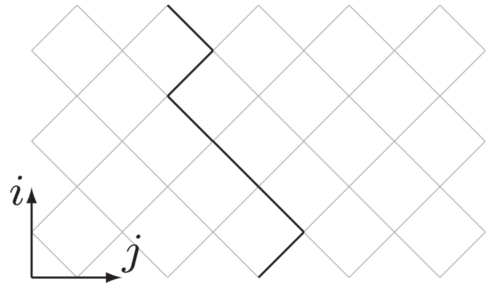
height $h(x,t)$



(Takeuchi-Sano 2010)

Directed polymer in random medium

free energy

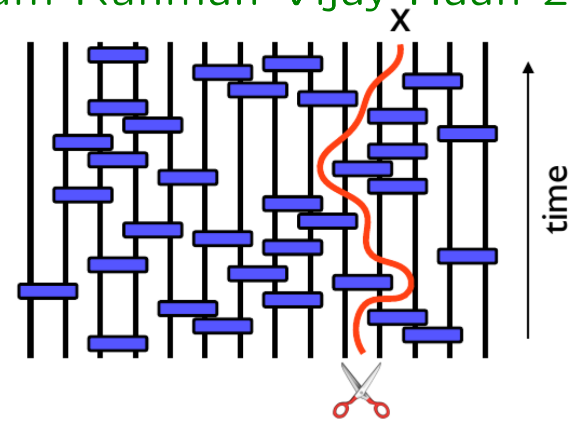


$$E = \sum_{(i,j)\in \text{path}} \varepsilon_{i,j}$$

Random unitary dynamics

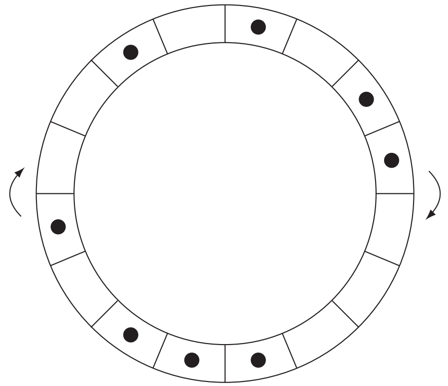
entanglement entropy

(Nahum-Ruhman-Vijay-Haah 2017)



Driven particles

current

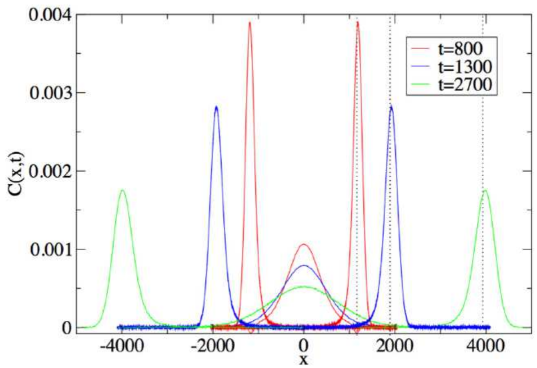


$$\partial_t \rho = \partial_x^2 \rho + \partial_x J(\rho) + \partial_x \xi$$

1D classical / quantum fluids with few conservation laws

normal modes hydrodynamics

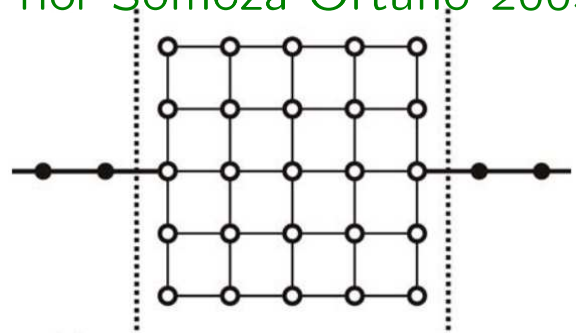
(Van Beijeren 2012, Spohn 2014)



Localization

conductance g

(Prior-Somoza-Ortuño 2005)



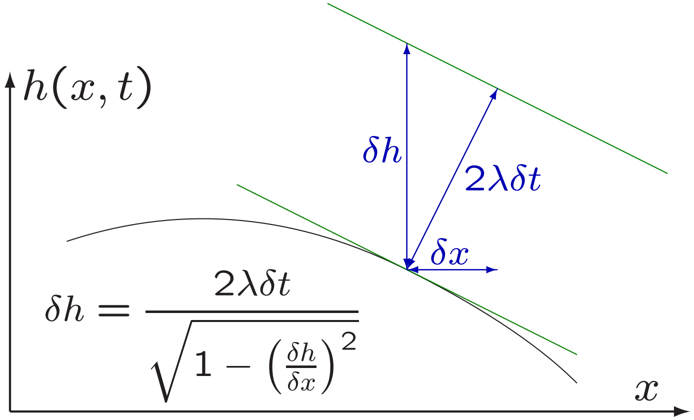
$$\log g \simeq -2L/\ell + \alpha (L/\ell)^{1/3} h$$

Kardar-Parisi-Zhang (KPZ) equation

Stable thermodynamic phase growing inside metastable phase
KPZ equation $\partial_t h(x,t) = \nu\,\partial_x^2 h(x,t) - \lambda\,(\partial_x h(x,t))^2 + \sqrt{D}\,\xi(x,t)$

Gaussian white noise $\xi \quad \langle \xi(x,t) \rangle = 0$
$\langle \xi(x,t)\xi(x',t') \rangle = \delta(x-x')\delta(t-t')$

Boundary conditions for system of size ℓ
- periodic $h(x+\ell,t) = h(x,t)$
- open $\partial_x h(0,t) = \rho_- \quad \partial_x h(\ell,t) = \rho_+$



Singular non-linear stochastic PDE (Hairer, Kupiainen, Gubinelli-Perkowski)

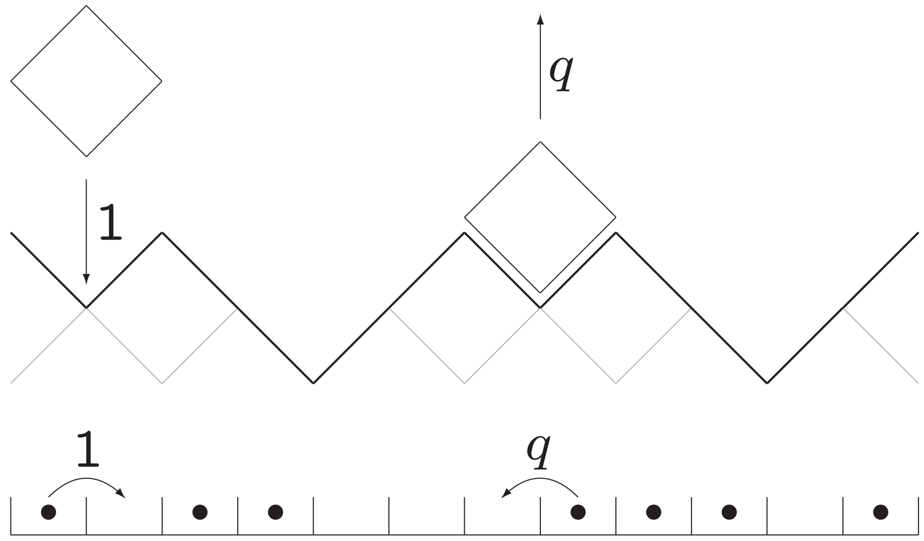
Only one parameter λ after rescaling space, time, height in finite volume

Large scale behaviour: two fixed points under renormalization group flow
- Edwards-Wilkinson $\lambda \to 0$ (repulsive) $z = 2$ interface at equilibrium
- KPZ fixed point $\lambda \to \infty$ (attractive) $z = 3/2$ irreversible evolution

Universality (at fixed points, but also RG flow EW $\to$ KPZ)

Asymmetric simple exclusion process (ASEP)

Continuous time Markov process
L sites, N particles, exclusion



Total time-integrated current
$Q(t) = \sum_{i=1}^{L} (H_i(t) - H_i(0))$

$$\left\langle \mathrm{e}^{\gamma Q(t)} \right\rangle = \sum_{\mathcal{C} \in \Omega} \langle \mathcal{C} | \mathrm{e}^{t M(\gamma)} | P_0 \rangle$$

$M(\gamma) \sim H_{\mathrm{XXZ}}$ twisted and non-Hermitian $\qquad \Delta = (q^{1/2} + q^{-1/2})/2 \geq 1$

KPZ equation at large scales for typical height fluctuations when $1 - q \sim \lambda/\sqrt{L}$

- Edwards-Wilkinson fixed point $\lambda \to 0$: SSEP $q = 1$ $\qquad \Delta - 1 \sim \lambda^2/L$
- KPZ fixed point $\lambda \to \infty$: TASEP $q = 0$ ($\Delta \to \infty$) sufficient

Conditioning on small / large height for ASEP beyond KPZ regime
$\Rightarrow$ crossover phase separation / conformal invariance (Karevski-Schütz 2017)

KPZ fluctuations in finite volume: several approaches

KPZ fixed point in finite volume: random field $h(x,t)$ $\quad x \equiv x+1$

Initial condition $h(x,0) = h_0(x)$: $\begin{cases} \text{flat } h_0(x) = 0 \\ \text{sharp wedge } h_0(x) = -|x|/0 \\ \text{stationary } h_0(x) = b(x) \text{ Brownian bridge} \end{cases}$

General n-point statistics $\mathbb{P}(h(x_1,t_1) > u_1, \ldots, h(x_n,t_n) > u_n)$

TASEP: expansion over Bethe eigenstates

- Euler-Maclaurin asymptotics: singularities, very tedious (P. 2016)
- Riemann surfaces: analytic continuation eigenstates ⇒ simpler expressions
  same kind of structures TASEP and KPZ (P. 2020)

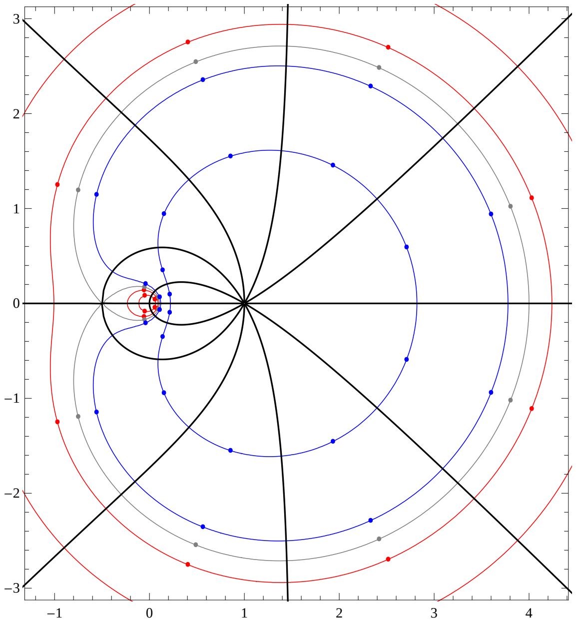
TASEP: integral formula for propagator ⇒ rigorous approach (Baik-Liu 2018)

Replica method: continuum ⇒ attractive δ-Bose gas (Brunet-Derrida 2000)

I ASEP and KPZ fluctuations

II Bethe ansatz and Riemann surfaces

$$C(1-y)^L = (-1)^{N-1} y^N \sim \widehat{\mathbb{C}}$$



III Height fluctuations of TASEP

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Bethe ansatz for TASEP

	ASEP $0 < q < 1 \quad (\Delta > 1)$	TASEP $q = 0 \quad (\Delta \to \infty)$
Bethe equations	$\mathrm{e}^{L\gamma}\left(\frac{1-y_j}{1-qy_j}\right)^L = -\prod_{k=1}^{N} \frac{y_j - qy_k}{qy_j - y_k}$	$C\,(1-y_j)^L = (-1)^{N-1} y_j^N$
Eigenvalue E	$\sum_{j=1}^{N}\left(\frac{1-q}{1-y_j} - \frac{1-q}{1-qy_j}\right)$	$\sum_{j=1}^{N} \frac{y_j}{1-y_j}$
Eigenvector $\psi(\vec{x})$	$\sum_{\sigma \in S_N} A_\sigma(\vec{y}) \prod_{j=1}^{N}\left(\mathrm{e}^{\gamma}\frac{1-y_j}{1-qy_j}\right)^{x_{\sigma(j)}}$	$\det\left(y_j^{-k}(1-y_j)^{x_k}\mathrm{e}^{\gamma x_k}\right)_{j,k}$
Gaudin det. $\langle\psi(\vec{x})\vert\psi(\vec{x})\rangle$	$\det\left(\partial_{y_i}\log\left(\left(\frac{1-y_j}{1-qy_j}\right)^L \prod_{k=1}^{N}\frac{qy_j - y_k}{y_j - qy_k}\right)\right)_{i,j}$	$\sum_{j=1}^{N}\frac{y_j}{N + (L-N)y_j}$

"Mean field" Bethe equations for TASEP: parameter $C = \mathrm{e}^{L\gamma}\prod_{k=1}^{N} y_k$

$\Rightarrow$ compact Riemann surface $\mathcal{R}_N$

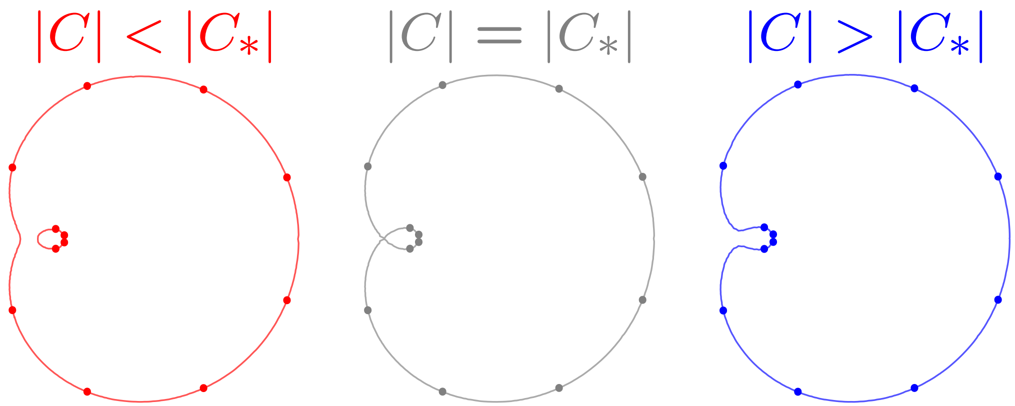
Symmetric functions of N Bethe roots $y_j \Rightarrow$ meromorphic functions on $\mathcal{R}_N$

Bethe root functions $y_j(C)$

$$C(1-y)^L + (-1)^N y^N = 0$$

Generalized Cassini ovals

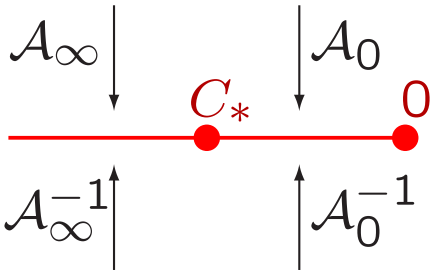
$$|C|\,|1-y|^L = |y|^N$$



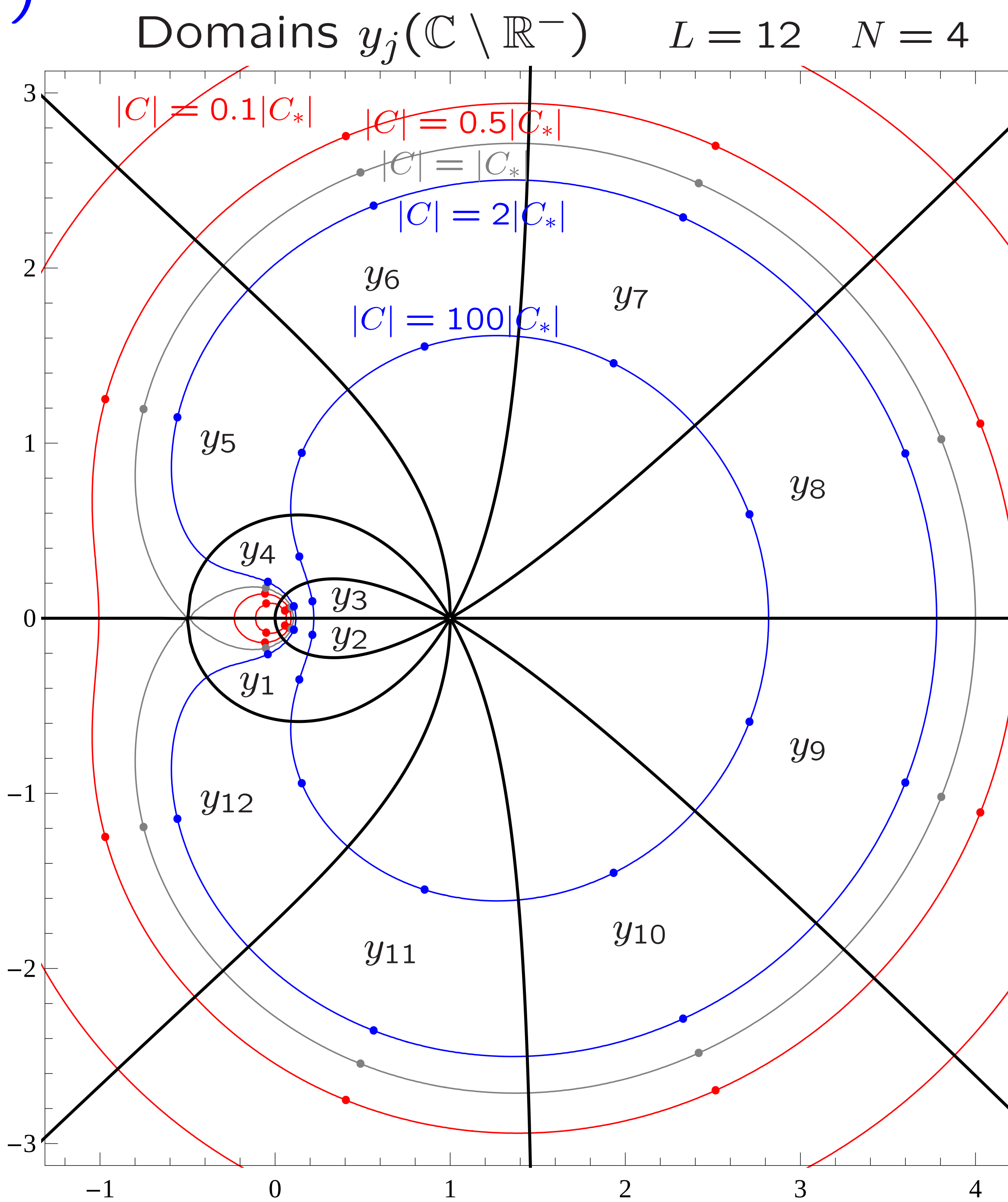
L solutions $y_j(C)$ analytic in $\mathbb{C} \setminus \mathbb{R}^-$

Generators of analytic continuations

$\mathcal{A}_0, \mathcal{A}_\infty$: $y_j \to y_k$



Group $G = S_L$
iff L, N co-prime



Riemann surface $\mathcal{R}_1 \sim \widehat{\mathbb{C}}$ for a single Bethe root

L sheets glued together along cuts $(-\infty, C_*)$, $(C_*, 0)$ according to analytic continuations of y_j $\Rightarrow$ Riemann surface $\mathcal{R}_1$ $[C, j]$, $C \in \mathbb{C}$, $j \in [\![1, L]\!]$

Meromorphic function y on $\mathcal{R}_1$

$$y([C, j]) = y_j(C)$$

Covering map $\pi_1 : \mathcal{R}_1 \to \widehat{\mathbb{C}}$

$$\pi_1([C, j]) = C$$

Identifications

$$y^{-1}(0) = [0, 1] = \ldots = [0, N]$$
$$y^{-1}(\infty) = [0, N+1] = \ldots = [0, L]$$
$$y^{-1}(1) = [\infty, 1] = \ldots = [\infty, L]$$
$$y^{-1}(-\tfrac{\rho}{1-\rho}) = [C_* - \mathrm{i}\epsilon, 1] = [C_* + \mathrm{i}\epsilon, N] = [C_* - \mathrm{i}\epsilon, N+1] = [C_* + \mathrm{i}\epsilon, L]$$

Riemann-Hurwitz formula: genus

$$g_{\mathcal{M}} = d(g_{\mathcal{N}} - 1) + 1 + \tfrac{1}{2}\textstyle\sum_{p \in \mathcal{M}}(e_p - 1)$$

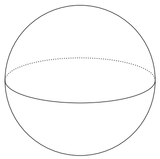
Euler charac. $\chi = 2 - 2g = V - E + F$ for graph on $\mathcal{M}$ linking ramif. points

Covering map $\pi : \mathcal{M} \to \mathcal{N}$ degree d

Ramification index e_p, $p \in \mathcal{M}$: winding number π(circle around p)

Ramification points $p \in \mathcal{M}$: $e_p \geq 2$ $\Rightarrow$ branch points $\pi(p) \in \mathcal{N}$

$\mathcal{R}_1$: genus $g = 0 \Leftrightarrow \mathcal{R}_1 \sim \widehat{\mathbb{C}}$ Riemann sphere



Riemann surface $\mathcal{R}_N$ for sym. functions N Bethe roots

Eigenstate: choice N Bethe roots among L
Symmetric functions $s(y_{j_1}(C), \ldots, y_{j_N}(C))$

$\Rightarrow$

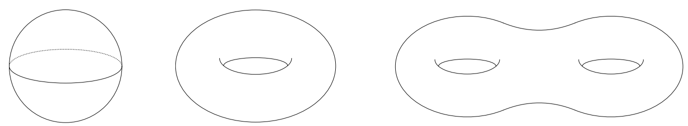
Riemann surface $\mathcal{R}_N$: $[C, J]$
$C \in \mathbb{C}$, $J \subset [\![1, L]\!]$, $|J| = N$

Covering map $\pi_N : \begin{array}{c} \mathcal{R}_N \to \widehat{\mathbb{C}} \\ [C, J] \mapsto C \end{array}$

TASEP height fluctuations: $\mathrm{tr}_{\pi_N} = \sum_J$

Several connected components
if L and N not co-prime

$L \backslash N$	1	2	3	4	5	6	7	8	9
2	1	·	·	·	·	·	·	·	·
3	1	1	·	·	·	·	·	·	·
4	1	2	1	·	·	·	·	·	·
5	1	1	1	1	·	·	·	·	·
6	1	2	3	2	1	·	·	·	·
7	1	1	1	1	1	1	·	·	·
8	1	2	1	6	1	2	1	·	·
9	1	1	4	1	1	4	1	1	·
10	1	2	1	3	11	3	1	2	1

Genus 

$L \backslash N$	1	2	3	4	5	6	7	8	9
2	0	·	·	·	·	·	·	·	·
3	0	0	·	·	·	·	·	·	·
4	0	0	0	·	·	·	·	·	·
5	0	0	0	0	·	·	·	·	·
6	0	0	0	0	0	·	·	·	·
7	0	0	1	1	0	0	·	·	·
8	0	0	2	1	2	0	0	·	·
9	0	0	1	7	7	1	0	0	·
10	0	0	4	8	7	8	4	0	0

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$$\mathbb{P}(H_{i_\ell}(t_\ell) \geq H_\ell, \ell \in [\![1,n]\!]) = \oint_{|C_n|<\ldots<|C_1|} \frac{\mathrm{d}C_1 \ldots \mathrm{d}C_n}{(2\mathrm{i}\pi)^n \, C_n} \Bigg(\prod_{\ell=1}^{n} \sum_{J_\ell \subset [\![1,L]\!],\, |J_\ell|=N}$$

$$\frac{\prod_{\ell=1}^{n} \mathrm{e}^{\int_O^{[C_\ell,J_\ell]} \frac{\mathrm{d}C}{C} \left(\frac{NL}{L-N} \mu([C,\cdot])^2 + \left(H_\ell - H_{\ell-1}\right) \frac{L\,\mu([C,\cdot])}{L-N} + (t_\ell - t_{\ell-1}) \left(\frac{\eta([C,\cdot])}{L} - \frac{N\,\mu([C,\cdot])}{L} \right) \right)}}{\prod_{\ell=1}^{n-1} \left((C_\ell - C_{\ell+1})\, \mathrm{e}^{\frac{NL}{L-N} \int_\gamma \frac{\mathrm{d}B}{B} \mu([C_\ell B,\cdot])\, \mu([C_{\ell+1} B,\cdot])} \right)} \Bigg)$$

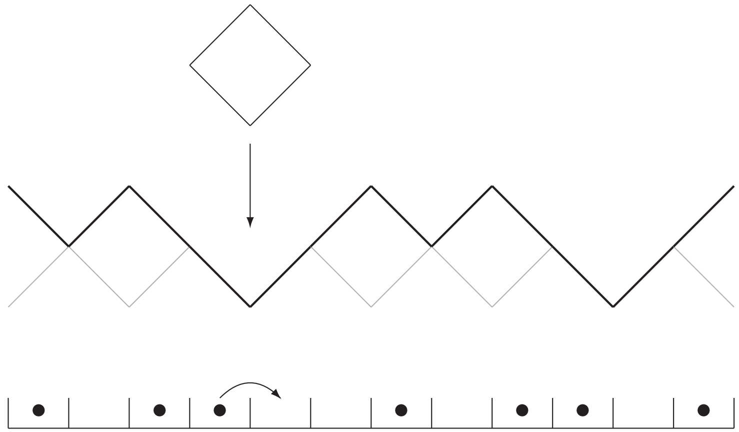
IV $g \to \infty \quad \Rightarrow \quad$ KPZ fluctuations

Generating function of TASEP height

Height $H_i(t)$ at site i and time t

Height increments $H_i(t) - H_i(0) \in \mathbb{N}$

Initial height $H_i(0) = \sum_{j=1}^{i}(\frac{N}{L} - n_i)$



One-point generating function average height

$$\left\langle e^{\gamma \sum_{i=1}^{L}(H_i(t)-H_i(0))} \right\rangle = \sum_{\mathcal{C}\in\Omega} \langle \mathcal{C}| e^{tM(\gamma)} |P_0\rangle \qquad M(\gamma) \sim H_{\mathrm{XXZ}}$$

Markov property (memoryless) $\Rightarrow$ generating function at times $0 < t_1 < \ldots < t_n$

$$\left\langle e^{\sum_{\ell=1}^{n} \gamma_\ell (H_{i_\ell}(t_\ell) - H_{i_\ell}(0))} \right\rangle = \sum_{\mathcal{C}\in\Omega} \langle \mathcal{C}| \Big(\prod_{\ell=n}^{1} (e^{-\gamma_\ell S_{i_\ell}} e^{(t_\ell - t_{\ell-1})\, M(\sum_{m=\ell}^{n} \gamma_m / L)}) \Big) e^{\sum_{\ell=1}^{n} \gamma_\ell S_{i_\ell}} |P_0\rangle$$

$S_i|\mathcal{C}\rangle = \left(\frac{1}{L}\sum_{j=1}^{N}[x_j]_i\right)|\mathcal{C}\rangle$ with $[x]_i$ positions counted from site i

Expansion over eigenstates of $M(\gamma)$

Eigenvectors $|\psi_r(\gamma)\rangle$ of $\quad M(\gamma) \qquad$ eigenvalue $E_r(\gamma)$
translation $\quad$ eigenvalue $\mathrm{e}^{\mathrm{i}P_r/L}$, $P_r \in 2\pi\mathbb{Z}$

$M_0(L\gamma) = \mathrm{e}^{L\gamma S_0} M(\gamma)\,\mathrm{e}^{-L\gamma S_0}$ counting current between sites L and 1 only:
Eigenvectors $|\psi_r^0(\gamma)\rangle = \mathrm{e}^{L\gamma S_0}|\psi_r(\gamma)\rangle$ with $S_0|\mathcal{C}\rangle = \left(\frac{1}{L}\sum_{j=1}^{N} x_j\right)|\mathcal{C}\rangle$

$$\left\langle \mathrm{e}^{\sum_{\ell=1}^{n}\gamma_\ell H_{i_\ell}(t_\ell)} \right\rangle_{P_0} = \sum_{r_1,\ldots,r_n=1}^{|\Omega|} \left(\prod_{\ell=1}^{n} \frac{\mathrm{e}^{(t_\ell - t_{\ell-1})E_{r_\ell}\left(\sum\limits_{m=\ell}^{n}\gamma_m/L\right) - \mathrm{i}(i_\ell - i_{\ell-1})P_{r_\ell}/L}}{\left\langle \psi_{r_\ell}^0\left(\sum\limits_{m=\ell}^{n}\frac{\gamma_m}{L}\right)\middle|\psi_{r_\ell}^0\left(\sum\limits_{m=\ell}^{n}\frac{\gamma_m}{L}\right)\right\rangle} \right.$$
$$\times \left(\sum_{\mathcal{C}\in\Omega} \left\langle \mathcal{C}\middle|\psi_{r_n}^0\left(\frac{\gamma_n}{L}\right)\right\rangle\right)\left\langle \psi_{r_1}^0\left(\sum_{m=1}^{n}\frac{\gamma_m}{L}\right)\middle|P_0\right\rangle$$
$$\left.\times \left(\prod_{\ell=1}^{n-1} \left\langle \psi_{r_{\ell+1}}^0\left(\sum_{m=\ell+1}^{n}\frac{\gamma_m}{L}\right)\middle|\psi_{r_\ell}^0\left(\sum_{m=\ell}^{n}\frac{\gamma_m}{L}\right)\right\rangle\right)\right)$$

Gaudin det. $\langle\psi(\vec{y})|\psi(\vec{y})\rangle$ + Slavnov det. $\langle\psi(\vec{w})|\psi(\vec{y})\rangle \Rightarrow$ scalar products
(Bogoliubov 2009, Motegi-Sakai 2013, P. 2016)

Bethe ansatz formula for the generating function

Initial condition $\mathcal{C}_0 : 1 \leq x_1^{(0)} < \ldots < x_N^{(0)} \leq L$

$(y_1^{(\ell)}, \ldots, y_N^{(\ell)})$ solution of the Bethe equations with fugacity $\sum_{m=\ell}^{n} \gamma_m / L$

$$\left\langle \mathrm{e}^{\sum_{\ell=1}^{n} \gamma_\ell H_{i_\ell}(t_\ell)} \right\rangle_{\mathcal{C}_0} = \Bigg(\prod_{\ell=1}^{n} \sum_{\vec{y}^{(\ell)}} \Bigg) \frac{\det \left((y_j^{(1)})^{k-1} (1 - y_j^{(1)})^{-x_k^{(0)}} \right)_{j,k \in [\![1,N]\!]}}{\prod_{j=1}^{N} \prod_{k=j+1}^{N} \left(y_j^{(1)} - y_k^{(1)} \right)} \frac{1}{\prod_{j=1}^{N} (y_j^{(n)})^N}$$

$$\times \left(\prod_{\ell=1}^{n-1} \frac{(-1)^{\frac{N(N-1)}{2}} \left(1 - \frac{\mathrm{e}^{\sum_{m=\ell+1}^{n} \gamma_m} \prod_{j=1}^{N} y_j^{(\ell+1)}}{\mathrm{e}^{\sum_{m=\ell}^{n} \gamma_m} \prod_{j=1}^{N} y_j^{(\ell)}} \right)^{N-1}}{\prod_{j=1}^{N} \prod_{k=1}^{N} \left(y_j^{(\ell)} - y_k^{(\ell+1)} \right)} \right) \left(\prod_{\ell=1}^{n} \prod_{j=1}^{N} \prod_{k=j+1}^{N} \left(y_j^{(\ell)} - y_k^{(\ell)} \right)^2 \right)$$

$$\times \prod_{\ell=1}^{n} \frac{(1 - \mathrm{e}^{-\gamma_\ell})\, \mathrm{e}^{\frac{N i_\ell \gamma_\ell}{L}} \left(\prod_{j=1}^{N} y_j^{(\ell)} (1 - y_j^{(\ell)})^{1 + i_\ell - i_{\ell-1}} \right) \exp\left((t_\ell - t_{\ell-1}) \sum_{j=1}^{N} \frac{y_j^{(\ell)}}{1 - y_j^{(\ell)}} \right)}{\left(\frac{L}{N} \sum_{j=1}^{N} \frac{y_j^{(\ell)}}{N + (L-N) y_j^{(\ell)}} \right) \prod_{j=1}^{N} \left(N + (L-N) y_j^{(\ell)} \right)}$$

Large L asymptotics in KPZ regime: doable using Euler-Maclaurin but tedious

Better approach: write before probability in terms of functions on $\mathcal{R}_N$

Probability of the height

Generating function ($g_\ell = \mathrm{e}^{\gamma_\ell}$) $\Rightarrow$ probability

$$\left\langle \prod_{\ell=1}^{n} g_\ell^{H_{i_\ell}(t_\ell)} \right\rangle = \left(\prod_{\ell=1}^{n} \sum_{U_\ell=0}^{\infty} \right) \mathbb{P}(H_{i_\ell}(t_\ell) = H_{i_\ell}^{(0)} + U_\ell, \ell \in [\![1,n]\!]) \prod_{\ell=1}^{n} g_\ell^{H_{i_\ell}^{(0)}+U_\ell}$$

$$\Rightarrow \mathbb{P}(H_{i_\ell}(t_\ell) = H_{i_\ell}^{(0)} + U_\ell, \ell \in [\![1,n]\!]) = \oint \left(\prod_{\ell=1}^{n} \frac{\mathrm{d}g_\ell}{g_\ell^{1+H_{i_\ell}^{(0)}+U_\ell}} \right) \left\langle \prod_{\ell=1}^{n} g_\ell^{H_{i_\ell}(t_\ell)} \right\rangle$$

Change of variable $g_\ell \to C_\ell = (\prod_{m=\ell}^{n} g_m) \prod_{j=1}^{N} y_j^{(\ell)}$

Jacobian $\det(\partial_{C_\ell} g_m)_{l,m\in[\![1,n]\!]} = \frac{C_1}{\prod_{j=1}^{n} y_j^{(1)}} \prod_{\ell=1}^{n} \left(\frac{1}{C_\ell} \frac{L}{N} \sum_{j=1}^{n} \frac{y_j^{(\ell)}}{N+(L-N)y_j^{(\ell)}} \right)$

Exponential representation:

$$y_j'(C) = \frac{1}{C} \frac{y_j(C)\,(1-y_j(C))}{N+(L-N)\,y_j(C)}$$

$$\frac{\prod_{j=1}^{N} f(y_j^{(\ell)})}{\prod_{j=1}^{N} \prod_{k=j+1}^{N} \left(y_j^{(\ell)} - y_k^{(\ell)}\right)^2 \prod_{j=1}^{N} \prod_{k=1}^{N} \left(y_j^{(\ell)} - y_k^{(\ell+1)}\right)} \quad \to \quad \exp(\textstyle\int \ldots) \text{ integration on Riemann surface } \mathcal{R}_N$$

Exponential representation 1: integrand

L and N co-prime $\Rightarrow \mathcal{R}_N$ has a single connected component

Domain wall initial condition $x_k^{(0)} = k + L - N$

$$\mathbb{P}(H_{i_\ell}(t_\ell) \geq H_\ell, \ell \in [\![1,n]\!]) = \oint_{|C_n|<\ldots<|C_1|} \frac{\mathrm{d}C_1 \ldots \mathrm{d}C_n}{(2\mathrm{i}\pi)^n\, C_n} \Bigg(\prod_{\ell=1}^{n} \sum_{J_\ell \subset [\![1,L]\!],\; |J_\ell|=N} \Bigg)$$

$$\frac{\prod_{\ell=1}^{n} \mathrm{e}^{\int_O^{[C_\ell, J_\ell]} \frac{\mathrm{d}C}{C} \left(\frac{NL}{L-N} \mu([C,\cdot])^2 + \left(H_\ell - H_{\ell-1}\right) \frac{L\,\mu([C,\cdot])}{L-N} + (t_\ell - t_{\ell-1}) \left(\frac{\eta([C,\cdot])}{L} - \frac{N\,\mu([C,\cdot])}{L} \right)\right)}}{\prod_{\ell=1}^{n-1} \Big((C_\ell - C_{\ell+1})\, \mathrm{e}^{\frac{NL}{L-N} \int_\gamma \frac{\mathrm{d}B}{B} \mu([C_\ell B,\cdot])\, \mu([C_{\ell+1}B,\cdot])} \Big)}$$

Arbitrary paths from $O = [0, [\![1,N]\!]]$ to $[C_\ell, J_\ell]$ on $\mathcal{R}_N$

Path $(O,O) \to ([C_\ell, J_\ell], [C_{\ell+1}, J_{\ell+1}])$ for $([C_\ell B, \cdot], [C_{\ell+1} B, \cdot]) \in \mathcal{R}_N \times \mathcal{R}_N$

Meromorphic functions on $\mathcal{R}_N$

$$\begin{cases} \mu([C,J]) = -1 + \displaystyle\sum_{j \in J} \frac{1}{N + (L-N) y_j(C)} \\ \eta([C,J]) = -N + \displaystyle\sum_{j \in J} \frac{1}{1 - y_j(C)} \end{cases}$$

$\exp(\int \ldots)$ also meromorphic on $\mathcal{R}_N$, $\mathcal{R}_N \times \mathcal{R}_N$: periods $\in 2\mathrm{i}\pi\mathbb{Z}$ and residues $\in \mathbb{Z}$

Exponential representation 2: trace over π_N

L and N co-prime $\Rightarrow \mathcal{R}_N$ has a single connected component

Domain wall initial condition $x_k^{(0)} = k + L - N$

$$\mathbb{P}(H_{i_\ell}(t_\ell) \geq H_\ell, \ell \in [\![1,n]\!]) = \oint_{|C_n|<\ldots<|C_1|} \frac{\mathrm{d}C_1 \ldots \mathrm{d}C_n}{(2\mathrm{i}\pi)^n\, C_n} \Bigg(\prod_{\ell=1}^{n} \sum_{J_\ell \subset [\![1,L]\!],\; |J_\ell|=N} \Bigg)$$

$$\frac{\prod_{\ell=1}^{n} \mathrm{e}^{\int_O^{[C_\ell,J_\ell]} \frac{\mathrm{d}C}{C} \left(\frac{NL}{L-N}\, \mu([C,\cdot])^2 + \left(H_\ell - H_{\ell-1}\right) \frac{L\,\mu([C,\cdot])}{L-N} + (t_\ell - t_{\ell-1}) \left(\frac{\eta([C,\cdot])}{L} - \frac{N\,\mu([C,\cdot])}{L} \right) \right)}}{\prod_{\ell=1}^{n-1} \left((C_\ell - C_{\ell+1})\, \mathrm{e}^{\frac{NL}{L-N} \int_\gamma \frac{\mathrm{d}B}{B}\, \mu([C_\ell B,\cdot])\, \mu([C_{\ell+1} B,\cdot])} \right)}$$

Contour integrals around 0 in the complex plane $\mathbb{C}$

Trace $\sum_{J_\ell \subset [\![1,L]\!],\; |J_\ell|=N}$ over π_N: $\mathcal{R}_N \to \mathbb{C}$

Ex.: $\mathrm{e}^{\sqrt{C}}$ branch cut in $\mathbb{C}$
$\mathrm{e}^{\sqrt{C}} + \mathrm{e}^{-\sqrt{C}}$ analytic in $\mathbb{C}$

Remark: Positions i_ℓ appear only through the condition $H_\ell - H_{i_\ell}(0) \in \mathbb{Z}$

Exponential representation 3: general initial condition

General initial condition $1 \leq x_1^{(0)} < \ldots < x_N^{(0)} \leq L$

Extra factor $\Theta_{\vec{x}_0}([C_1, J_1])$: symmetric Grothendieck polynomial

$$\Theta_{x_1^{(0)},\ldots,x_N^{(0)}}([C,\{j_1,\ldots,j_N\}]) = \frac{\det\Big((y_{j_\lambda}(C))^{k-1}(1-y_{j_\lambda}(C))^{L-x_k^{(0)}}\Big)_{k,\lambda\in[\![1,N]\!]}}{\prod_{\kappa=1}^{N}\prod_{\lambda=\kappa+1}^{N}\Big(y_{j_\lambda}(C)-y_{j_\kappa}(C)\Big)}$$

Domain wall initial condition $x_k^{(0)} = k + i$

$$\Theta_{\mathrm{dw}}([C,J]) = \exp\left(\frac{N(L-N-i)}{L-N}\int_O^{[C,J]}\frac{\mathrm{d}B}{B}\,\mu([B,\cdot\,])\right)$$

Stationary initial condition: same weight for each $\{x_k^{(0)}, k \in [\![1,N]\!]\}$

$$\Theta_{\mathrm{stat}}([C,J]) = \frac{-1+\exp\Big(-\frac{L}{L-N}\int_O^{[C,J]}\frac{\mathrm{d}B}{B}\,\mu([B,\cdot\,])\Big)}{\binom{L}{N}\,C}$$

Pole structure on $\mathcal{R}_N$ of the integrand

Poles of function $g([C,J])$ on $\mathcal{R}_N \neq$ poles of differential $g([C,J])\,\mathrm{d}C$ on $\mathcal{R}_N$

Ex.: loc. param. $B = \sqrt{C}$ around branch point 0 $\Rightarrow$ $\frac{1}{\sqrt{C}} = \frac{1}{B}$ pole; $\frac{\mathrm{d}C}{\sqrt{C}} = 2\,\mathrm{d}B$ not pole

$$\mathbb{P}(H_{i_\ell}(t_\ell) \geq H_\ell, \ell \in [\![1,n]\!]) = \oint_{|C_n|<\ldots<|C_1|} \frac{\mathrm{d}C_1 \ldots \mathrm{d}C_n}{(2\mathrm{i}\pi)^n} \Big(\prod_{\ell=1}^{n} \sum_{\substack{J_\ell \subset [\![1,L]\!] \\ |J_\ell|=N}} \Big) f([C_1,J_1],\ldots,[C_n,J_n])$$

	Function f	Diff. $\frac{\mathrm{d}C_1 \ldots \mathrm{d}C_n}{(2\mathrm{i}\pi)^n} f([C_1,J_1],\ldots,[C_n,J_n])$
$C_\ell = 0$	Multiple pole	Multiple pole
$C_\ell = \infty$	Essential singularity	Essential singularity
$C_\ell = C_*$	Simple pole	Regular point $\rightsquigarrow$ also after trace
$[C_\ell, J_\ell] = [C_{\ell+1}, J_{\ell+1}]$	Simple pole	Simple pole

Open question: How much does pole structure constrain function f ?

I ASEP and KPZ fluctuations

II Bethe ansatz and Riemann surfaces

III Height fluctuations of TASEP

IV $g \to \infty \quad \Rightarrow \quad$ KPZ fluctuations

$$\mathbb{P}_{\mathsf{flat}}(h(y,3t) > x) = \int_{c-\mathsf{i}\pi}^{c+\mathsf{i}\pi} \frac{\mathsf{d}\nu}{2\mathsf{i}\pi}\,\tau_{\mathsf{KdV}}(x,t;\nu)$$

KPZ scaling limit $(t_\ell, i_\ell, H_\ell) \to (\tau_\ell, x_\ell, h_\ell)$

Infinite genus limit
$\mathcal{R}_N \to \mathcal{R}_{\mathrm{KPZ}}$
disconnected space

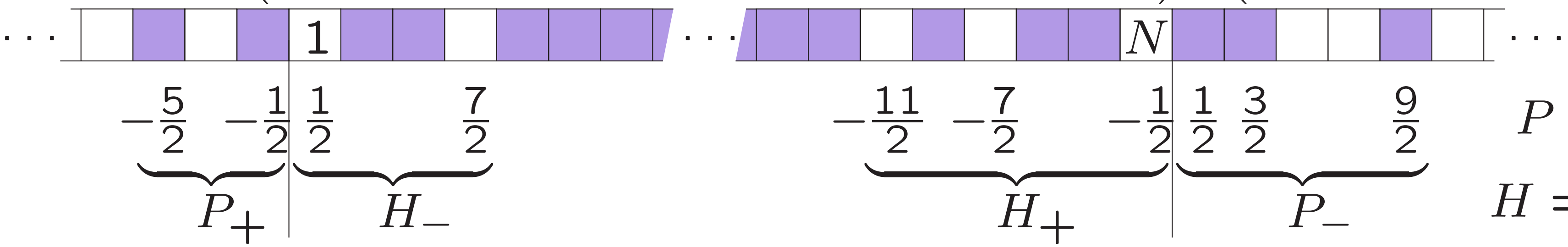
$L \to \infty$
$N \to \infty$
$N/L \simeq \rho$

with

$$t_\ell = \frac{\tau_\ell L^{3/2}}{\sqrt{\rho(1-\rho)}}$$
$$i_\ell = (1-2\rho)t_\ell + x_\ell L$$
$$H_\ell = \rho(1-\rho)t_\ell + \mathcal{H}_0 L + \sqrt{\rho(1-\rho)L}\, h_\ell$$

Only sheets $J = J_{P,H}$ contribute: particle-hole excitations at edge Fermi sea

$$J_{P,H} = \Big([\![1,N]\!] \setminus ((1/2 - H_-) \cup (N + 1/2 - H_+))\Big) \cup \Big((N+1/2-P_-) \cup (L+1/2-P_+)\Big)$$



$$P = P_+ \cup P_- \subset \mathbb{Z} + \tfrac{1}{2}$$
$$H = H_+ \cup H_- \subset \mathbb{Z} + \tfrac{1}{2}$$

$$\rho^N (1-\rho)^{L-N} C = \mathrm{e}^\nu \quad \Rightarrow \quad \mu([C, J_{P,H}]) \simeq -\sqrt{\frac{1-\rho}{\rho L}}\, \chi''_{P,H}(\nu)$$

$$\mathrm{Li}_s(z) = \textstyle\sum_{k=1}^\infty \frac{z^k}{k^s}$$

$$\chi_{P,H}(\nu) = -\frac{\mathrm{Li}_{5/2}(-\mathrm{e}^\nu)}{\sqrt{2\pi}} + \sum_{a\in P} \frac{(4\mathrm{i}\pi a)^{3/2}(1-\frac{\nu}{2\mathrm{i}\pi a})^{3/2}}{3} + \sum_{a \in H} \frac{(4\mathrm{i}\pi a)^{3/2}(1-\frac{\nu}{2\mathrm{i}\pi a})^{3/2}}{3}$$

Connected components $\mathcal{R}_{\mathrm{KPZ}}$ labelled by symmetric difference $\Delta = P \ominus H$

KPZ in finite volume with sharp wedge initial condition

Connected components $\mathcal{R}^{\Delta}_{\mathrm{KPZ}} \quad \to \quad \chi^{\Delta} = $ "$\mathrm{Li}_{5/2}$ minus some branch points"

$$\mathbb{P}(h(x_1,t_1) > u_1, \ldots, h(x_n,t_n) > u_n)$$

$$= \Bigg(\prod_{\ell=1}^{n} \sum_{\Delta_\ell \sqsubset \mathbb{Z}+1/2} \sum_{\substack{P_\ell \sqsubset \mathbb{Z}+1/2 \\ P_\ell \cap \Delta_\ell = \emptyset}} \Bigg) \int_{c_1-\mathrm{i}\pi}^{c_1+\mathrm{i}\pi} \frac{\mathrm{d}\nu_1}{2\mathrm{i}\pi} \ldots \int_{c_n-\mathrm{i}\pi}^{c_n+\mathrm{i}\pi} \frac{\mathrm{d}\nu_n}{2\mathrm{i}\pi} \, \Xi_{x_1,\ldots,x_n}^{\Delta_1,\ldots,\Delta_n}(\nu_1,\ldots,\nu_n)$$

$$\frac{\prod_{\ell=1}^{n} \mathrm{e}^{\fint_{[-\infty,\emptyset]}^{[\nu_\ell,P_\ell]} \mathrm{d}v \, \big((t_\ell - t_{\ell-1}) \chi'^{\Delta_\ell}([v,\cdot]) - (u_\ell - u_{\ell-1}) \chi''^{\Delta_\ell}([v,\cdot]) + \chi''^{\Delta_\ell}([v,\cdot])^2 \big)}}{\prod_{\ell=1}^{n-1} \mathrm{e}^{\fint_{\beta_{\ell,\ell+1}} \mathrm{d}v \, \chi''^{\Delta_\ell}([v+\nu_\ell,\cdot]) \chi''^{\Delta_{\ell+1}}([v+\nu_{\ell+1},\cdot])}}$$

Integrals from $O \to$ integrals from O^{Δ}, properly regularized $\Rightarrow$ momentum

$$\Xi_{x_1,\ldots,x_n}^{\Delta_1,\ldots,\Delta_n}(\nu_1,\ldots,\nu_n)$$

$$= \Bigg(\prod_{\ell=1}^{n} \sum_{\substack{A_\ell \subset \Delta_\ell \\ |A_\ell| = |\Delta_\ell \setminus A_\ell|}} \Bigg) \prod_{\ell=1}^{n} \Big(V_{A_\ell}^2 V_{\Delta_\ell \setminus A_\ell}^2 \, \mathrm{e}^{2\mathrm{i}\pi (x_\ell - x_{\ell-1}) \big(\sum_{a \in A_\ell} a - \sum_{a \in \Delta_\ell \setminus A_\ell} a \big)} \Big)$$

$$\times \prod_{\ell=1}^{n-1} \frac{(1 - \mathrm{e}^{\nu_{\ell+1} - \nu_\ell})^{|\Delta_\ell|/2} (1 - \mathrm{e}^{\nu_\ell - \nu_{\ell+1}})^{|\Delta_{\ell+1}|/2}}{(1 - \mathrm{e}^{\nu_{\ell+1} - \nu_\ell}) \, V_{A_\ell, A_{\ell+1}}(\nu_\ell, \nu_{\ell+1}) \, V_{\Delta_\ell \setminus A_\ell, \Delta_{\ell+1} \setminus A_{\ell+1}}(\nu_\ell, \nu_{\ell+1})}$$

KdV solitons ???

Flat initial condition: only connected component $\Delta = \emptyset$ of $\mathcal{R}_{\mathsf{KPZ}}$ contributes
Same particle-hole excitations at both edges Fermi sea $\Rightarrow$ zero // momentum

One-point distribution $\quad \mathbb{P}_{\mathsf{flat}}(h(y,3t) > x) = \int_{c-\mathrm{i}\pi}^{c+\mathrm{i}\pi} \frac{\mathrm{d}\nu}{2\mathrm{i}\pi}\,\tau(x,t;\nu)$

$u(x,t) = 2\partial_x^2 \log \tau(x,t)$ solution Korteweg-de Vries equation $4\partial_t u = 6u\partial_x u + \partial_x^3 u$

$$\tau(x,t;\nu) = \frac{\mathrm{e}^{3t\chi(\nu) - x\chi'(\nu) + \frac{1}{2}\int_{-\infty}^{\nu} \mathrm{d}v\,\chi''(v)^2}}{(1+\mathrm{e}^{\nu})^{1/4}} \det(1 - M(x,t;\nu)) \qquad \chi(\nu) = -\frac{\mathrm{Li}_{5/2}(-\mathrm{e}^{\nu})}{\sqrt{2\pi}}$$

Kernel $M(x,t;\nu)_{a,b} = \dfrac{\mathrm{e}^{2x\kappa_a(\nu) + 2t\kappa_a^3(\nu) + 2\int_{-\infty}^{\nu} \mathrm{d}v\,\frac{\chi''(v)}{\kappa_a(v)}}}{\kappa_a(\nu)\,(\kappa_a(\nu) + \kappa_b(\nu))} \qquad \kappa_a(\nu) = \sqrt{4\mathrm{i}\pi a}\sqrt{1 - \frac{\nu}{2\mathrm{i}\pi a}}$

Infinitely many solitons in interaction, with velocities $-\kappa_a(\nu)^2 = 2\nu - 4\mathrm{i}\pi a$

Other initial conditions: KP τ functions (Baik-Liu-Silva 2020)
KPZ fluctuations on $\mathbb{R}$: KdV / KP τ functions (Quastel-Remenik 2019)

Classical integrability hidden within KPZ: completely unexpected ???

Short time limit: KPZ fixed point on $\mathbb{R}$

Correlation length $\begin{cases} \text{grows as } t^{1/3} \text{ at short time} \\ \to \text{ system size when } t \to \infty \end{cases}$ $\underset{t\to 0}{\Rightarrow}$ recover KPZ on $\mathbb{R}$

Essential sing. $\mathrm{e}^{-ct^{-2/3}}$

Baik-Liu-Silva 2020

One-point Tracy-Widom distributions (= largest eigenvalue random matrices)

- Flat interface $h_0 = 0$ $\Rightarrow F_{\mathsf{GOE}}$
- Droplet $h_0 = |x|/0$ $\Rightarrow F_{\mathsf{GUE}}$
- Stationary h_0 Brownian $\Rightarrow F_{\mathsf{BR}}$

$$F_{\mathsf{GUE}}(s) = \exp\Big(-\int_s^\infty \mathrm{d}u\,(u-s)q^2(u)\Big)$$

Painlevé II $\quad q''(u) = 2q^3(u) + u\,q(u)$

$$F_{\mathsf{GUE}}(s) = \det(1 - P_s K_{\mathsf{Ai}} P_s) \text{ Airy kernel}$$

$$K_{\mathsf{Ai}}(u,v) = \int_0^\infty \mathrm{d}s\ \mathrm{Ai}(u+s)\,\mathrm{Ai}(v+s)$$

Spatial correlations droplet $\Rightarrow$ $h(x) = \mathcal{A}_2(x) - \frac{x^2}{4}$ Airy$_2$ process

$N \times N$ GUE Dyson's Brownian motion
$\equiv$ non-intersecting Wiener processes

$$\frac{\mathrm{d}\lambda_j}{\mathrm{d}u} = -\frac{\lambda_j}{2N} + \frac{1}{N}\sum_{k\neq j}^{N}\frac{1}{\lambda_j - \lambda_k} + \frac{\xi_j}{\sqrt{N}}$$

Eigenvalues $\lambda_1(u) \le \ldots \le \lambda_N(u)$

$$\mathcal{A}_2(u) \equiv \lim_{N\to\infty} \frac{\lambda_N(2uN^{2/3}) - 2\sqrt{N}}{N^{-1/6}}$$

Long time limit: large deviations

$$\frac{h(x,t)}{t} \underset{t\to\infty}{\to} 1 \quad \text{a.s.} \quad \rightsquigarrow \quad \begin{cases} \text{Typical fluctuations Gaussian} & \frac{h(x,t)-t}{\pi^{1/4}\sqrt{t/2}} \to \mathcal{N}_{0,1} \\ \text{Large deviations} & \mathbb{P}(h(x,t)=jt) \sim \mathrm{e}^{-tg(j)} \\ & \log\langle \mathrm{e}^{sh(x,t)}\rangle \underset{t\to\infty}{\simeq} te(s) + \log\theta(s;h_0) \end{cases}$$

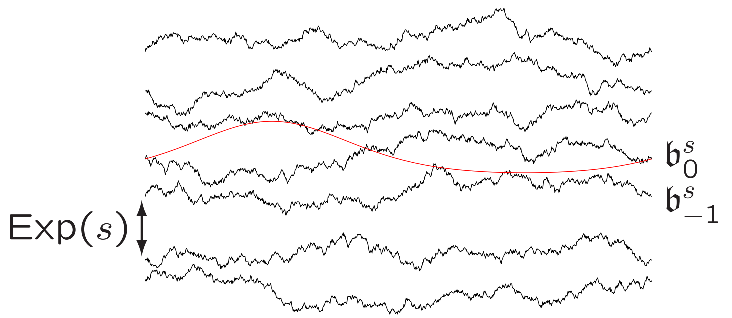
Essential sing. e^{-ct}

Exact formulas $e(s) = \chi(\nu(s))$

$$\theta(s;0) = \frac{s\,\exp(\frac{1}{2}\int_{-\infty}^{\nu(s)}\mathrm{d}v\,\chi''(v)^2)}{(1+\mathrm{e}^{\nu(s)})^{1/4}\chi''(\nu(s))} \quad \text{with} \quad \begin{array}{l} \chi(v) = -\,\mathrm{Li}_{5/2}(-\mathrm{e}^v)/\sqrt{2\pi} \\ \chi'(\nu(s)) = s \end{array}$$

Matrix product representation stat. state $\Rightarrow$ non-intersecting Brownian bridges

$$\theta(s;h_0) = \frac{\mathbb{P}(\mathfrak{b}^s_{-1} < h_0 | \ldots < \mathfrak{b}^s_{-2} < \mathfrak{b}^s_{-1})}{\mathbb{P}(\mathfrak{b}^s_{-1} < \mathfrak{b}^s_0 | \ldots < \mathfrak{b}^s_{-2} < \mathfrak{b}^s_{-1}, \mathfrak{b}^s_0 < \mathfrak{b}^s_1 < \ldots)}$$



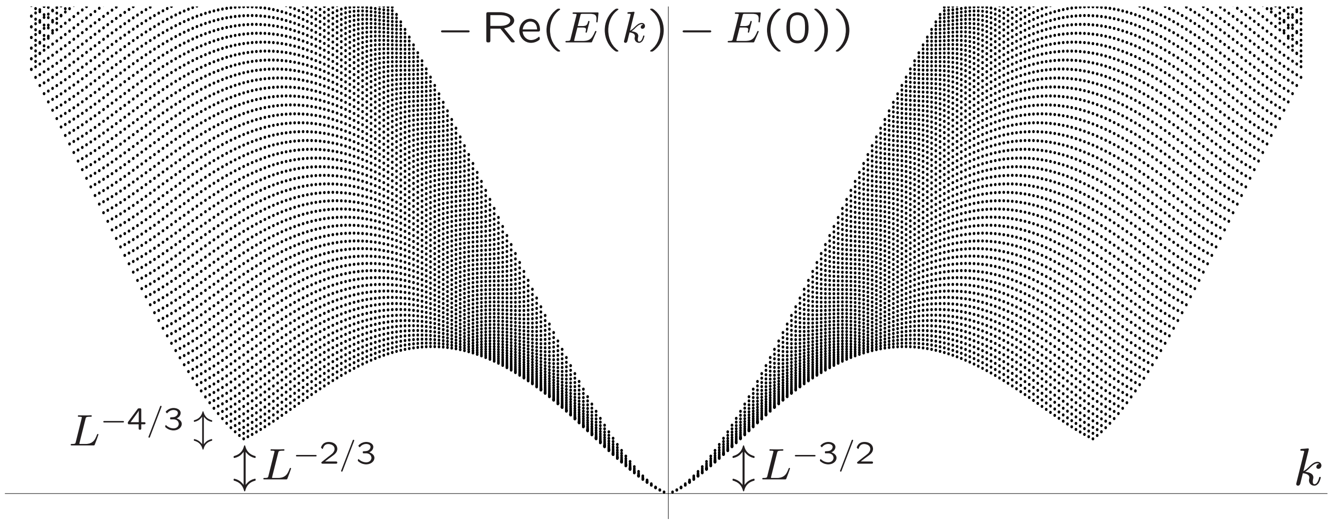
(Mallick-P. 2018)

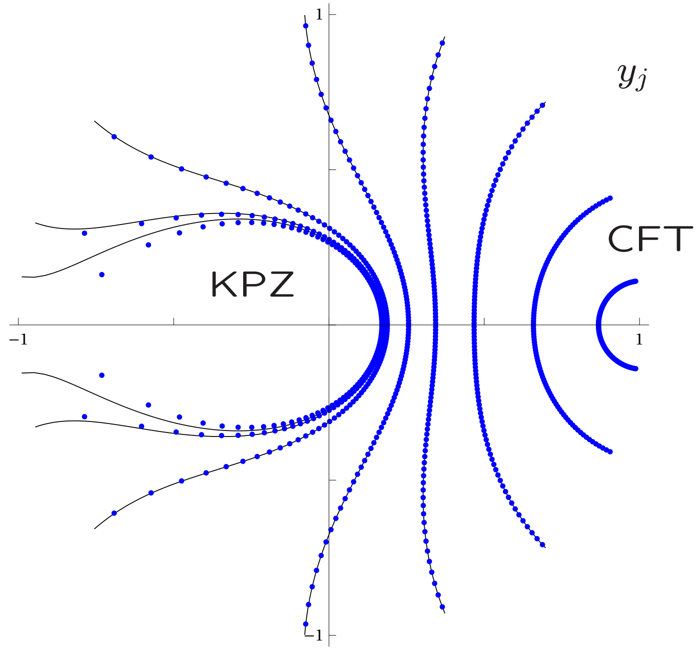
Direct derivation exact formulas $\theta(s;h_0)$ from Brownian bridges ?

Higher eigenstates and Riemann surface $\mathcal{R}_{\mathrm{KPZ}}$ for Brownian bridges ?

Further directions

Transition KPZ/CFT when $H \gg 1$





Open boundaries $\partial_x h = \pm\infty$: spectrum $\Rightarrow \mathcal{R}_L^{\mathrm{MC}} \to \mathcal{R}_{\mathrm{KPZ}}^{\mathrm{MC}}$ (Godreau-P. 2020)

Eigenvectors ?

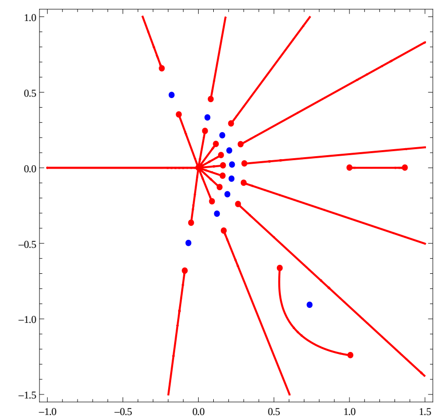
↰ Large deviations Lazarescu-Mallick 2011

↰ Bethe equations Crampé-Nepomechie 2018

Crossover EW-KPZ: ASEP $1 - q \simeq \lambda/\sqrt{L}$

- Riemann surfaces ?
- Duality for large deviations EW fixed point $\lambda \to 0$

  $\mathrm{Li}_{5/2}$ KPZ fixed point $\lambda \to \infty$



Conclusions

Bethe ansatz TASEP

Covering map $\mathcal{R}_N \to \widehat{\mathbb{C}}$

$P(y, C) = 0 \quad + \quad \text{sym. } y_j$

$L \to \infty \quad \Downarrow \quad g \to \infty$

KPZ fixed point in finite volume

Riemann surface $\mathcal{R}_{\text{KPZ}}$ $\quad \text{Li}_{5/2}$

Probability of the height

$\oint \mathrm{d}C \, \mathrm{tr}_\pi \, \mathrm{e}^{\int_O^{[C,\cdot]} \omega}$

Hidden classical integrability

KdV / KP solitons ???