Quantum Separation of Variables for Higher Rank Models

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Based on J. Math. Phys. 59, 091417 (2018) (special issue: in memory of LUDWIG FADDEEV) and subsequent papers in collaboration with Jean-Michel Maillet (ENS-Lyon) and Louis Vignoli (ENS-Lyon).

Some historical summary on quantum Separation of Variables (SoV)

- The quantum version of SoV has been invented by E. Sklyanin (1985) and applied to some important integrable quantum models, like Toda model and XXZ spin chain.

- Until 2008 quantum SoV applied only by some key researchers: **Gutzwiller, Kharchev, Lebedev, Babelon, Smirnov (Toda model), Babelon, Bernard and Smirnov (sine-Gordon model), Derkachov, Korchemsky and Manashov (non-compact XXX chain), Lukyanov, Bytsko & Teschner (sinh-Gordon model), von Gehlen, Iorgov, Pakuliak, and Shadura (Bazhanov-Stroganov model), Frahm, Grelik, Seel and Wirth (functional SoV for spin 1/2 XXX chain) etc.**

- For rank 1, together with **Teschner, Grosjean, Maillet, Faldella, Kitanine, Levy-Bencheton, Terras, Pezelier**, we have generalized Sklyanin's SoV to large variety of compact integrable quantum models.

- Several important developments and completions of previous results in the SoV framework for rank 1 non-compact models, **e.g. by Kozlowski (Toda model) and Derkachov, Kozlowski and Manashov (non-compact XXZ chain and sinh-Gordon model), etc.**.

- SoV for higher rank case first findings and open problems:

  ○ **Introduced by Sklyanin for $Y(gl_3)$ (1996), see also Smirnov for $U_q(sl_n)$ (2001), several open problems left (construction of the shift operator and quantum spectral curve).**

  ○ **Some first understanding of SoV spectrum in compact representations and non-Nested Algebraic Bethe Ansatz conjecture on transfer matrix eigenvectors by Gromov et al (2016), see Liashyk & Slavnov (2018) for NABA proof of it.**

  ○ **Important results on the SoV spectrum in non-compact representations by Derkachov and Valinevich (2018).**

- Need to overcome open problems and systematically develop SoV method.

Some historical summary on quantum Separation of Variables (SoV)

- Our new SoV approach based on the abelian algebra of the commuting conserved charges.
  - It is potentially universal as it exclusively relays on the quantum integrable structure.
  - It bypasses mentioned problems for higher rank cases.
  - It allows to prove the spectrum characterization by quantum spectral curve equation.
  - By using it, with Maillet and Vignoli, we have solved the spectral problems of:
    - **Fundamental quasi-periodic $Y(gl_n)$ models**
    - **Fundamental quasi-periodic $U_q(gl_n)$ models**
    - **Higher-spin $Y(gl_2)$ models**
    - **$Y(gl_n)$ models with integrable boundaries**
    - **First steps toward quasi-periodic super-symmetric $Y(gl_{n,m})$ and inhomogeneous Hubbard model**
  - Ryan $\&$ Volin (2018-2019) have implemented this SoV for any $Y(gl_n)$ quasi-periodic compact rep.

- Toward the dynamics by SoV approaches
  - The rank 1 cases:
    - **Universal determinant form for scalar products of separate states and form factors in compact models with Maillet and Grosjean (2012) and subsequent papers.**
    - **First computation of correlation functions in SoV framework with Terras (2020), the quasi-periodic XXX chain.**
  - The higher rank cases:
    - **First argued formulae for gl(3) Sklyanin's SoV measure in Cavaglia et al (2019-07) and Gromov et al (2019-10).**
    - **Together with Maillet $\&$ Vignoli (2020), we have determined our gl(3) SoV measure by recursion formulae and identified the commuting conserved charges generating SoV bases leading to scalar products of gl_2 type.**

Plan of the seminar:

- Quantum analogous of separation of variables.
- Recall of Sklyanin's quantum separation of variables[1] in quantum inverse scattering[2].
- Our new approach to quantum SoV by only using commuting conserved charges.
- The quasi-periodic $Y(gl_2)$ fundamental models:
  - **Comparison with Sklyanin's SoV basis**
  - **Transfer matrix spectrum by our SoV approach**
  - **Quantum spectral curve characterization of transfer matrix spectrum by our SoV approach**
- The quasi-periodic $Y(gl_3)$ fundamental models:
  - **Transfer matrix spectrum by our SoV approach**
  - **Quantum spectral curve characterization of transfer matrix spectrum by our SoV approach**
- Scalar products towards correlation functions:
  - **For $Y(gl_2)$: scalar products and correlation functions.**
  - **For $Y(gl_3)$: the SoV scalar product form depends from the choice of the generating commuting charges.**
- Conclusion and projects

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[1] E. K. Sklyanin, Lect. Notes Phys. 226 (1985) 196.

[2] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, Teor. Mat. Fiz. 40 (1979) 194.

I. Quantum analogous of separation of variables

Quantum separation of variables (SoV): a definition

- Let $T(\lambda)$ be one-parameter family of commuting conserved charges:

  $\exists\, T(\lambda) \in$ End$(\mathcal{H})$: i) $[T(\lambda), T(\lambda')]=0 \quad \forall\lambda, \lambda'$, ii) $[T(\lambda), H]=0 \quad \forall\lambda \in \mathbb{C}$, H Hamiltonian and $\mathcal{H}$ quantum space.

- Let $\mathsf{Y}_n \in$ End$(\mathcal{H})$ be simultaneous diagonalizable operators with simple spectrum:

  That is for any N-upla of eigenvalues $\{y_1, \ldots, y_{\mathsf{N}}\}$ there exists only one common eigenvector

$$Y_n \; |y_1, \ldots, y_{\mathsf{N}}\rangle = y_n \; |y_1, \ldots, y_{\mathsf{N}}\rangle .$$

- **Definition**: Y_n are quantum separate variables for $T(\lambda)$ iff its eigenvectors $|t\rangle$ have the form:

$$|t\rangle = \sum_{\text{over spectrum of } \{Y_n\}} \prod_{n=1}^{\mathsf{N}} \mathsf{Q}_t^{(n)}(y_n) \; |y_1, \ldots, y_{\mathsf{N}}\rangle ,$$

  where its eigenvalue $t(\lambda)$ and $Q_t^{(n)}(\lambda)$ are solutions of separate equations in y_n of the type

$$F_n(y_n, \frac{i}{2\pi}\frac{d}{dy_n}, t(y_n))\, Q_t^{(n)}(y_n) = 0, \quad \text{for all the } \; n \in \{1, \ldots, \mathsf{N}\}.$$

- **The N quantum separate relations are the natural quantum analogue of the classical ones in the Hamilton-Jacobi's approach.**

- **The Hydrogen atom Hamiltonian represents one natural example of integrable quantum system to which quantum SoV applies, here the y_n are the spherical coordinates r, θ, ϕ.**

II. Recall of Sklyanin's SoV approach

Sklyanin's approach to SoV:

Two commutative operator families $A(\lambda)$, $B(\lambda)$ written by Yang-Baxter generators satisfying:

- $B(\lambda)$ diagonalizable with its operator zeroes Y_n having simultaneous simple spectrum.
- $A(\mu)$ and $B(\lambda)$ satisfy the following commutation relations:

$$i) \quad (\lambda-\mu)A(\mu)B(\lambda) = (\lambda-(\mu-\eta))B(\lambda)A(\mu) + B(\mu)\Xi_0(\lambda,\mu).$$

- $A(\lambda)$, $B(\lambda)$ and the (higher) transfer matrices of the quantum integrable models $T_i(\lambda)$ (the quantum spectral invariants) satisfy a difference functional equation of finite order:

$$ii) \quad \sum_{j=0}^{r+1} \prod_{a=0}^{j-1} A(\lambda-a\eta)T_{r+1-j}(\lambda) = B(\lambda)\,\Xi(\lambda).$$

Sklyanin's SoV works iff $\Xi_0(\lambda,\mu)$ & $\Xi(\lambda)$ are finite in the Y_n spectrum. Then, Y_n are the quantum separate variables and $ii)$ are the quantum separate relations (quantum spectral curve equations) when computed in the Y_n spectrum.

Works perfectly for gl_2-models, problems arise e.g. with the gl_3 fundamental representations due to $\Xi_0(\lambda,\mu)$ & $\Xi(\lambda)$ poles over the spectrum of the Y_n.

III. Our New SoV Approach:

III.1 SoV basis generated by commuting conserved charges

Definition: The commuting conserved charges $T(\lambda)$ are "basis generating" if $\exists\, \langle L| \in \mathcal{H}^*$ and N sets $\{y_n^{(1)}, \ldots, y_n^{(d_n-1)}\}$:

$$*)\ \langle h_1, \ldots, h_N| \equiv \langle L| \prod_{a=1}^{N} \prod_{k_a=1}^{h_a} T(y_a^{(k_a)}) \text{ with } \{h_1, \ldots, h_N\} \in \otimes_{n=1}^{N}\{0, \ldots, d_n - 1\},$$

is a basis of $\mathcal{H}^*$, co-vector space of finite dimension $d = \prod_{n=1}^{N} d_n$.

∘ Note that for any common eigenvector $|t\rangle$ of $T(\lambda)$, the wave function $\Psi_t(h_1, \ldots, h_N)$ is separated in terms of the same conserved charge eigenvalue $t(\lambda)$:

$$\Psi_t(h_1, \ldots, h_N) \equiv \langle h_1, \ldots, h_N|t\rangle = \langle L|t\rangle \prod_{a=1}^{N} \prod_{k_a=1}^{h_a} t(y_a^{(k_a)}),$$

eigenvector fixed by the corresponding eigenvalue.

∘ $*)$ is our natural choice for SoV basis if we are able to identify the separate relations (closure conditions) allowing to compute the T-action in the basis and to characterize the T-spectrum.

∘ **In all compact quantum integrable lattice models so far considered this set of charges can be provided by the hierarchy of fused transfer matrices and Q-operators and the separate relations are direct consequences of the fusion relations satisfied by these commuting conserved charges.**

III.2 Example for matrix with simple spectrum: The **characteristic polynomial** as example of **closure relation**

Let X be a $d \times d$ simple matrix, then $V_X X V_X^{-1} = C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{d-2} & -a_{d-1} \end{pmatrix}$

Lemma The $\langle f_j| = \langle f_1|X^{j-1}$ with $1 \leq j \leq d$ is a co-vector basis, for $\langle f_1| \equiv \langle e_1|V_X^{-1}$, and the unique X-eigenvector $|\Lambda\rangle$ associated to the X-eigenvalue λ has coordinates (wavefunctions):

$$*) \quad \langle f_n|\Lambda\rangle = \lambda^{n-1} \langle f_1|\Lambda\rangle, \quad \text{with } \langle f_1|\Lambda\rangle \neq 0.$$

Proof From $\langle e_j| \equiv \langle e_1|C^{j-1}$ and $\langle f_j| = \langle e_j|V_X^{-1}$ then $\langle f_{1 \leq j \leq d}|$ is a basis.
So, if $|\Lambda\rangle$ is eigenvector $*)$ holds by def., vice versa, if $*)$ holds we want to prove:

$$\langle f_n|X|\Lambda\rangle = \lambda\langle f_n|\Lambda\rangle, \text{ for any } 1 \leq n \leq d.$$

i) If $n \leq d-1$, then by def. it holds $\langle f_n|X|\Lambda\rangle = \langle f_{n+1}|\Lambda\rangle = \lambda\langle f_n|\Lambda\rangle$.

ii) For $n = d$, $\langle f_d|X = \langle f_1|X^d$ is not in the basis, we need a **closure relations**:

$$\langle f_d|X|\Lambda\rangle = \langle f_1|X^d|\Lambda\rangle \underset{P_X(X)=0}{=} -\langle f_1|\sum_{n=0}^{d-1} a_n X^n|\Lambda\rangle = -\langle f_1|\Lambda\rangle \sum_{n=0}^{d-1} a_n \lambda^n \underset{P_X(\lambda)=0}{=} \langle f_1|\Lambda\rangle\lambda^d = \lambda\langle f_d|\Lambda\rangle$$

with $P_X(t) = a_0 + ta_1 + t^2 a_2 + \cdots + t^{d-1}a_{d-1} + t^d$ the **characteristic polynomial**.

IV. Our SoV basis for quasi-periodic fundamental $Y(gl_n)$ models

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The gl_n invariant R-matrix $R_{ab}(\lambda_a - \lambda_b) = (\lambda_a - \lambda_b)\mathbb{I}_{ab} + \eta\mathbb{P}_{ab}$ satisfies:

$$R_{ab}(\lambda_a - \lambda_b)K_a K_b = K_b K_a R_{ab}(\lambda_a - \lambda_b), \quad \forall K \in \mathrm{End}(\mathbb{C}^n),$$

The K-twisted monodromy matrix, on $\mathcal{H} \equiv \otimes_{l=1}^{N} V_l$ of dimension $d = n^N$, reads:

$$M_a^{(K)}(\lambda, \{\xi_1, ..., \xi_N\}) \equiv K_a R_{aN}(\lambda - \xi_N) \cdots R_{a1}(\lambda - \xi_1),$$

it satisfies the Yang-Baxter equation and defines a commuting family of transfer matrices:

$$T^{(K)}(\lambda) \equiv tr_{V_a} M_a^{(K)}(\lambda).$$

Theorem 1. $\langle h_1, ..., h_N| \equiv \langle L| \prod_{a=1}^{N} (T^{(K)}(\xi_a))^{h_a} \; \forall \{h_1, ..., h_N\} \in \{0, ..., n-1\}^{\otimes N}$

is a co-vector basis of $\mathcal{H}^*$ for almost any choice of the co-vector $\langle L|$ and of the inhomogeneity parameters $\{\xi_1, ..., \xi_N\}$ if $K \in \mathrm{End}(\mathbb{C}^n)$ has simple spectrum.

a) The tensor product choice $\langle L| \equiv \bigotimes_{a=1}^{N} \langle L_a|$, with $\langle L_a|$ local co-vector in V_a^*, can be done if $\langle L_a|K_a^h$ with $h \in \{0, ..., n-1\}$ is a co-vector basis for V_a for any $a \in \{1, ..., N\}$.

b) If K is diagonalizable with simple spectrum then $T^{(K)}(\lambda)$ has the same property.

c) The quantum determinant defines the separate relations for the fundamental representations.

○ **The proof given yesterday by Louis, it uses the polynomial properties of the transfer matrix and the form of the R-matrix.**

○ **If Sklyanin's B-operator is diagonalizable with simple spectrum, our SoV basis can be made coinciding with the Sklyanin one: we have proven it for rank 1 and verified for finite chains for rank 2, for rank higher than 1 proven by Ryan and Volin.**

V. The quasi-periodic $Y(gl_2)$ fundamental models

Our SoV basis in the quasi-periodic $Y(gl_2)$ fundamental models reads:

$$\langle h_1, \dots, h_N| \equiv \langle L| \prod_{a=1}^{N} \left(\frac{T^{(K)}(\xi_a)}{a(\xi_a)}\right)^{h_a} \text{ for any } \{h_1, \dots, h_N\} \in \{0,1\}^{\otimes N}$$

where:

$$a(\lambda - \eta) = d(\lambda) = \prod_{a=1}^{N} (\lambda - \xi_a)$$

We can take $\langle L|$ as $\langle L| = \bigotimes_{a=1}^{N} (x, y)_a$. For $K \in \mathrm{End}(\mathbb{C}^2)$ not proportional to the identity matrix:

$$(x, y)K^i \text{ for } i = 0, 1$$

is a covector basis for almost any $x, y \in \mathbb{C}$.

Indeed:

$$\det|| \left((x, y)K^{i-1}|e_j\rangle\right)_{i,j\in\{1,2\}} || = \det \begin{pmatrix} x & y \\ ax + cy & bx + dy \end{pmatrix} = bx^2 + (d-a)xy + cy^2$$

is non-zero for almost all the values of $x, y \in \mathbb{C}$ if $K \neq \alpha I$.

V.2 Comparison with Sklyanin's SoV basis

V.2 Comparison with Sklyanin's SoV basis

Sklyanin's approach to SoV applies iff the twist matrix satisfies the condition:

$$K = \begin{pmatrix} a & b \neq 0 \\ c & d \end{pmatrix}$$

and $\xi_a \neq \xi_b + r\eta \ \ \forall a \neq b \in \{1, \ldots, N\}$ and $r \in \{-1, 0, 1\}$. Defined:

$$A^{(K)}(\lambda) = aA(\lambda) + bC(\lambda), \quad B^{(K)}(\lambda) = aB(\lambda) + bD(\lambda),$$
$$C^{(K)}(\lambda) = cA(\lambda) + dC(\lambda), \quad D^{(K)}(\lambda) = cB(\lambda) + dD(\lambda),$$

then Sklyanin's SoV basis is the eigenbasis of $B^{(K)}(\lambda)$ and it reads:

$$\underline{\langle h_1, \ldots, h_N|} \equiv \langle 0| \prod_{a=1}^{N} \left(\frac{A^{(K)}(\xi_a)}{a(\xi_a)}\right)^{h_a} \text{ for any } \{h_1, \ldots, h_N\} \in \{0,1\}^{\otimes N}$$

with $\langle 0| = \bigotimes_{a=1}^{N} (1,0)_a$ and

$$\underline{\langle h_1, \ldots, h_N|} B^{(K)}(\lambda) \equiv b \prod_{a=1}^{N} (\lambda - \xi_a + h_a\eta) \underline{\langle h_1, \ldots, h_N|}$$

Proposition 1. The two basis coincide under the identification $\langle L| \equiv \langle 0|$:

$$\langle h_1, \ldots, h_N| = \underline{\langle h_1, \ldots, h_N|} \text{ for any } \{h_1, \ldots, h_N\} \in \{0,1\}^{\otimes N}$$

Proof done by induction on $l = \sum_{a=1}^{N} h_a$ by transfer matrix action on the B-eigenbasis.

V.3 Fundamental $Y(gl_2)$ transfer matrix spectrum by our SoV

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The fusion relations (closure relations):

$$T^{(K)}(\xi_a)T^{(K)}(\xi_a-\eta)=q\text{-det}M^{(K)}(\xi_a),\ \forall a\in\{1,\ldots,N\}$$

where the quantum determinant reads

$$q\text{-det}M^{(K)}(\lambda)=A^{(K)}(\lambda)D^{(K)}(\lambda-\eta)-B^{(K)}(\lambda)C^{(K)}(\lambda-\eta)=a(\lambda)d(\lambda-\eta)\text{det}K\ ,$$

$T^{(K)}(\lambda)$ is a polynomial of degree N in λ with leading central coefficient given by tr$K$.

Theorem 2.

$$\Sigma_{T(K)}=\left\{t(\lambda):t(\lambda)=\text{tr}\,K\ \prod_{a=1}^{N}(\lambda-\xi_a)+\sum_{a=1}^{N}g_a(\lambda)x_a,\quad\forall\{x_1,\ldots,x_N\}\in\Sigma_T\right\}$$

with $g_a(\lambda)=\prod_{b\neq a,b=1}^{N}(\lambda-\xi_b)/(\xi_a-\xi_b)$ and Σ_T is the set of solutions $\{x_1,\ldots,x_N\}$ to:

$$x_n[\text{tr}\,K\ \prod_{a=1}^{N}(\xi_n-\xi_a-\eta)+\sum_{a=1}^{N}g_a(\xi_n-\eta)x_a]=a(\xi_n)d(\xi_n-\eta)\text{det}K,\ \forall n\in\{1,\ldots,N\},$$

Moreover, $T^{(K)}(\lambda)$ has simple spectrum and for any $t(\lambda)\in\Sigma_{T(K)}$ the unique (up-to normalization) eigenvector $|t\rangle$ has a factorized wave-function in the SoV basis:

$$\langle h_1,\ldots,h_N|t\rangle=\prod_{n=1}^{N}(t(\xi_n)/a(\xi_n))^{h_n}$$

V.4 Sketch of the proof of Theorem 2.

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The system of N quadratic equations in N unknown x_i coincides wit the fusion relations:

$$t(\xi_a)t(\xi_a - \eta) = q\text{-det}M^{(K)}(\xi_a), \ \forall a \in \{1, ..., N\}.$$

Any eigenvalue satisfies it and the associated eigenvector $|t\rangle$ has factorized wave function.
Reverse statement:

$$\langle h_1, ..., h_N|T^{(K)}(\lambda)|t\rangle = t(\lambda)\langle h_1, ..., h_N|t\rangle, \ \forall\{h_1, ..., h_N\} \in \{0, 1\}^{\otimes N}.$$

Defined $\xi_a^{(h)} = \xi_a - h\eta, \ h \in \{0, 1\}$, we have:

$$\langle h_1, ..., h_a, ..., h_N|T^{(K)}(\xi_a^{(h_a)})|t\rangle = \begin{cases} a(\xi_a)\langle h_1, ..., h'_a = 1, ..., h_N|t\rangle & \text{if} \quad h_a = 0 \\ q\text{-det}M^{(K)}(\xi_a)\frac{\langle h_1, ..., h'_a = 0, ..., h_N|t\rangle}{a(\xi_a)} & \text{if} \quad h_a = 1 \end{cases}$$

Using the definition of the state $|t\rangle$ and the fusion relation it can be rewritten as

$$\langle h_1, ..., h_a, ..., h_N|T^{(K)}(\xi_a^{(h_a)})|t\rangle = \begin{cases} t(\xi_a)\prod_{n\neq a, n=1}^{N}\left(\frac{t(\xi_n)}{a(\xi_n)}\right)^{h_n} & \text{if} \quad h_a = 0 \\ t(\xi_a - \eta)\prod_{n=1}^{N}\left(\frac{t(\xi_n)}{a(\xi_n)}\right)^{h_n} & \text{if} \quad h_a = 1 \end{cases},$$

and so:

$$\langle h_1, ..., h_a, ..., h_N|T^{(K)}(\xi_a^{(h_a)})|t\rangle = t(\xi_a^{(h_a)})\langle h_1, ..., h_a, ..., h_N|t\rangle$$

from which the result follows using the polynomial interpolation for the transfer matrix.

V.5 The quantum spectral curve for the $Y(gl_2)$ case

Theorem 3. If $K \neq \alpha I$ has a non-zero eigenvalue k_0 and $t(\lambda)$ is entire in λ, then $t(\lambda)$ is an element of the spectrum of $T^{(K)}(\lambda)$ if and only if there exists a unique polynomial:

$$Q_t(\lambda) = \prod_{a=1}^{\mathsf{M}} (\lambda - \lambda_a), \text{ with } \mathsf{M} \leq N \text{ with } \lambda_a \neq \xi_b, \forall\, (a,b) \in \{1, \dots, \mathsf{M}\} \times \{1, \dots, N\}$$

such that $t(\lambda)$ and $Q_t(\lambda)$ are solutions of the quantum spectral curve functional equation:

$$\alpha(\lambda) Q_t(\lambda - 2\eta) - \beta(\lambda) t(\lambda - \eta) Q_t(\lambda - \eta) + q\text{-det} M^{(K)}(\lambda) Q_t(\lambda) = 0.$$

where we have defined $\alpha(\lambda) = \beta(\lambda)\beta(\lambda - \eta)$, $\beta(\lambda) = \mathsf{k}_0 a(\lambda)$.

Moreover, up to a normalization the associated transfer matrix eigenvector $|t\rangle$ admits the following rewriting in the SoV basis:

$$\langle h_1, \dots, h_N | t \rangle = \mathsf{k}_0^{\sum_{n=1}^{N} h_n} \prod_{n=1}^{N} Q_t(\xi_n^{(h_n)})$$

All this scheme extends also to the trigonometric case.

V.6 Sketch of the proof of Theorem 3.

If $Q_t(\lambda)$ and $t(\lambda)$ satisfy the functional equation, then from $t(\lambda)$ entire it follows that it is a polynomial of degree N with leading coefficient tr$K$.

Moreover, the functional equation implies:

$$- t(\xi_a - \eta)Q_t(\xi_a - \eta) + \mathsf{k}_1 d(\xi_a - \eta)Q_t(\xi_a) = 0$$
$$\mathsf{k}_0 a(\xi_a)Q_t(\xi_a - \eta) - t(\xi_a)Q_t(\xi_a) = 0$$

which implies

$$t(\xi_a - \eta)\frac{t(\xi_a)Q_t(\xi_a)}{\mathsf{k}_0 a(\xi_a)} = \mathsf{k}_1 d(\xi_a - \eta)Q_t(\xi_a)$$

and so $t(\lambda)$ satisfies the fusion relations for any $a \in \{1, \ldots, N\}$ being $Q_t(\xi_a) \neq 0$.

Hence, $t(\lambda)$ is a transfer matrix eigenvalue by the previous Theorem 2.

The functional equation has to be satisfied in $3N+1$ different points to be proven.

Indeed, if $t(\lambda)$ is a transfer matrix eigenvalue then the l.h.s. of the equation is a polynomial in λ of maximal degree $2N+\mathsf{M}$, with $\mathsf{M} \leq N$.

∘ The leading coefficient of this polynomial is zero for the asymptotic behaviour of $t(\lambda)$.

∘ At any $\xi_a - \eta$ the functional equation is directly satisfied as it factorizes $a(\lambda)$.

∘ It is moreover satisfied in the $2N$ points ξ_a and $\xi_a + \eta$, if we impose:

$$\mathsf{k}_0 a(\xi_a) Q_t(\xi_a - \eta) = t(\xi_a) Q_t(\xi_a),$$

since $t(\lambda)$ satisfies the fusion relations.

∘ We prove $\exists!$ the $Q_t(\lambda)$ of stated polynomial form that satisfies the above characterization.

∘ Finally, from the identities:

$$\prod_{n=1}^{N} Q_t(\xi_n) \prod_{n=1}^{N} \left(\frac{t(\xi_n)}{a(\xi_n)} \right)^{h_n} = \mathsf{k}_0^{\sum_{n=1}^{N} h_n} \prod_{n=1}^{N} Q_t(\xi_n^{(h_n)}),$$

we get the representation of the transfer matrix eigenstate in the left SoV basis.

VI. Fundamental $Y(gl_3)$ transfer matrix spectrum by our SoV

VI.1 Definitions and first properties of the transfer matrices

○ The commuting fused transfer matrices (quantum spectral invariants) reads:

$$T_1^{(K)}(\lambda) \equiv \text{tr}_a M_a^{(K)}(\lambda) \quad T_2^{(K)}(\lambda) \equiv \text{tr}_{ab} P_{ab}^- M_a^{(K)}(\lambda) M_b^{(K)}(\lambda-\eta)$$

$$q\text{-det} M^{(K)}(\lambda) \equiv \text{tr}_{abc}(P_{abc}^- M_a^{(K)}(\lambda) M_b^{(K)}(\lambda-\eta) M_c^{(K)}(\lambda-2\eta))$$

○ They have the following polynomial form:

i) $T_1^{(K)}(\lambda)$ is a degree N polynomial in λ with tr$K$ as leading coefficient.

ii) $T_2^{(K)}(\lambda)$ is a degree $2N$ polynomial in λ with the following N central zeros and asymptotic:

$$T_2^{(K)}(\xi_a+\eta) = 0, \ \lim_{\lambda\to\infty} \lambda^{-2\mathsf{N}} T_2^{(K)}(\lambda) = ((\text{tr}K)^2 - \text{tr}K^2)/2,$$

iii) the quantum determinant is central q-det$M^{(K)}(\lambda) = \det K \ a(\lambda) d(\lambda-\eta) d(\lambda-2\eta)$.

○ Fusion identities:

$$T_1^{(K)}(\xi_a) T_2^{(K)}(\xi_a - \eta) = q\text{-det} M^{(K)}(\xi_a)$$

$$T_1^{(K)}(\xi_a-\eta) T_1^{(K)}(\xi_a) = T_2^{(K)}(\xi_a).$$

○ Then, the $T_2^{(K)}(\lambda)$ transfer matrix is completely determined by:

$$T_2^{(K)}(\lambda) = T_{2,\mathbf{h}=\mathbf{0}}^{(K,\infty)}(\lambda) + \sum_{a=1}^{N} f_{a,\mathbf{h}=\mathbf{0}}(\lambda) T_1^{(K)}(\xi_a-\eta) T_1^{(K)}(\xi_a),$$

where $f_{a,\mathbf{h}}(\lambda) = d(\lambda-\eta)/d(\xi_a^{(h_a)}-\eta) \prod_{b\neq a,b=1}^{\mathsf{N}} (\lambda - \xi_b^{(h_b)})/(\xi_a^{(h_a)} - \xi_b^{(h_b)})$.

VI.2 $Y(gl_3)$ transfer matrix spectrum by discrete SoV characterization

Theorem 4. The spectrum of $T_1^{(K)}(\lambda)$ is characterized by:

$$\Sigma_{T(K)} = \left\{ t_1(\lambda) : t_1(\lambda) = \text{tr}\, K \prod_{a=1}^{N}(\lambda - \xi_a) + \sum_{a=1}^{N} g_a(\lambda) x_a, \quad \forall \{x_1, ..., x_\mathsf{N}\} \in \Sigma_T \right\},$$

Σ_T is the set of solutions to the following inhomogeneous system of N cubic equations:

$$x_a[T_{2,\mathbf{h=0}}^{(K,\infty)}(\xi_a - \eta) + \sum_{n=1}^{N} f_{n,\mathbf{h=0}}(\xi_a - \eta) t_1(\xi_n - \eta) x_n] = q\text{-det} M^{(K)}(\xi_a),$$

in N unknown $\{x_1, ..., x_N\}$. Moreover, $T_1^{(K)}(\lambda)$ has simple spectrum and for any $t_1(\lambda) \in \Sigma_{T(K)}$ the associated unique (up-to normalization) eigenvector $|t\rangle$ has the following wave-function in the left SoV basis:

$$\langle h_1, ..., h_N | t \rangle = \prod_{n=1}^{N} t_1^{h_n}(\xi_n)$$

The main idea of the proof is like in the gl_2 case. Namely to use the fusion relations for the transfer matrices and their eigenvalues until we get the quantum determinant which acts trivially on any covector. While doing so we use interpolation formulae for the transfer matrices and their eigenvalues.

VI.3 $Y(gl_3)$ transfer matrix spectrum by quantum spectral curve

Theorem 5. If the K simple twist matrix has a nonzero eigenvalue k_0, then the entire functions $t_1(\lambda)$ is a $T_1^{(K)}(\lambda)$ transfer matrix eigenvalue iff there exists a unique polynomial:

$$Q_t(\lambda) = \prod_{a=1}^{\mathsf{M}} (\lambda - \lambda_a) \text{ with } \mathsf{M} \leq N \text{ and } \lambda_a \neq \xi_n \; \forall (a, n) \in \{1, ..., \mathsf{M}\} \times \{1, ..., N\}$$

satisfying with

$$t_2(\lambda) = T_{2,\mathbf{h=0}}^{(K,\infty)}(\lambda) + \sum_{n=1}^{N} f_{n,\mathbf{h=0}}(\lambda) t_1(\xi_n - \eta) t_1(\xi_n),$$

the following **quantum spectral curve equation:**

$$\begin{aligned} &\alpha(\lambda) Q_t(\lambda - 3\eta) - \beta(\lambda) t_1(\lambda - 2\eta) Q_t(\lambda - 2\eta) \\ &+ \gamma(\lambda) t_2(\lambda - \eta) Q_t(\lambda - \eta) - q\text{-}det\, M_a^{(K)}(\lambda - 2\eta) Q_t(\lambda) = 0, \end{aligned}$$

$$\alpha(\lambda) = \gamma(\lambda)\gamma(\lambda - \eta)\gamma(\lambda - 2\eta), \beta(\lambda) = \gamma(\lambda)\gamma(\lambda - \eta), \gamma(\lambda) = k_0 a(\lambda).$$

Moreover, up to a normalization the common transfer matrix eigenstate $|t\rangle$ admits the following separate representation:

$$\langle h_1, ..., h_N | t\rangle = \prod_{a=1}^{N} \gamma^{h_a}(\xi_a) Q_t^{h_a}(\xi_a - \eta) Q_t^{2-h_a}(\xi_a).$$

VII Toward quantum dynamics in SoV

Theorem 6.

a) Universal characterization of scalar products and form factors by SOV-method[3]:

For rank 1 integrable quantum models associated to finite dimensional quantum space, there exists a basis $\mathbb{B}_{\mathcal{H}}$ in End($\mathcal{H}$) such that for any $\mathsf{O} \in \mathbb{B}_{\mathcal{H}}$ the matrix elements read:

$$\langle t'|\mathsf{O}|t\rangle = \det_{\mathsf{N}} ||\Phi_{a,b}^{(\mathsf{O},t,t)}||, \quad \Phi_{a,b}^{(\mathsf{O},t,t)} \equiv \sum_{c=1}^{p} F_{\mathsf{O},b}(y_a^{(c)}) Q_t(y_a^{(c)}) Q_{t'}(y_a^{(c)}) (y_a^{(c)})^{2(b-1)}.$$

$F_{\mathsf{O},b}()$ characterize the operator O and are computed algebraically by SOV reconstruction of O.

b) Rewriting of SoV scalar products by ABA like formulae[4]: The XXX/XXZ spin 1/2 chains with closed or open integrable boundaries admit scalar products rewriting of Izergin's and Slavnov's types[5].

c) First SoV computations of correlation functions[6]: For finite quasi-periodic XXX spin 1/2 chains, multiple integral representations of correlation functions proven by taking the thermodynamic limit of the SoV results.

[3] Grosjean, Maillet, Niccoli (2012-2016) and subsequent papers.
[4] Kitanine, Maillet, Niccoli, Terras (2015-2018)
[5] For closed XXX chain they are reminiscent of ones first obtained by Kostov (2012) by another approach.
[6] Niccoli and Terras (2020)

Theorem[7] 7. **a)** The following set of co-vectors and vectors are SoV bases:

$$*)\ \langle \underline{\mathbf{h}}| = \langle \underline{\mathbf{1}}| \prod_{n=1}^{\mathsf{N}} T_2^{(K)\delta_{h_n,0}}(\xi_n^{(1)}) T_1^{(K)\delta_{h_n,2}}(\xi_n), \quad |\underline{\mathbf{h}}\rangle \equiv \prod_{n=1}^{\mathsf{N}} T_2^{(K)\delta_{h_n,1}}(\xi_n) T_1^{(K)\delta_{h_n,2}}(\xi_n)|\underline{\mathbf{0}}\rangle,$$

with $\langle \underline{\mathbf{1}}|$, $|\underline{\mathbf{0}}\rangle$ of known tensor product forms and the following *pseudo-orthogonality* relations:

$$\frac{\langle \underline{\mathbf{h}}|\underline{\mathbf{k}}\rangle}{\langle \underline{\mathbf{k}}|\underline{\mathbf{k}}\rangle} = \delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}} + C_{\underline{\mathbf{h}}}^{\underline{\mathbf{k}}} \sum_{r=1}^{n_{\underline{\mathbf{k}}}} (\det K)^r \sum_{\substack{\alpha\cup\beta\cup\gamma=\mathbf{1}_{\underline{\mathbf{k}}}, \\ \alpha,\beta,\gamma \text{ disjoint}, \#\alpha=\#\beta=r}} \delta_{\underline{\mathbf{h}},\underline{\mathbf{k}}_{\alpha,\beta}^{(\mathbf{0},\mathbf{2})}},$$

$$\langle \underline{\mathbf{h}}|\underline{\mathbf{h}}\rangle = \frac{V^2(\xi_1,\dots,\xi_{\mathsf{N}}) \prod_{a=1}^{\mathsf{N}} (d(\xi_a^{(1)})/d(\xi_a^{(1+\delta_{h_a,1}+\delta_{h_a,2})})}{V\left(\xi_1^{(\delta_{h_1,2}+\delta_{h_1,1})},\dots,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},1}+\delta_{h_{\mathsf{N}},2})}\right) V\left(\xi_1^{(\delta_{h_1,2})},\dots,\xi_{\mathsf{N}}^{(\delta_{h_{\mathsf{N}},2})}\right)},$$

$C_{\underline{\mathbf{h}}}^{\underline{\mathbf{k}}} \neq 0$ are fixed by recursions independent w.r.t. $\det K$, $n_{\underline{\mathbf{k}}}$ integer part of $(\sum_{a=1}^{\mathsf{N}} \delta_{k_a,1})/2$.

Selection rules: $\underline{\mathbf{k}}_{\alpha,\beta}^{(\mathbf{0},\mathbf{2})} \equiv (k_1(\alpha,\beta),\dots,k_{\mathsf{N}}(\alpha,\beta)) \in \{0,1,2\}^{\mathsf{N}}$, $\mathbf{1}_{\underline{\mathbf{k}}} \equiv \{a \in \{1,\dots,\mathsf{N}\} : k_a = 1\}$,

$$k_a(\alpha,\beta) = 0, \quad k_b(\alpha,\beta) = 2,\ \forall a \in \alpha, b \in \beta,\ k_c(\alpha,\beta) = k_c,\ \forall c \in \{1,\dots,\mathsf{N}\}\backslash\{\alpha\cup\beta\}.$$

b) We construct two commuting families $\hat{T}_1(\lambda)$ and $\hat{T}_2(\lambda)$ (commuting with the original transfer matrix) such that $*)$ defines orthogonal SoV bases and the scalar product of a generic separate co-vector with a transfer matrix eigenvector admits the gl(2) type universal form.

[7] Maillet, Niccoli, Vignoli 2020.

Conclusion and projects

Results

- ◦ SoV basis are defined by a complete sets of commuting conserved charges
- ◦ Our SoV construction applies to all lattice compact higher rank quantum integrable models
- ◦ Complete characterization of transfer matrix spectrum in SoV basis:
  - Proof of the quantum spectral curve equations from fusion relations.
  - Complete characterization of the transfer matrix eigenvalues and eigenvectors.
- ◦ First steps toward the dynamics of integrable quantum models in our SoV framework

What's next?

- ⊛ Understand better the various possible choices of SoV basis, in particular for higher rank.
- ⊛ Understand better the role of the Q-operator in general models and representations
- ⊛ Solve the spectrum of important integrable quantum models, e.g. the Hubbard model.
- ⊛ Correlation functions in this new SoV scheme for rank 1 and higher