Non-compact spin chains, stochastic particle processes and hidden equilibrium

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based on collaborations with C. Giardinà and J. Kurchan

RAQIS'20, 3. September 2020

Hommage to RAQIS’18

This talk exists because of RAQIS’18

	Monday 10.09	Tuesday 11.09	Wednesday 12.09	Thursday 13.09	Friday 14.09
8:30 - 9:00	*Registration* *9:10 Welcome*				
9:15 - 10:15	B. McCoy	F. Göhmann	J. S. Caux	G. Schütz	C. Giardinà
10:15 - 10:45	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*	*Coffee break*
10:45 - 11:45	V. Terras	N. Slavnov	T. Prosen	J. Jacobsen	B. Derrida
11:45 - 12:15	A. Gerrard	A. Hutsalyuk	E. Vernier	M. Vanicat	J. De Nardis
12:30 - 14:30	*Lunch*		*Buffet*	*Lunch*	
14:30 - 15:00	K. Kozlowski	T. Imoto		S. Sotiriadis	*End*
15:00 - 15:30	S. Lacroix	A. Liashyk		B. Pozsgay	
15:30 - 16:00	*Coffee break*	*Coffee break*	*Free afternoon*	*Coffee break*	
16:00 - 16:30	R. Frassek	J. Roussillon		A. Morin-Duchesne	
16:30 - 17:00	M. Preti			O. Gamayun	
			20:00 Banquet		

- Thanks a lot to the organisers
- Thanks to the excellent introductions to integrable stochastic processes [Crampé, Ragoucy, Vanicat; Derrida; Mallick; Schütz; ...]

Overview

Content

1. Non-compact spin chains as stochastic particle process
    - Non-compact XXX chain [1]
    - Non-compact XXZ chain[2]
    - Non-compact XXX chain with open boundaries [1]
2. Hidden equilibrium in SSEP [3]
3. Outlook

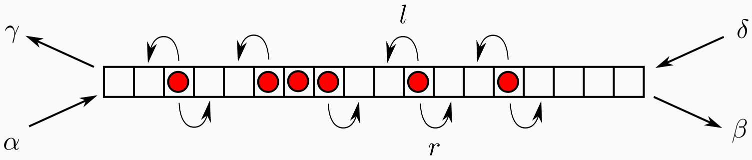
[1] [arXiv:1904.01048] with C. Giardinà and J. Kurchan
[2] [arXiv:1904.02191]
[3] [arXiv:1910.13163] and [arXiv:2004.12796] with C. Giardinà and J. Kurchan

Integrable simple exclusion models

Simple Exclusion Process

Most famous stochastic particle processes are: ASEP and SSEP

- Integrable
- Nearest-neighbor hopping model
- One particle per site (exclusion)
- Closed or open boundary conditions



Hopping rates: r and l and α, β, γ and δ

SSEP: $r = l = 1$

Markov matrix of ASEP/SSEP

Exclusion process is generated by Markov matrix

$$M = B_L + \sum_{i=1}^{N-1} \omega_{i,i+1} + B_R$$

Bulk:

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -l & r & 0 \\ 0 & l & -r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Boundary:

$$B_L = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}, \qquad B_R = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}$$

Stochastic process:

Sum over columns vanishes

Off-diagonal entries have opposite sign of diagonal entries

Relation to integrable spin chains

Stochastic process can be mapped to integrable spin chain

- ASEP $\leftrightarrow$ XXZ spin chain
- SSEP $\leftrightarrow$ XXX spin chain

Hamiltonian is related to Markov generator

$$M = SHS^{-1}$$

Particle process can be studied using integrability tools: Coordinate Bethe ansatz, algebraic Bethe ansatz, ...

Multi-species generalisations from higher rank spin chains

Traffic jam

ASEP/SSEP produces traffic jams!

Multi-particle generalisations

Put particles on top of each other

Naive observation

- Higher spin integrable model: Hamiltonian not stochastic
- Higher spin stochastic model: Hamiltonian not integrable

→ Non-compact integrable spin chains [Giardinà, Kurchan, RF]

Non-compact integrable spin chains

Non-compact spin chains

Quantum space of non-compact chains with hws

$$V = |m_1\rangle \otimes |m_2\rangle \otimes \ldots \otimes |m_N\rangle\,, \qquad m_i = 0, 1, 2, \ldots$$

For spin s generators of $\mathfrak{sl}(2)$ act locally as

$$S_+|m\rangle = (m+2s)|m+1\rangle\,, \quad S_-|m\rangle = m|m-1\rangle \quad S_0|m\rangle = (m+s)|m\rangle$$

Nearest-neighbor Hamiltonian density [Faddeev et al.]

$$\mathcal{H} = 2\left(\psi(\mathbb{S}) - \psi(2s)\right)$$

where $\psi(x)$ is Digamma function and $\mathbb{S}$ is related to the two-site Casimir operator via $C_{[2]} = \mathbb{S}(\mathbb{S}-1)$

- First studied in high energy QCD [Lipatov;Faddeev,Korchemsky]
- Important subsector of the $\mathcal{N}=4$ SYM spin chain! ($s = \frac{1}{2}$)
- Integrable models [Derkachov, Manashov]

The operator $\mathbb{S}$

Consider tensor product decomposition

$$D_s \otimes D_s = \bigoplus_{j=0}^{\infty} D_{2s+j}$$

Operator $\mathbb{S}$ acts diagonally on the irreps on the rhs

$$\mathbb{S}|D_{2s+j}\rangle = (2s+j)|D_{2s+j}\rangle$$

Eigenvalues of Hamiltonian density are harmonic numbers h_s

$$\mathcal{H}|D_{2s+j}\rangle = 2\sum_{k=1}^{j} \frac{1}{2s+k-1}|D_{2s+j}\rangle$$

- Can't tell if process is stochastic from eigenvalues
- A priory not known how $\mathcal{H}$ acts on the lhs...
  $\rightsquigarrow$ Clebsch Gordan decomposition

Harmonic action as stochastic process

Nearest neighbor hopping model for $s = \frac{1}{2}$ [Beisert]

$$\begin{aligned}\mathcal{H}|m\rangle \otimes |m'\rangle = (h(m) + h(m'))\,|m\rangle \otimes |m'\rangle \\ - \sum_{k=1}^{m} \frac{1}{k}|m-k\rangle \otimes |m'+k\rangle \\ - \sum_{k=1}^{m'} \frac{1}{k}|m+k\rangle \otimes |m'-k\rangle\end{aligned}$$

with the harmonic numbers $h(m)$.

Hamiltonian density $\mathcal{H}$ is generator of Markov process!

[Giardinà, Kurchan, RF]

E.g. $m + m' = 2$:

$$\mathcal{H}_2 = \begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -1 & 2 & -1 \\ -\frac{1}{2} & -1 & \frac{3}{2} \end{pmatrix}$$

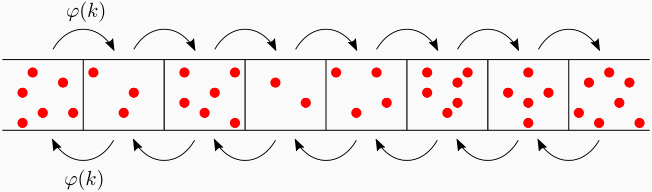
Harmonic action as stochastic process

Hamiltonian defined on N sites as

$$H = \sum_{i=1}^{N-1} \mathcal{H}_{i,i+1}$$

Symmetric stochastic process without exclusion!

$\rightarrow$ k particles jump with the rate $\varphi(k) = \frac{1}{k}$



Harmonic action as stochastic process

Hopping rates generalise to arbitrary spin s [Martins,Melo]

$$\varphi_s(m,k) = \frac{1}{k}\frac{\Gamma(m+1)\Gamma(m-k+2s)}{\Gamma(m-k+1)\Gamma(m+2s)}$$

Again we find a symmetric particle process!
$\hookrightarrow$ Rates depend on number of particles at departing site

Up to now only reinterpreting results of others...

Add a particle current (non-equilibrium models):

- q-analog/XXZ-analog $\longrightarrow$ asymmetric (drift) process
- Rational case with boundary reservoirs

Non-compact XXZ chain

Harmonic action was unknown for the trigonometric case

Analog of Faddeev formula in XXZ chain [Bytsko]

$$\mathcal{H} = \frac{\psi_q(\mathbb{S}) - \psi_q(2s)}{-q^{4s}\log(q)}$$

with q-Digamma function ψ_q and $\mathbb{S}$ is related to the co-product of the Casimir operator via $\Delta(C) = [\mathbb{S}][\mathbb{S}-1]$.

Undiagonalise $\mathcal{H}$ using Clebsch-Gordan coefficients! [RF]

Hamiltonian can be related to q-Hahn process by a similarity transformation (as in ASAP)

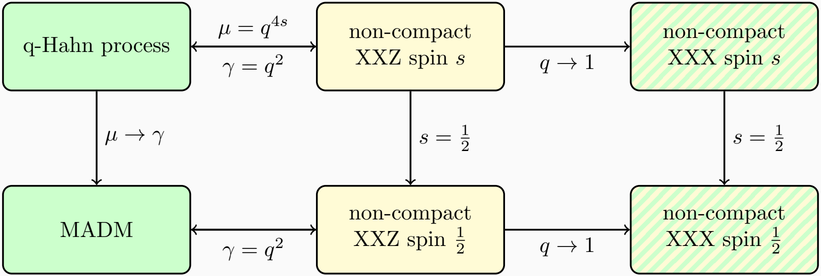
q-Hahn: Asymmetric hopping process

[Sasamoto-Wadati,Povolotsky;Barraquand-Corwin]

Belong to KPZ universility class and have interesting relations to theory of random matrices

Summary

Non-compact spin chains and stochastic particle processes



Probability Theory

High Energy Theory

Non-compact XXX chain with boundaries

Stochastic process with boundary reservoirs

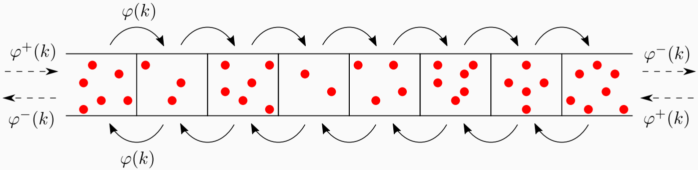
Add stochastic boundary conditions to rational process

$$\mathcal{H} = \mathcal{H}_1 + \sum_{i=1}^{N-1} \mathcal{H}_{i,i+1} + \mathcal{H}_N \,.$$

Guess boundary terms for $0 < \beta_i < 1$ and $s = \frac{1}{2}$ [Giardinà, Kurchan, RF]

$$\mathcal{H}_i|m_i\rangle = \left(h(m_i) + \sum_{k=1}^{\infty} \frac{\beta_i^k}{k}\right)|m_i\rangle - \sum_{k=1}^{m_i} \frac{1}{k}|m_i - k\rangle - \sum_{k=1}^{\infty} \frac{\beta_i^k}{k}|m_i + k\rangle$$

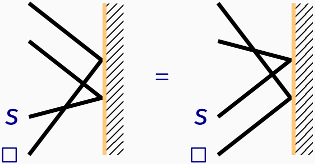
Introduces reservoirs at left and right end of the chain:



Is this process integrable? Find the corresponding K-matrix

[Giardinà, Kurchan, RF]

BYBE relevant for universal K-matrix



$$\mathcal{L}(x-y)\hat{\mathcal{K}}(x)\mathcal{L}(x+y)\hat{K}(y) = \hat{K}(y)\mathcal{L}(x+y)\hat{\mathcal{K}}(x)\mathcal{L}(x-y)$$

Lax matrix and K-matrix in fundamental representation

$$\mathcal{L}(x) = \begin{pmatrix} x+\frac{1}{2}+S_0 & -S_- \\ S_+ & x+\frac{1}{2}-S_0 \end{pmatrix}, \quad \hat{K}(x) = \begin{pmatrix} q_1+xq_2 & xq_3 \\ xq_4 & q_1-xq_2 \end{pmatrix}$$

Solve for $\hat{\mathcal{K}}(x)$...

Universal solution to BYBE

1. Introduce useful parametrisation of boundary variables

$$q_1 = \delta\,, \quad q_2 = \frac{1}{2}(1 + 2\alpha\beta)\gamma\,, \quad q_3 = -(1 + \alpha\beta)\beta\gamma\,, \qquad q_4 = \alpha\gamma$$

2. Make the ansatz

$$\hat{\mathcal{K}}(x) = e^{\beta S_+}\, e^{-\alpha S_-}\, \hat{\mathcal{K}}_0(S_0; x)\, e^{\alpha S_-}\, e^{-\beta S_+}$$

Yields difference equation for $\hat{\mathcal{K}}_0(S_0; x)$ which can be solved

$$\hat{\mathcal{K}}_0(S_0; x) = \frac{\Gamma(\frac{1}{2} + s + 2\frac{\delta}{\gamma} - x)}{\Gamma(\frac{1}{2} + s + 2\frac{\delta}{\gamma} + x)} \frac{\Gamma\left(\frac{1}{2} + S_0 + 2\frac{\delta}{\gamma} + x\right)}{\Gamma\left(\frac{1}{2} + S_0 + 2\frac{\delta}{\gamma} - x\right)}$$

Relation to stochastic boundary

To derive stochastic boundary conditions for Hamiltonian fix

$$2\frac{\delta}{\gamma} = s - \frac{1}{2}, \qquad \alpha = \frac{1}{1-\beta}$$

and compute the logarithmic derivative of the transfer matrix

$$\frac{\partial}{\partial x} \ln T(x)\Big|_{x=0} = \frac{\mathrm{tr}_a \mathcal{K}'_a(0)}{\mathrm{tr}_a \mathcal{K}_a(0)} + 2\frac{\mathrm{tr}_a \mathcal{K}_a(0)\mathcal{H}_{a,1}}{\mathrm{tr}_a \mathcal{K}_a(0)} + \frac{\hat{\mathcal{K}}'_N(0)}{\hat{\mathcal{K}}_N(0)} + 2\sum_{k=1}^{N-1} \frac{\partial}{\partial x} \ln \mathcal{R}_{k,k+1}(x)\Big|_{x=0},$$

A longer computation shows that we obtain the desired boundary terms!

$$\mathcal{H}_i|m_i\rangle = \left(h^{(s)}(m_i) + \sum_{k=1}^{\infty} \frac{\beta_i^k}{k}\right)|m_i\rangle - \sum_{k=1}^{m_i} \frac{1}{k} \frac{\Gamma(m_i+1)\Gamma(m_i-k+2s)}{\Gamma(m_i-k+1)\Gamma(m_i+2s)}|m_i-k\rangle - \sum_{k=1}^{\infty} \frac{\beta_i^k}{k}|m_i+k\rangle,$$

- Process is integrable!
- Derived stochastic boundaries for arbitrary spin s

More

Further interesting properties of the model:

- Dual (integrable) stochastic process of finite particles
- Long wavelength limit
  - Rotating string in AdS/CFT [Kruczenski] [Stefanski,Tseytlin] [Bellucci et al.]
  - Hydrodynamic limit of particle process

Difficult to generalise full K-matrix to XXZ case [Tsuboi; Mangazeev-Lu]

- Still we can write stochastic boundaries for q-Hahn process [to appear]

Steady state for non-compact models?

- Not clear how DEHP ansatz works
- Come up with something different... Here for SSEP

Hidden equilibrium of SSEP

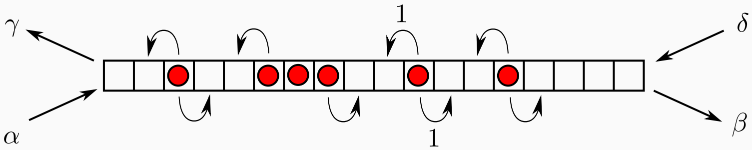
Definition of steady state

Back to the SSEP chain

$$H = H_1^L + \sum_{k=1}^{N-1} (P_{k,k+1} - I_{k,k+1}) + H_N^R$$

with boundary terms

$$H^L = \begin{pmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{pmatrix}, \qquad H^R = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix}$$



A theorem

Spin chain:

$\exists$ a similarity transformation *W* which maps the XXX spin chain with lower/upper triangular boundaries to the spin chain with diagonal boundaries

$\updownarrow$

Particle process:

$\exists$ a similarity transformation *W′* which maps the Markov generator *H* of the open SSEP with arbitrary $\alpha, \beta, \gamma, \delta$ to its equilibrium with $\alpha = \delta$ and $\gamma = \beta$.

Similarity transformation for SSEP

Similarity transformation W for SSEP

$$H = W\,H_{equil}\,W^{-1}$$

In SSEP the perturbation series can be written explicitly in terms of non-local charge Q_-

$$W \sim \sum_{k=0}^{N} \frac{1}{k!}\left(\frac{Q_-}{\Gamma_1\Gamma_N}\right)^k \frac{\Gamma\left(\sigma^3_{tot} + \Gamma_1^{-1} + \Gamma_N^{-1} - k\right)}{\Gamma\left(\sigma^3_{tot} + \Gamma_1^{-1} + \Gamma_N^{-1}\right)}$$

with first nonlocal charge at order x^{2N} of transfer matrix

$$Q_- = \Gamma_1(\rho_N - \rho_1)\left(\sigma^-_{tot} + \Gamma_N \sum_{i=1}^{N} \sigma^-_i \left(\frac{\sigma^3_i - 1}{2} + \sum_{k=i+1}^{N} \sigma^3_k\right)\right).$$

Here $\Gamma_1 = \alpha + \gamma\,, \Gamma_N = \delta + \beta\,, \rho_1 = \alpha/(\alpha+\gamma)\,, \rho_N = \delta/(\delta+\beta)$.

Similarity transformation for SSEP

W maps any equilibrium eigenstate to non-equilibrium!

Steady state obtained via from equilibrium via

$$|\psi\rangle = W|\psi_{equil}\rangle$$

- Observed macroscopically by [Tailleur,Kurchan,Lecomte; Bertini et al.]

Closed-form expression for probabilities in steady state

$$P(\underline{m}) = \sum_{h=0}^{N} (1-\rho_N)^{N-h} (\rho_1 - \rho_N)^h \sum_{1\le i_1<\ldots<i_h\le N} \mathfrak{M}(i_1\ldots,i_h;\underline{m}),$$

with the coefficients

$$\mathfrak{M}(i_1\ldots,i_h;\underline{m}) = (-1)^{\sum_{k=1}^h m_{i_k}} \left(\frac{\delta}{\beta}\right)^{m-\sum_{k=1}^h m_{i_k}} \prod_{k=1}^{h} \frac{i_k + h - k - N - \frac{1}{\beta+\delta}}{N - k + \frac{1}{\alpha+\gamma} + \frac{1}{\beta+\delta}}.$$

Closed-form expression

k-point correlation function

$$\langle i_1 \cdots i_k \rangle = \sum_{m=0}^{k} (\rho_N - \rho_1)^m (\rho_N)^{k-m} \sum_{1 \le l_1 < \ldots < l_m \le k} \prod_{r=1}^{m} \frac{i_{l_r} + m - r - N - \frac{1}{\beta+\delta}}{N - r + \frac{1}{\alpha+\gamma} + \frac{1}{\beta+\delta}} .$$

Steady state:

- No DEHP algebra needed
- Solving of Bethe equations not necessary
- Works the same way for non-compact model [to appear]

Conclusion & Outlook

Conclusion

Take home messages:

- Very interesting connections between high energy physics, quantum groups, condensed matter and probability theory
- q-Hahn type models are described by non-compact spin chains!
- QISM is useful to study integrable stochastic processes
  - Introduced integrable boundaries
  - Obtained steady state
- Open SSEP can be mapped to equilibrium

Outlook

Outlook and open problems

Non-compact

- TAZRPs and Baxter Q-operators? Relation to [Lazarescu,Pasquier]?
- Generalisation to $\mathfrak{su}_q(n,1)$ and relation to stochastic R-matrix (multi-species generalisation)

  [Kuniba,Mangazeev,Maruyama,Okado]
- Implications for AdS/CFT? Hopping of fermions?

Compact chains

- Hidden equilibrium and steady state in ASEP?
- Apply off-diagonal Bethe ansatz or SoV?
- Implications for Quantum SSEP (Tony Jin's talk)? Duality

  [Giardiná,Kurchan,RF]

Thank you!

Relation to stochastic q-Hahn process

As in ASEP, Hamiltonian density $\mathcal{H}$ is not a Markov matrix!

Similarity transformation yields Markov matrix

$$\mathcal{M} = \begin{pmatrix} \alpha_+(n)+\alpha_-(0) & -\beta_-(1,1) & -\beta_-(2,2) & \cdots & -\beta_-(n,n) \\ -\beta_+(n,1) & \alpha_+(n-1)+\alpha_-(1) & -\beta_-(2,1) & \cdots & -\beta_-(n,n-1) \\ -\beta_+(n,2) & -\beta_+(n-1,1) & \alpha_+(n-2)+\alpha_-(2) & \cdots & -\beta_-(n,n-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_+(n,n) & -\beta_+(n-1,n-1) & -\beta_+(n-2,n-2) & \cdots & \alpha_+(0)+\alpha_-(n) \end{pmatrix}.$$

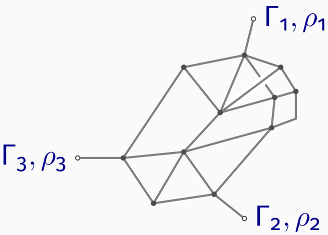
with

$$\beta_\pm(m,k) = \frac{\mu^{\frac{1}{2}k(1\pm 1)}}{\mu\left(1-\gamma^k\right)} \frac{(\gamma;\gamma)_m(\mu;\gamma)_{m-k}}{(\gamma;\gamma)_{m-k}(\mu;\gamma)_m},$$

where $\gamma = q^2$ and $\mu = q^{4s}$

Coincides with rates of q-Hahn process studied by [Povolotsky;Barraquand-Corwin;Sasamoto-Wadati] without reference to XXZ chain!

Generalisation



- Models with a dual absorption process can be solved in a similar way using perturbation theory, e.g. non-integrable SEP$_j$-type models on arbitrary graph. ($H = H_0 + \Delta H_1$)
- In integrable models we can use one of the non-local charges to sum up perturbation theory
- Idea of perturbing around diagonal chain

[Alcaraz, Droz, Henkel, Rittenberg '97]