Free-Fermion entanglement and Leonard pairs

Nicolas CRAMPE

Institut Denis-Poisson, Tours

based on work done in collaboration with

Pierre-Antoine Bernard (CRM)
Krystal Guo (CRM)
Rafael Nepomechie (U. Miami)
Luc Vinet (CRM)

Introduction

- **Physical interest:**

Free-Fermion models on 1D system or graph

$$\Downarrow$$

Computation of entanglement entropy

$$=$$

Spectrum of the chopped correlation matrix C

- **Numerical issue:** C is hard to be diagonalized numerically

- **Surprising fact (V. Eisler and I. Peschel):**

Tridiagonal matrix T commutes with C and is easy to be diagonalized

- **Goals:**
  - Identify T as an algebraic Heun operator of Leonard pairs
  - Classify the models where T exists

Outline

- Ground state of free-Fermion Hamiltonian
- Chopped correlation matrix and entanglement entropy
- Leonard pairs and algebraic Heun operators
- Algebraic Heun operator and chopped correlation matrix
- Example: Uniform chain
- Further results and concluding remarks

Ground state of free-Fermion Hamiltonian

Open quadratic free-Fermion inhomogeneous Hamiltonian

$$\widehat{\mathscr{H}} = \sum_{n=0}^{N-1} J_n(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) - \sum_{n=0}^{N} B_n c_n^\dagger c_n = \sum_{m,n=0}^{N} c_m^\dagger \widehat{H}_{mn} c_n \ ,$$

J_n and B_n real parameters,

$$\{c_m^\dagger, c_n\} = \delta_{m,n} \ , \quad \{c_m^\dagger, c_n^\dagger\} = \{c_m, c_n\} = 0 \ .$$

To diagonalize $\widehat{\mathscr{H}}$, first diagonalize $(N+1)\times(N+1)$ matrix

$$\widehat{H} = |\widehat{H}_{mn}|_{0\le m,n\le N} = \begin{pmatrix} B_0 & J_0 & 0 & & & \\ J_0 & B_1 & J_1 & 0 & & \\ 0 & J_1 & B_2 & J_2 & 0 & \\ & & \ddots & \ddots & \ddots & \\ & & 0 & J_{N-2} & B_{N-1} & J_{N-1} \\ & & & 0 & J_{N-1} & B_N \end{pmatrix}$$

Ground state of free-Fermion Hamiltonian

Two orthonormal basis

- **Position basis** $\{|0\rangle, |1\rangle, \dots, |N\rangle\}$

$$\widehat{H}|n\rangle = J_{n-1}|n-1\rangle - B_n|n\rangle + J_n|n+1\rangle\,,$$

- **Momentum basis** $\{|\omega_k\rangle\}$

$$\widehat{H}|\omega_k\rangle = \omega_k|\omega_k\rangle \quad \textit{with} \quad |\omega_k\rangle = \sum_{n=0}^{N} \phi_n(\omega_k)|n\rangle$$

We order the $N+1$ eigenvalues $\omega_0 < \omega_1 < \cdots < \omega_N$

$\phi_n(\omega_k)$, the eigenfunctions, are related to orthogonal polynomials.

Ground state of free-Fermion Hamiltonian

Having diagonalized $\widehat{H}$, we see that Hamiltonian $\widehat{\mathscr{H}}$ can be rewritten as

$$\widehat{\mathscr{H}} = \sum_{k=0}^{N} \omega_k \tilde{c}_k^\dagger \tilde{c}_k \,,$$

where the annihilation operators $\tilde{c}_k$ are given by

$$\tilde{c}_k = \sum_{n=0}^{N} \phi_n(\omega_k)\, c_n \,, \qquad c_n = \sum_{k=0}^{N} \phi_n(\omega_k)\, \tilde{c}_k \,,$$

and creation operators $\tilde{c}_k^\dagger$ obtained by Hermitian conjugation.

These obey

$$\{\tilde{c}_k^\dagger, \tilde{c}_p\} = \delta_{k,p} \,, \qquad \{\tilde{c}_k^\dagger, \tilde{c}_p^\dagger\} = \{\tilde{c}_k, \tilde{c}_p\} = 0$$

Ground state of free-Fermion Hamiltonian

Eigenvectors of $\widehat{\mathscr{H}}$ given by

$$|\Psi\rangle\rangle = \tilde{c}^\dagger_{k_1} \dots \tilde{c}^\dagger_{k_r} |0\rangle\rangle\,,$$

with $k_1, \dots, k_r \in \{0, \dots, N\}$ pairwise distinct

Vacuum state $|0\rangle\rangle$ is annihilated by all the annihilation operators

$$\tilde{c}_k |0\rangle\rangle = 0\,, \qquad k = 0\,, \dots\,, N$$

The energy eigenvalues are given by

$$E = \sum_{i=1}^{r} \omega_{k_i}$$

Ground state of free-Fermion Hamiltonian

Ground state $|\Psi_0\rangle\rangle$ is constructed by filling the Fermi sea:

$$|\Psi_0\rangle\rangle = \tilde{c}_0^\dagger \dots \tilde{c}_K^\dagger |0\rangle\rangle\,,$$

where $K \in \{0, 1, \dots, N\}$ is the greatest integer below the Fermi momentum, such that

$$\omega_K < 0\,, \qquad \omega_{K+1} > 0\,.$$

K can be modified by adding a constant term in the external magnetic field B_n.

Chopped correlation matrix and entanglement entropy

The 1- particle correlation matrix $\widehat{C}$ in the ground state is the $(N+1)\times(N+1)$ matrix with entries

$$\widehat{C}_{mn} = \langle\langle\Psi_0|c_m^\dagger c_n|\Psi_0\rangle\rangle\,.$$

It is seen

$$\widehat{C} = \sum_{k=0}^{K} |\omega_k\rangle\langle\omega_k|\,,$$

i.e. $\widehat{C}$ is projector onto subspace of $\mathbb{C}^{N+1}$ spanned by vectors $|\omega_k\rangle$ with $k=0,\ldots,K$

Chopped correlation matrix and entanglement entropy

To discuss entanglement, one needs **bipartition**:

- Part 1: sites $\{0, 1, ..., \ell\}$
- Part 2: sites $\{\ell+1, \ell+2, ..., N\}$

Entanglement properties in ground state is provided by reduced density matrix

$$\rho_1 = tr_2|\Psi_0\rangle\rangle\langle\langle\Psi_0| \qquad (2^{\ell+1} \times 2^{\ell+1})$$

Observation (Peschel, Vidal et al.):
ρ_1 is determined by "chopped" correlation matrix C
$(\ell+1) \times (\ell+1)$ submatrix of $\widehat{C}$:

$$C = |\widehat{C}_{mn}|_{0 \le m,n \le \ell}$$

Chopped correlation matrix and entanglement entropy

Introduce the projectors

$$\pi_1 = \sum_{n=0}^{\ell} |n\rangle\langle n| \quad \text{and} \quad \pi_2 = \sum_{k=0}^{K} |\omega_k\rangle\langle\omega_k| = \widehat{C}\,,$$

the chopped correlation matrix can be written as

$$C = \pi_1\pi_2\pi_1$$

To calculate entanglement entropies one has to compute the eigenvalues of C

Not easy to do numerically because the eigenvalues of that matrix are exponentially close to 0 and 1

Parallel between study of entanglement properties of finite free-Fermion chains and *time and band limiting problems* will indicate how this can be circumvented

Leonard pairs and Algebraic Heun operators

Definition Leonard pairs (A, A^*)
A and A^* are linear transformation of V ($\dim V < +\infty$) such that

- In a basis $\mathscr{B}$ of V, A is diagonal and A^* is irreducible tridiagonal
- In a basis $\mathscr{B}^*$ of V, A^* is diagonal and A is irreducible tridiagonal

Remarks

- Leonard pairs have been classified
- Leonard pairs satisfy Askey–Wilson algebra
- Bispectral problems and orthogonal polynomials

Tridiagonalization The operator

$$W = r_0 + r_1 A + r_2 A^* + r_3\{A, A^*\} + r_4[A, A^*]$$

is the more general operator tridiagonal in both bases $\mathscr{B}$ and $\mathscr{B}^*$.
W is called algebraic Heun operator.

Why “Heun” ?

Let us choose

$$A = x(x-1)\frac{d^2}{dx^2} + (\alpha+1-(\alpha+\beta+2)x)\frac{d}{dx}$$
$$A^* = x$$

Then the operator W becomes

$$W \sim \frac{d^2}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\varepsilon}{x-d}\right)\frac{d}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-d)},$$

and is the standard differential Heun operator (Fuchsian 2nd order differential equation with four regular singularities).

Algebraic Heun operator and chopped correlation matrix

Strategy

- Pick J_n and B_n in the Hamiltonian so that $\widehat{H}$ is one element of a Leonard pair $(\widehat{H}, \widehat{X})$
- Construct an algebraic Heun operator and prove that it commutes with the chopped correlation matrix

Recall

$$\widehat{H}|\omega_k\rangle = \omega_k|\omega_k\rangle, \qquad \widehat{H}|n\rangle = J_{n-1}|n-1\rangle - B_n|n\rangle + J_n|n+1\rangle$$

By definition of Leonard pairs

$$\widehat{X}|\omega_k\rangle = \overline{J}_{k-1}|\omega_{k-1}\rangle - \overline{B}_k|\omega_k\rangle + \overline{J}_k|\omega_{k+1}\rangle, \qquad \widehat{X}|n\rangle = \lambda_n|n\rangle$$

Algebraic Heun operator and chopped correlation matrix

Introduce algebraic Heun operator:

$$\widehat{T} = \{\widehat{X}, \widehat{H}\} + \mu \widehat{X} + \nu \widehat{H}$$

In position basis

$$\begin{aligned} \widehat{T}|n\rangle &= J_{n-1}(\lambda_{n-1} + \lambda_n + \nu)|n-1\rangle + (\mu\lambda_n - 2B_n\lambda_n - \nu B_n)|n\rangle \\ &\quad + J_n(\lambda_n + \lambda_{n+1} + \nu)|n+1\rangle \ , \end{aligned}$$

Recall that $\pi_1 = \sum_{n=0}^{\ell} |n\rangle\langle n|$.
One gets

$$[\widehat{T}, \pi_1] = 0$$

if

$$\nu = -(\lambda_\ell + \lambda_{\ell+1})$$

Algebraic Heun operator and chopped correlation matrix

In momentum basis

$$\begin{aligned}\widehat{T}|\omega_k\rangle &= \overline{J}_{k-1}(\omega_{k-1}+\omega_k+\mu)|\omega_{k-1}\rangle + (\nu\omega_k - 2\overline{B}_k\omega_k - \mu\overline{B}_k)|\omega_k\rangle \\ &\quad + \overline{J}_k(\omega_k+\omega_{k+1}+\mu)|\omega_{k+1}\rangle\end{aligned}$$

and

$$\pi_2 = \sum_{k=0}^{K} |\omega_k\rangle\langle\omega_k|$$

One gets

$$[\widehat{T},\pi_2] = 0$$

if

$$\mu = -(\omega_K + \omega_{K+1})$$

Since $C = \pi_1\pi_2\pi_1$, one gets

$$[T,C] = 0$$

Algebraic Heun operator and chopped correlation matrix

The tridiagonal matrix

$$T = \begin{pmatrix} d_0 & t_0 & & & & \\ t_0 & d_1 & t_1 & & & \\ & t_1 & d_2 & t_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & t_{\ell-2} & d_{\ell-1} & t_{\ell-1} \\ & & & & t_{\ell-1} & d_\ell \end{pmatrix}$$

with

$$t_n = J_n(\lambda_n + \lambda_{n+1} - \lambda_\ell - \lambda_{\ell+1})$$

$$d_n = -B_n(2\lambda_n - \lambda_\ell - \lambda_{\ell+1}) - \lambda_n(\omega_K + \omega_{K+1})\,.$$

commutes with the chopped correlation matrix C

The homogeneous chain

Let us choose

$$J_0 = \ldots = J_{N-1} = -\frac{1}{2}\,, \qquad B_n = 0\,.$$

The associated eigenvalues of the Hamiltonian $\widehat{H}$ are

$$\omega_k = -\cos\left(\frac{\pi(k+1)}{N+2}\right)\,, \qquad k = 0, 1, \ldots, N\,.$$

The matrix T is then given by with

$$t_n = \frac{1}{2}\left[\cos(\theta_n) + \cos(\theta_{n+1}) - \cos(\theta_\ell) - \cos(\theta_{\ell+1})\right]$$

$$d_n = -\cos(\theta_n)\left[\cos(\theta_K) + \cos(\theta_{K+1})\right]$$

This readily recovers recent results of Eisler & Peschl.

Further results and concluding remarks

- Shown that for chains associated to Leonard pairs (bispectral orthogonal polynomials), algebraic Heun operator readily provides a tridiagonal matrix that commutes with correlation matrix
- The approach provides such commuting matrices for the many chains corresponding to finite discrete polynomials of Askey scheme

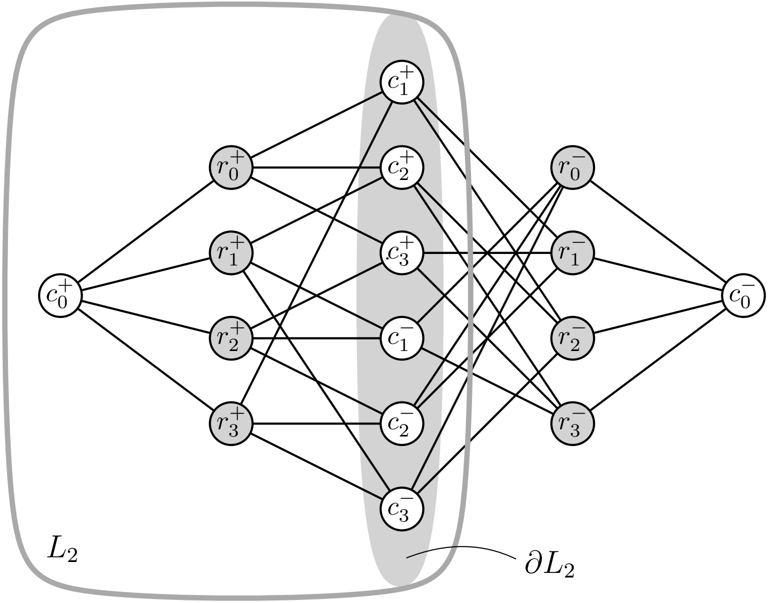
N. Crampé, R. Nepomechie, L. Vinet, *Entanglement in Fermionic Chains and Bispectrality*, Roman Jackiw 80th Birthday Festschrift, arXiv:2001.10576
N. Crampé, R. Nepomechie, L. Vinet, *Free-Fermion entanglement and orthogonal polynomials*, J. Stat. Mech. (2019) arXiv:1907.00044

Further results and concluding remarks

- Generalization to graphs belonging to a P- and Q- association schemes
  Leonard pair $\longrightarrow$ Tridiagonal pair

N. Crampé, K. Guo, L. Vinet, *Entanglement of Free Fermions on Hadamard Graphs*, NPB and arXiv:2008.04925
P.-A. Bernard, N. Crampé, K. Guo, L. Vinet, *Free Fermions on Hamming Graphs*, to appear

- Algebraic Heun operator is one conserved quantity associated to integrable models (Gaudin, XXZ)
  $\Rightarrow$ Diagonalization by Bethe ansatz or other methods



THANK YOU FOR YOUR ATTENTION !