The U(1)-invariant Potts model and its symmetries

Eric Vernier (Oxford University)
with Paul Fendley and Edward O'Brien

RAQIS, Annecy, 12/09/2018

Background: degeneracies at root of unity

XXZ Hamiltionian

$$H = \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cos\gamma\, \sigma_i^z \sigma_{i+1}^z \qquad \gamma = \pi \frac{m}{n} \qquad q = e^{i\gamma}$$

Bethe equations :

$$\left(\frac{\sinh(\lambda_k + i\gamma/2)}{\sinh(\lambda_k - i\gamma/2)} \right)^L = \prod_{l(\neq k)} \frac{\sinh(\lambda_k - \lambda_l + i\gamma)}{\sinh(\lambda_k - \lambda_l - i\gamma)} \qquad E = \sum_k \frac{4\sin^2\gamma}{\cosh 2\lambda_k + \cos\gamma}$$

Degeneracies between states of magnetizations $S^z, S^z + n, S^z + 2n, \ldots$

These are due to **exact n-strings** : $x, x + i\gamma, \ldots, x + i(n-1)\gamma$ **Fabricius McCoy'01**

$$\frac{4\sin^2\gamma}{\cosh 2x + \cos\gamma} + \ldots + \frac{4\sin^2\gamma}{\cosh 2(x + i(n-1)\gamma) + \cos\gamma} = 0$$ zero energy

$$\frac{\sinh(\lambda_k - x + i\gamma)}{\sinh(\lambda_k - x - i\gamma)} \times \ldots \times \frac{\sinh(\lambda_k - x + i(n-1)\gamma + i\gamma)}{\sinh(\lambda_k - x + i(n-1)\gamma - i\gamma)} = 1$$ do not change BAE for other roots

$$\left(\frac{\sinh(x + i\gamma/2)}{\sinh(x - i\gamma/2)} \right)^L = \frac{0}{0}$$ "0/0 solutions" (even though, the original BAE are really 0=0)

Algebraic structure behind this : $L(sl_2)$ loop algebra **Deguchi Fabricius McCoy'01**

Some questions :

→ are exact n-strings quantized ? **Fabricius McCoy'01, Baxter '02**

→ do they have some more physics ?
(do they play the role of quasiparticles for another Hamiltonian?)

→ implications for CFT ?

To investigate this, we will look at a seemingly different model

The U(1)-invariant Potts model

Self-dual $\mathbb{Z}_n$ quantum chain :

$$(\mathbb{C}^n)^{\otimes L} \qquad H = \sum_{j=1}^{L} \sum_{a=1}^{n-1} \alpha_n \left(\tau_j^n + (\sigma_j^\dagger \sigma_{j+1})^n\right)$$

$$\tau = \begin{pmatrix} 1 & & & \\ & e^{\frac{2i\pi}{n}} & & \\ & & \ddots & \\ & & & e^{\frac{2i\pi(n-1)}{n}} \end{pmatrix}$$

Duality : $\tau_j \longrightarrow \sigma_j^\dagger \sigma_{j+1} \longrightarrow \tau_{j+1}$

$$\sigma = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$$

We look for a new model which preserves **nearest-neighbour** interaction and **self-duality**, but in addition has **U(1)-invariance**

$$\boxed{H = \sum_{j=1}^{L} \sum_{a,b=1}^{n-1} \frac{\sin \frac{ab\pi}{n}}{\sin \frac{a\pi}{n} \sin \frac{b\pi}{n}} \left(\tau_j^a (\sigma_j^\dagger \sigma_{j+1})^b + (\sigma_j^\dagger \sigma_{j+1})^b \tau_{j+1}^a\right) - \sum_{j=1}^{L} \sum_{a=1}^{n-1} \frac{n-2a}{\sin \frac{a\pi}{n}} \left(\tau_j^a + (\sigma_j^\dagger \sigma_{j+1})^a\right)}$$

Conserved U(1) :

$$Q = \sum_{j=1}^{L} \sum_{a=1}^{n-1} \frac{\tau_j^a}{1 - e^{-\frac{2i\pi a}{n}}} = \sum_{j=1}^{L} q_j$$

$$q = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & & n-1 \end{pmatrix}$$

Coordinate Bethe ansatz

Because of U(1)-invariance, we can try CBA (for illustration $n = 3$)

$$|\{\}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^{\otimes L} \equiv |- - \ldots -\rangle$$

$$|\{k\}\rangle = \sum_{x=1}^{L} e^{ikx} \equiv |- - \ldots 0_x \ldots -\rangle$$

$$|\{k_1, k_2\}\rangle = \sum_{x_1 \leq x_2} \left(A_{12} e^{i(k_1 x_1 + k_2 x_2)} + A_{21} e^{i(k_2 x_1 + k_1 x_2)} \right) | \ldots 0_{x_1} \ldots 0_{x_2} \ldots \rangle \qquad |0_x 0_x\rangle \equiv |+_x\rangle$$

etc...

$$\boxed{e^{iLk_j} = \prod_{l \neq j} \frac{1 + e^{ik_l} + e^{i(k_l + k_j)}}{1 + e^{ik_j} + e^{i(k_l + k_j)}} \qquad E = -L + \sum_j (2 \cos k_j - 1)}$$

Reparametrization : $e^{ik} = -\frac{e^{i\frac{\pi}{3}} + e^{2\lambda}}{e^{i\frac{\pi}{3}} e^{2\lambda} + 1}$

$$\boxed{\left(\frac{\sinh\left(\lambda_j - i\frac{\pi}{3}\right)}{\sinh\left(\lambda_j + i\frac{\pi}{3}\right)}\right)^L = \prod_{l \neq j} \frac{\sinh\left(\lambda_j - \lambda_l - i\frac{\pi}{3}\right)}{\sinh\left(\lambda_j - \lambda_l + i\frac{\pi}{3}\right)} \qquad E = -L + \sum_j \frac{6}{2\cosh\lambda_j + 1}}$$

Same as spin-1 XXZ chain at $\gamma = \frac{\pi}{3}$ **Fateev Zamolodchikov' 80**

And indeed, the Hamiltonians coincide ! In spin language :

$$H = \sum_j \left[S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+ - (S_j^+)^2 (S_{j+1}^-)^2 - (S_j^-)^2 (S_{j+1}^+)^2 - S_j^z (S_j^z + 1)\right]$$

$$S^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad S^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad S^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

→ **Degeneracies** due to **exact 3-strings** $x - i\frac{\pi}{3},\ x,\ x + i\frac{\pi}{3}$

→ continuum limit: antiferro: c=3/2 superCFT **Alcaraz Martins '89**

ferro: c=1 boson **Baranowski Rittenberg '90**

→ generally,

n-state U(1) Potts = spin-$\frac{n-1}{2}$ XXZ at $\gamma = \frac{\pi}{n}$

So far so good, but we haven’t learned much new about (higher spin) XXZ

Now, we will use the U(1) Potts formulation to access some interesting new (or not so new) features :

1. Onsager algebra symmetry

2. “chiral decomposition”

Onsager symmetry

A simple observation to start with

H self-dual and commutes with Q, so it should also commute with the dual of Q !

$$\hat{Q} = \sum_{j=1}^{L}\sum_{a=1}^{n-1} \frac{(\sigma_j^{\dagger}\sigma_{j+1})^a}{1-e^{-\frac{2i\pi a}{n}}} = \hat{Q}^- + \hat{Q}^0 + \hat{Q}^+$$

changes Q by -n

U(1)-neutral

changes Q by +n

For n=3,

$$\hat{Q}^0 = \frac{i}{\sqrt{3}}\sum_j \left(S_j^+ S_{j+1}^- - S_j^- S_{j+1}^+ - (S_j^+ S_{j+1}^-)^2 + (S_j^- S_{j+1}^+)^2\right)$$

$$\hat{Q}^- = \frac{i}{\sqrt{3}}\sum_j S_j^-(S_{j+1}^- - S_j^-)S_{j+1}^-$$

$$\hat{Q}^+ = \frac{i}{\sqrt{3}}\sum_j S_j^+(S_{j+1}^+ - S_j^+)S_{j+1}^-$$

Easy to check that in fact,

$$[H, \hat{Q}^{0,+,-}] = 0$$

2nd observation :

$$[Q,[Q,[Q,\hat{Q}]]] = n^2[Q,\hat{Q}]$$
$$[\hat{Q},[\hat{Q},[\hat{Q},Q]]] = n^2[\hat{Q},Q]$$

Dolan-Grady relations

Dolan Grady' 82

Therefore, $Q, \hat{Q}$ generate the **Onsager algebra** (algebra of fermion bilinears)

$$A_m = \frac{4}{n}\left(\hat{Q}^0_m + \hat{Q}^+_m + \hat{Q}^-_m\right) \qquad A_0 = \frac{4}{n}Q \qquad A_1 = \frac{4}{n}\hat{Q}$$

Onsager '44, Davies' 90

$$A_{-m} = \frac{4}{n}\left(\hat{Q}^0_m - \hat{Q}^+_m - \hat{Q}^-_m\right)$$

Duality : $A_m \leftrightarrow A_{1-m}$

Conclusions :

→ $[H, \mathrm{Onsager}] = 0$ **so Onsager related to degeneracies**

already noticed in XXZ from loop algebra : **Deguchi Fabricius McCoy'01**
Nishino Deguchi '06

→ connection with superintegrable Chiral Potts : $H_{\mathrm{si}} = Q + J\hat{Q}$

von Gehlen Rittenberg '85, Albertini McCoy Perk Tang '89

Chiral decomposition

Originally (before noticing the spin-1 correspondence) we were wondering whether we could deform H into something chiral, but still integrable.

Chiral decomposition (n=3 for the example)

Originally (before noticing the spin-1 correspondence) we were wondering whether we could deform H into something chiral, but still integrable.

the answer is yes:

$$H = \sum_j \left[S_j^+ S_{j+1}^- - (S_j^+)^2 (S_{j+1}^-)^2 - \frac{1}{2} S_j^z (S_j^z + 1) + S_j^- S_{j+1}^+ - (S_j^-)^2 (S_{j+1}^+)^2 - \frac{1}{2} S_j^z (S_j^z + 1) \right]$$

$$= H_+ + H_- \qquad [H_+, H_-] = 0 \quad \text{(for periodic bc)}$$

→ $H_\pm$ are purely left (right) moving

→ energy of 1 particle : $\epsilon_\pm = e^{\pm ik} - \frac{1}{2} = -\frac{e^{\pm i\frac{\pi}{3}} + e^{2\lambda}}{e^{\pm i\frac{\pi}{3}} e^{2\lambda} + 1} - \frac{1}{2}$

→ energy of one exact 3-string $\epsilon_\pm \left(\{ x - i\frac{\pi}{3}, x, x + i\frac{\pi}{3} \} \right) = \mp 3\sqrt{3} \tanh(3x)$

so $H_\pm$ lift the degeneracies due to exact 3-strings

therefore exact 3-strings should be quantized through CBA for H_+ (or H_-)

→ relation with Onsager : $\hat{Q}^0 = -\frac{i}{\sqrt{3}} (H_+ - H_-)$

Coordinate Bethe ansatz for H_+

→ Ordinary particles satisfy the same BAE as for the full H

→ But in addition we find a quantization for the exact 3-strings:

$$\boxed{1 + e^{ikL}\prod_j \left(- \frac{1+e^{ik}+e^{i(k+k_j)}}{1+e^{ik_j}+e^{i(k+k_j)}} \right) + \left(\frac{e^{ik}}{1+e^{ik}}\right)^L \prod_j \left(- \frac{-e^{ik}+e^{ik_j}}{1+e^{ik_j}+e^{i(k+k_j)}} \right) = 0}$$

k characterizes the exact 3-string

scattering with ordinary particles

Exact 3-strings **do not feel one another**, only through some **exclusion principle**

eg: in the absence of ordinary particles, they are quantized by the solutions of

$$1 + e^{ikL} + \left(\frac{e^{ik}}{1+e^{ik}}\right)^L = 0$$

In terms of Chebyshev polynomials :

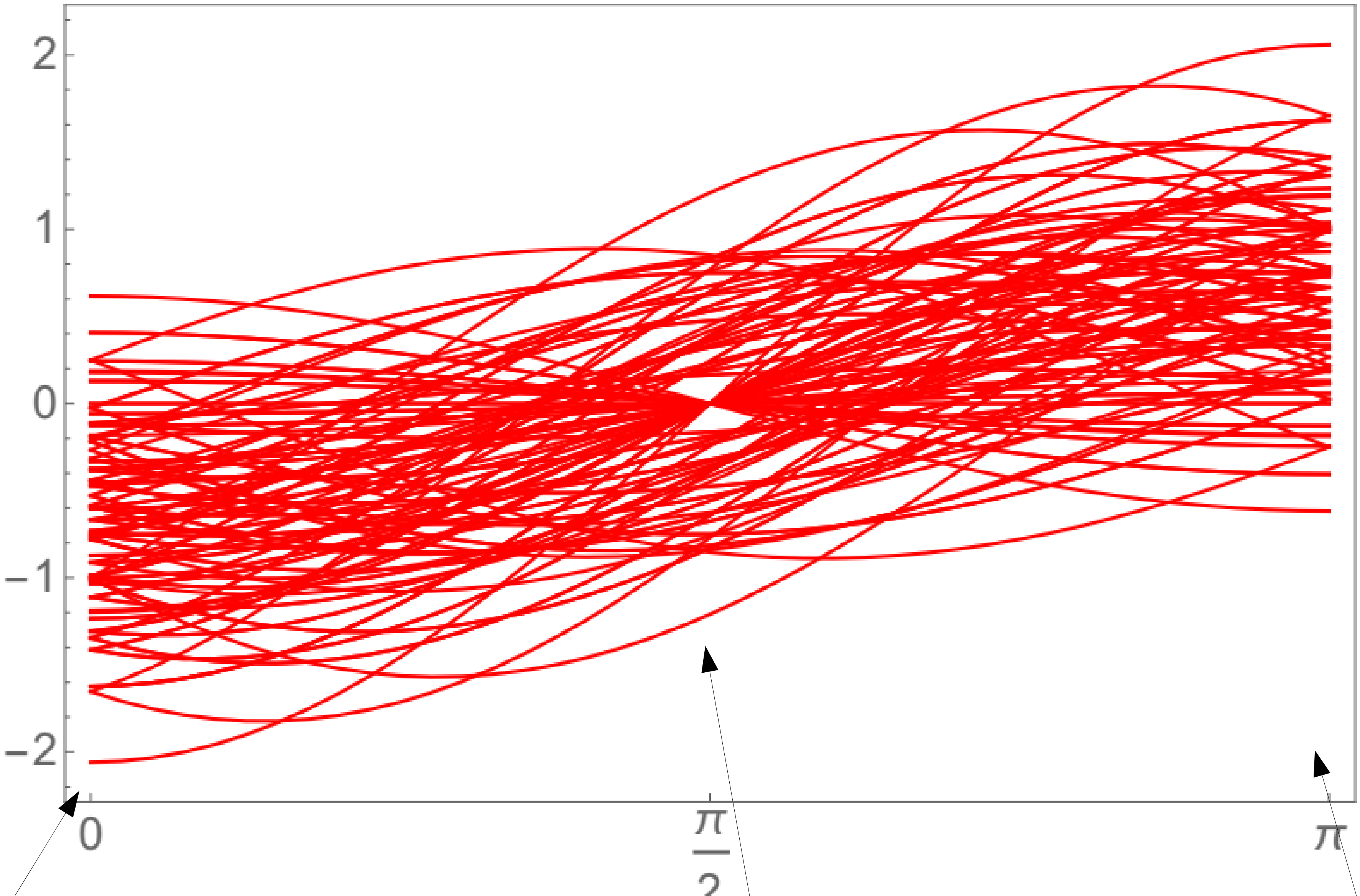
$$\left(\cos\frac{k}{2}\right) T_L\left(\cos\frac{k}{2}\right) = (-2)^{-L-1}$$

Remarks :

→ not the same quantization as that of **Fabricius McCoy'01,** obtained from a limit (but the latter is unrelated with $H_\pm$)

→ **Crampé Frappat Ragoucy '13** study 3-states models solvable by CBA, but it looks like $H_\pm$ are not directly related to their classification

Spectrum of the family $e^{i\alpha}H_+ + e^{-i\alpha}H_-$



H_{XXZ} spin 1

antiferro, c=3/2
ground state = L/2 2-strings

$i\,(H_+ - H_-)$

ground state = L/3 exact 3-strings

$-H_{\mathrm{XXZ}}$ spin 1

ferro, c=1
ground state = L (1-)-strings

To finish :

Can we fit all that into a quantum group construction ?

Quantum group construction I (once again n=3 for illustration)

The XXZ spin-1 Hamiltonian is generated by a family of transfer matrices:

$$T(u) = \mathrm{Tr}_{\mathcal{A}}\left(\mathcal{L}_L(u)\ldots\mathcal{L}_1(u)\right)$$

$$\mathcal{L}_i(u) = \begin{pmatrix} \frac{\sin(u+\gamma\mathbf{S}^z)}{\sin\gamma}\frac{\sin(u+\gamma\mathbf{S}^z+\gamma)}{\sin\gamma} & \sqrt{[2]}\mathbf{S}^-\frac{\sin(u+\gamma\mathbf{S}^z)}{\sin\gamma} & (\mathbf{S}^-)^2 \\ \sqrt{[2]}\mathbf{S}^+\frac{\sin(u+\gamma\mathbf{S}^z+\gamma)}{\sin\gamma} & \mathbf{S}^+\mathbf{S}^- + \frac{\sin(u+\gamma\mathbf{S}^z+\gamma)}{\sin\gamma}\frac{\sin(u-\gamma\mathbf{S}^z)}{\sin\gamma} & \sqrt{[2]}\mathbf{S}^-\frac{\sin(u-\gamma\mathbf{S}^z+\gamma)}{\sin\gamma} \\ (\mathbf{S}^+)^2 & \sqrt{[2]}\mathbf{S}^+\frac{\sin(u-\gamma\mathbf{S}^z)}{\sin\gamma} & \frac{\sin(u-\gamma\mathbf{S}^z+\gamma)}{\sin\gamma}\frac{\sin(u-\gamma\mathbf{S}^z)}{\sin\gamma} \end{pmatrix}_i$$

$\mathbf{S}^z, \mathbf{S}^\pm$ are the $U_q(sl_2)$ spin-1 generators on the auxiliary space $\mathcal{A}$

Now, at $\gamma = \frac{\pi}{3}$, there are other 3-dimensional representations one can put on the aux space :

$$\mathbf{S}^z = \begin{pmatrix} v+1 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & v-1 \end{pmatrix} \qquad \mathbf{S}^- = \begin{pmatrix} 0 & 0 & \beta \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\mathbf{S}^+ = \begin{pmatrix} 0 & \alpha\beta + \frac{2\sin\left(\frac{4}{3}\pi(v+1)\right)}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{2\sin\left(\frac{2}{3}\pi(2(v+1)-1)\right)}{\sqrt{3}} - \alpha\beta \\ \alpha & 0 & 0 \end{pmatrix}$$

$v = \alpha = \beta = 0$ spin-1

$\alpha = \beta = 0$ *nilpotent*

$\alpha(\beta) = 0$ *semicyclic*

$\alpha, \beta \neq 0$ *cyclic*

This fact has been used in the last few years by the quench community to construct conserved quasilocal charges **Prosen, Ilievski, Medenjak, Zadnik ‘13, ‘14, ‘15**

But it has remained apparently unnoticed that one can also build local charges

Quantum group construction II : the chiral Hamiltonians

Focus on the nilpotent representations : 2 parameters u (spectral parameter), and v

→ new parameters $w, \bar{w} = u \pm \frac{\pi}{3} v$

→ we consider the transfer matrix $T(w)$ at $\bar{w} = 0$. It is generated by :

$$\check{R}(w) = \begin{pmatrix} 2\cos(4w)+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\cos(4w)+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\cos(4w)+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\sin(2w)\sin\left(2w+\frac{\pi}{3}\right) & 0 & 2\sqrt{3}\sin\left(2w+\frac{\pi}{3}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\sin(2w)\cos\left(\frac{\pi}{6}-2w\right) & 0 & 2\sqrt{3}\sin\left(2w+\frac{\pi}{3}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{3}\sin\left(2w+\frac{\pi}{3}\right) & 0 & 0 & 0 \\ 0 & 0 & 1-2\sin\left(4w+\frac{\pi}{6}\right) & 0 & 2\sqrt{3}\sin(2w) & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\sqrt{3}\sin(2w) & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Lower triangular, so purely right-moving !

and indeed,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}w} \log T(w)|_{w=0} = H_+}$$

→ TQ equation **De Luca, Collura, De Nardis '17**

$$T(u,v) = Q\left(u+(2-v)\frac{\pi}{3}\right) Q\left(u-(1-v)\frac{\pi}{3}\right) \sum_{m=-1}^{1} \frac{\sin\left(u+(m+v)\frac{\pi}{3}\right)^L \sin\left(u+(m+1+v)\frac{\pi}{3}\right)^L}{Q\left(u+(m+v)\frac{\pi}{3}\right) Q\left(u+(m+1+v)\frac{\pi}{3}\right)}$$

From there we can rederive the quantization of exact 3-strings !

Quantum group construction III : (semi)cyclic reps

$T(u, v, \alpha, \beta)$ do not commute with one another, but they do commute with the fundamental $T(u,0,0,0)$, and therefore with H

From there we can construct conserved charges which change Q by ± 3 . Eg :

$$\frac{d}{d\alpha} \log T(0,0,\alpha,0)\Big|_{\alpha=0} = \sum_j S_j^- (S_{j+1}^- - S_j^-) S_{j+1}^- \propto \hat{Q}^-$$

Onsager generators !

However it is not yet clear which precise scheme connects the $\hat{Q}_m^{\pm}$ to the successive derivatives of the TM

Conclusions

→ U(1) invariant Potts models = spin- $\frac{n-1}{2}$ XXZ at $\gamma = \frac{\pi}{n}$

→ chiral structure $H = H_+ + H_-$ $\quad [H_+, H_-] = 0$

Gives energy to the exact n-strings, lifts degeneracies

→ all of this hidden in quantum group auxiliary representations

Perspectives & aspects I didn't have time to tell about :

→ Implications for the continuum limit (symmetries of CFT characters)

→ Onsager generators in the CFT ?

→ supersymmetry of the n=3 chain **Hagendorf '17** **Bernard Pasquier ' 89**

→ Physics of the $H_+ - H_-$ model ?

→ Phase diagram of the 1 parameter Potts/U(1) Potts model **[to appear]**

Thank you for your attention !