Heisenberg XXX/XXZ spin chains by Separation of Variables

recent advances

Véronique TERRAS

CNRS - LPTMS, Univ. Paris Sud

Scalar products of separate states:

- N Kitanine, JM Maillet, G Niccoli, VT, J. Phys. A **49** (2016) 104002
- N Kitanine, JM Maillet, G Niccoli, VT, J. Phys. A **50** (2017) 224001
- N Kitanine, JM Maillet, G Niccoli, VT, arXiv:1807.05197

Recent Advances in Quantum Integrable Systems
Annecy, 10-14 September 2018

The Heisenberg spin-1/2 chain: an archetype of quantum integrable models

The XXZ spin-1/2 Heisenberg chain

$$H_{XXZ} = \sum_{m=1}^{N} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\, \sigma_m^z \sigma_{m+1}^z \right\}$$

- space of states: $\mathcal{H} = \otimes_{n=1}^{N} \mathcal{H}_n$ with $\mathcal{H}_n \simeq \mathbb{C}^2$
- $\sigma_m^{x,y,z} \in \mathrm{End}(\mathcal{H}_n)$: local spin-1/2 operators (Pauli matrices) at site m
- $\Delta = \cosh \eta$: anisotropy parameter $\to \Delta = 1$ for XXX (isotropic) chain
- usually periodic boundary conditions are considered: $\sigma_{N+1}^\alpha = \sigma_1^\alpha$

⋆ First model solved via Bethe ansatz [Bethe, 1931]

⋆ More algebraic solution in the framework of the Quantum Inverse Scattering Method (QISM) [Faddeev, Sklyanin, Takhtajan, 1979]

$\rightsquigarrow$ solution based on the representation theory of the Yang-Baxter algebra

QISM framework for quantum integrable models

Yang-Baxter algebra $\mathcal{A}_R$:

- generators $T_{ij}(\lambda)$, $1 \le i,j \le n$ $\quad\leftarrow$ elements of the monodromy matrix $T(\lambda)$
- commutation relations given by the R-matrix of the model:

$$R(\lambda-\mu)\,\big(T(\lambda)\otimes 1\big)\big(1\otimes T(\mu)\big) = \big(1\otimes T(\mu)\big)\big(T(\lambda)\otimes 1\big)\,R(\lambda-\mu)$$

- $R(\lambda) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ satisfies the Yang-Baxter equation (on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$):

$$R_{12}(\lambda_1-\lambda_2)\,R_{13}(\lambda_1-\lambda_3)\,R_{23}(\lambda_2-\lambda_3) = R_{23}(\lambda_2-\lambda_3)\,R_{13}(\lambda_1-\lambda_3)\,R_{12}(\lambda_1-\lambda_2)$$

$\hookrightarrow$ abelian subalgebra generated by $\quad t(\lambda) = \mathrm{tr}\, T(\lambda) \quad\leftarrow$ transfer matrix

$$[t(\lambda), t(\mu)] = 0 \quad \forall \lambda, \mu$$

A quantum integrable model with Hamiltonian H in the framework of the Quantum Inverse Scattering Method (QISM, Faddeev, Sklyanin, Takhtajan, 1979) is such that

- the space of states $\mathcal{H}$ of the model is constructed as a representation space of $\mathcal{A}_R$
- H is obtained in terms of the transfer matrix $\rightarrow$ $[H, t(\lambda)] = 0$
- the Yang-Baxter commutation relations are used to characterize the transfer matrix spectrum and eigenstates ($\rightarrow$ spectrum and eigenstates of H)

QISM framework for quantum integrable models

Yang-Baxter algebra $\mathcal{A}_R$:

- generators $T_{ij}(\lambda)$, $1 \leq i,j \leq n$ $\quad \leftarrow$ elements of the monodromy matrix $T(\lambda)$
- commutation relations given by the R-matrix of the model:
  $$R(\lambda-\mu)\,\big(T(\lambda)\otimes 1\big)\big(1\otimes T(\mu)\big) = \big(1\otimes T(\mu)\big)\big(T(\lambda)\otimes 1\big)\,R(\lambda-\mu)$$
- $R(\lambda) \in \mathrm{End}(\mathbb{C}^n\otimes\mathbb{C}^n)$ satisfies the Yang-Baxter equation (on $\mathbb{C}^n\otimes\mathbb{C}^n\otimes\mathbb{C}^n$):
  $$R_{12}(\lambda_1-\lambda_2)\,R_{13}(\lambda_1-\lambda_3)\,R_{23}(\lambda_2-\lambda_3) = R_{23}(\lambda_2-\lambda_3)\,R_{13}(\lambda_1-\lambda_3)\,R_{12}(\lambda_1-\lambda_2)$$

$\hookrightarrow$ abelian subalgebra generated by $\quad t(\lambda) = \mathrm{tr}\, T(\lambda) \quad \leftarrow$ transfer matrix

$$[t(\lambda), t(\mu)] = 0 \quad \forall \lambda, \mu$$

A quantum integrable model with Hamiltonian H in the framework of the Quantum Inverse Scattering Method (QISM, Faddeev, Sklyanin, Takhtajan, 1979) is such that

- the space of states $\mathcal{H}$ of the model is constructed as a representation space of $\mathcal{A}_R$
- H is obtained in terms of the transfer matrix $\rightarrow$ $[H, t(\lambda)] = 0$
- the Yang-Baxter commutation relations are used to characterize the transfer matrix spectrum and eigenstates ($\rightarrow$ spectrum and eigenstates of H)

The Yang-Baxter algebra for the Heisenberg spin-1/2 chain

$\sigma_m^\alpha \longrightarrow$ **quantum Lax operator at site** m

$$L_m(\lambda) = \begin{pmatrix} \varphi(\lambda + \eta\sigma_m^z) & \varphi(\eta)\,\sigma_m^- \\ \varphi(\eta)\,\sigma_m^+ & \varphi(\lambda - \eta\sigma_m^z) \end{pmatrix} \quad \in \mathrm{End}(V_a \otimes \mathcal{H}_m)$$

$V_a \simeq \mathbb{C}^2$: auxiliary space

$\mathcal{H}_m \simeq \mathbb{C}^2$: local quantum spin space at site m

such that it satisfies the quadratic relation

$$R(\lambda-\mu)\,\big(L_m(\lambda)\otimes 1\big)\big(1\otimes L_m(\mu)\big) = \big(1\otimes L_m(\mu)\big)\big(L_m(\lambda)\otimes 1\big)\,R(\lambda-\mu)$$

where the R-matrix of the model is the following solution of the Yang-Baxter equation:

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\varphi(\lambda)}{\varphi(\lambda+\eta)} & \frac{\varphi(\eta)}{\varphi(\lambda+\eta)} & 0 \\ 0 & \frac{\varphi(\eta)}{\varphi(\lambda+\eta)} & \frac{\varphi(\lambda)}{\varphi(\lambda+\eta)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \varphi(\lambda) = \begin{cases} \lambda & \text{(XXX chain)} \\ \sinh(\lambda) & \text{(XXZ chain)} \end{cases}$$

The Yang-Baxter algebra for the Heisenberg spin-1/2 chain

$\sigma_m^\alpha \longrightarrow$ **quantum Lax operator at site m**

$$L_m(\lambda) = \begin{pmatrix} \varphi(\lambda + \eta\sigma_m^z) & \varphi(\eta)\,\sigma_m^- \\ \varphi(\eta)\,\sigma_m^+ & \varphi(\lambda - \eta\sigma_m^z) \end{pmatrix} \quad \in \mathrm{End}(V_a \otimes \mathcal{H}_m)$$

$V_a \simeq \mathbb{C}^2$: auxiliary space
$\mathcal{H}_m \simeq \mathbb{C}^2$: local quantum spin space at site m

such that it satisfies the quadratic relation

$$R(\lambda-\mu)\,\big(L_m(\lambda)\otimes 1\big)\big(1\otimes L_m(\mu)\big) = \big(1\otimes L_m(\mu)\big)\big(L_m(\lambda)\otimes 1\big)\,R(\lambda-\mu)$$

$\rightsquigarrow$ **monodromy matrix for a quasi-periodic chain with twist K ($K \in GL_2(\mathbb{C})$):**

$$\begin{aligned} T_K(\lambda) &= K\,L_N(\lambda)\ldots L_2(\lambda)\,L_1(\lambda) \\ &= \begin{pmatrix} A_K(\lambda) & B_K(\lambda) \\ C_K(\lambda) & D_K(\lambda) \end{pmatrix}, \qquad A_K(\lambda), B_K(\lambda), C_K(\lambda), D_K(\lambda) \in \mathrm{End}(\mathcal{H}) \end{aligned}$$

It satisfies the same quadratic relation as $L_m(\lambda)$ (cf. defining relation of the Yang-Baxter algebra) provided that $[R(\lambda), K\otimes K] = 0$

The Yang-Baxter algebra for the Heisenberg spin-1/2 chain

$\sigma_m^\alpha \longrightarrow$ **quantum Lax operator at site m**

$$L_m(\lambda) = \begin{pmatrix} \varphi(\lambda + \eta\sigma_m^z) & \varphi(\eta)\,\sigma_m^- \\ \varphi(\eta)\,\sigma_m^+ & \varphi(\lambda - \eta\sigma_m^z) \end{pmatrix} \quad \in \mathrm{End}(V_a \otimes \mathcal{H}_m)$$

$V_a \simeq \mathbb{C}^2$: auxiliary space
$\mathcal{H}_m \simeq \mathbb{C}^2$: local quantum spin space at site m

such that it satisfies the quadratic relation

$$R(\lambda-\mu)\left(L_m(\lambda)\otimes 1\right)\left(1\otimes L_m(\mu)\right) = \left(1\otimes L_m(\mu)\right)\left(L_m(\lambda)\otimes 1\right)R(\lambda-\mu)$$

$\rightsquigarrow$ **monodromy matrix for a quasi-periodic chain with twist K ($K \in GL_2(\mathbb{C})$):**

$$T_K(\lambda) = K\,L_N(\lambda)\dots L_2(\lambda)\,L_1(\lambda)$$
$$= \begin{pmatrix} A_K(\lambda) & B_K(\lambda) \\ C_K(\lambda) & D_K(\lambda) \end{pmatrix}, \qquad A_K(\lambda), B_K(\lambda), C_K(\lambda), D_K(\lambda) \in \mathrm{End}(\mathcal{H})$$

It satisfies the same quadratic relation as $L_m(\lambda)$ (cf. defining relation of the Yang-Baxter algebra) provided that $[R(\lambda), K\otimes K] = 0$

$\rightsquigarrow$ **commuting transfer matrices:** $t_K(\lambda) = \mathrm{tr}\,T_K(\lambda) = A_K(\lambda) + D_K(\lambda)$

$H_K \propto \frac{\partial}{\partial\lambda}\log t_K(\lambda)\big|_{\lambda=0}$: Hamiltonian of the spin chain with twisted boundary conditions: $\sigma_{N+1}^\alpha = K\sigma_1^\alpha K^{-1}$

Algebraic Bethe ansatz for the periodic chain

$$T(\lambda) = L_N(\lambda)\dots L_2(\lambda)\, L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

- there exists a reference state (the state $|\,0\,\rangle \equiv |\uparrow\uparrow\dots\uparrow\rangle$) such that

$$\begin{cases} C(\lambda)\,|\,0\,\rangle = 0 \\ A(\lambda)\,|\,0\,\rangle = a(\lambda)|\,0\,\rangle \\ D(\lambda)\,|\,0\,\rangle = d(\lambda)|\,0\,\rangle \end{cases}$$

- The eigenstates of the transfer matrix $t(\lambda)$ (and of the Hamiltonian) are constructed as Bethe states:

$$|\,\{\lambda\}\,\rangle = \prod_{k=1}^{n} B(\lambda_k)|\,0\,\rangle \in \mathcal{H}, \quad \langle\,\{\lambda\}\,| = \langle\,0\,|\prod_{k=1}^{n} C(\lambda_k) \in \mathcal{H}^*$$

$\rightarrow$ eigenstates ("on-shell" Bethe states) if $\{\lambda\}$ solution of the Bethe equations:

$$a(\lambda_j)\prod_{k\neq j}\varphi(\lambda_j-\lambda_k-\eta) = d(\lambda_j)\prod_{k\neq j}\varphi(\lambda_j-\lambda_k+\eta), \quad 1\leq j\leq n$$

$\rightarrow$ "off-shell" Bethe states otherwise

It is possible to have access to correlation functions from the study of the periodic XXZ chain by algebraic Bethe Ansatz

- either numerically [Caux et al. 2005...]
- either analytically: large distance asymptotic behavior at the thermodynamic limit... [Kitanine, Kozlowski, Maillet, Slavnov, VT 2008, 2011...]

Both approaches are based

- on the form factor decomposition of the correlation functions:
$$\langle \psi_g | \sigma_n^\alpha \, \sigma_{n'}^\beta | \psi_g \rangle = \sum_{\substack{\text{eigenstates} \\ |\,\psi_i\,\rangle}} \langle \psi_g | \sigma_n^\alpha | \psi_i \rangle \cdot \langle \psi_i | \sigma_{n'}^\beta | \psi_g \rangle$$
- on the exact determinant representations for the form factors $\langle \psi_i | \sigma_n^\alpha | \psi_j \rangle$ in finite volume [Kitanine, Maillet, VT 1999] , obtained from
  - the action of local operators on Bethe states (using the solution of the quantum inverse problem, e.g. $\sigma_n^- = t(0)^{n-1} \, B(0) \, t(0)^{-n}$)
  - the use of Slavnov's determinant representation for the scalar products of Bethe states [Slavnov 89]
$$\langle \{\mu\}_{\text{off-shell}} | \{\lambda\}_{\text{on-shell}} \rangle \propto \det_{1 \le j,k \le n} \left[\frac{\partial \tau(\mu_j | \{\lambda\})}{\partial \lambda_k} \right]$$
where $t(\mu_j) \, |\{\lambda\}\rangle = \tau(\mu_j|\{\lambda\}) \, |\{\lambda\}\rangle$

Generalizations to more complicated integrable models ?

Limitations of the ABA approach:

- it requires the clear identification of a reference state $|\,0\,\rangle$
  $\rightsquigarrow$ there are some interesting models for which ABA cannot be applied

- even if ABA is a priori applicable, the completeness of the eigenstate construction is a delicate issue

- the ABA Bethe states have a complicated combinatorial structure
  $\rightsquigarrow$ the generalization of Slavnov's formula for the scalar products of Bethe states may be a very difficult problem

Integrable generalizations of the XXZ Heisenberg chain

It has several interesting generalizations which are still integrable (in the sense that one can still define a family of commuting transfer matrices):

- ⋆ XYZ model (related to 8-vertex model):

$$H_{\mathrm{XYZ}} = \sum_{m=1}^{N} \left\{ J_x\, \sigma_m^x \sigma_{m+1}^x + J_y\, \sigma_m^y \sigma_{m+1}^y + J_z\, \sigma_m^z \sigma_{m+1}^z \right\}$$

- ⋆ Open spin chains (with boundary magnetic fields):

$$H_{\mathrm{XXZ}}^{\mathrm{open}} = \sum_{m=1}^{N-1} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\, \sigma_m^z \sigma_{m+1}^z \right\} + h_-^x \sigma_1^x + h_-^y \sigma_1^y + h_-^z \sigma_1^z + h_+^x \sigma_N^x + h_+^y \sigma_N^y + h_+^z \sigma_N^z$$

- ⋆ higher spins or higher ranks. . .

Integrable generalizations of the XXZ Heisenberg chain

It has several interesting generalizations which are still integrable (in the sense that one can still define a family of commuting transfer matrices):

⋆ XYZ model (related to 8-vertex model):

$$H_{XYZ} = \sum_{m=1}^{N} \left\{ J_x\, \sigma_m^x \sigma_{m+1}^x + J_y\, \sigma_m^y \sigma_{m+1}^y + J_z\, \sigma_m^z \sigma_{m+1}^z \right\}$$

⋆ Open spin chains (with boundary magnetic fields):

$$H_{XXZ}^{\text{open}} = \sum_{m=1}^{N-1} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta\, \sigma_m^z \sigma_{m+1}^z \right\} + h_-^x \sigma_1^x + h_-^y \sigma_1^y + h_-^z \sigma_1^z + h_+^x \sigma_N^x + h_+^y \sigma_N^y + h_+^z \sigma_N^z$$

⋆ higher spins or higher ranks. . .

The reflection algebra for the XXZ open spin chain

The open spin chains are solvable in the framework of the representation theory of the reflection algebra (or boundary Yang-Baxter algebra) [Sklyanin 88]

- generators $\mathcal{U}_{ij}(\lambda)$, $1 \leq i, j \leq n$ $\leftarrow$ elements of the boundary monodromy matrix $\mathcal{U}(\lambda)$
- commutation relations given by the reflection equation:

$$R_{12}(\lambda-\mu)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda+\mu-\eta)\,\mathcal{U}_2(\mu) = \mathcal{U}_2(\mu)\,R_{12}(\lambda+\mu-\eta)\,\mathcal{U}_1(\lambda)\,R_{12}(\lambda-\mu)$$

$\hookrightarrow$ most general 2×2 solution of the refl. eq [de Vega, Gonzalez-Ruiz; Ghoshal, Zamolodchikov 93] :

$$K(\lambda;\zeta,\kappa,\tau) = \frac{1}{\sinh\zeta}\begin{pmatrix} \sinh(\lambda-\frac{\eta}{2}+\zeta) & \kappa\, e^{\tau}\sinh(2\lambda-\eta) \\ \kappa\, e^{-\tau}\sinh(2\lambda-\eta) & \sinh(\zeta-\lambda+\frac{\eta}{2}) \end{pmatrix}$$

$\rightsquigarrow$ boundary matrices $K^-(\lambda) \equiv K(\lambda;\zeta_-,\kappa_-,\tau_-)$ and $K^+(\lambda) \equiv K(\lambda+\eta;\zeta_+,\kappa_+,\tau_+)$ describing most general boundary fields in left/right boundaries:

$$h^x_\pm = 2\kappa_\pm \cosh\tau_\pm, \quad h^y_\pm = 2i\kappa_\pm \sinh\tau_\pm, \quad h^z_\pm = \sinh\eta \coth\zeta_\pm$$

$\rightsquigarrow$ $\mathcal{U}(\lambda) = T(\lambda)\,K_-(\lambda)\,\sigma^y\,T^t(-\lambda)\,\sigma^y = \begin{pmatrix} \mathcal{A}(\lambda) & \mathcal{B}(\lambda) \\ \mathcal{C}(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}$

$\rightsquigarrow$ transfer matrix: $\mathcal{T}(\lambda) = \mathrm{tr}\{K^+(\lambda)\,\mathcal{U}(\lambda)\}$ $\qquad [\mathcal{T}(\lambda),\mathcal{T}(\mu)] = 0$

$$H^{\text{open}}_{XXZ} \propto \frac{d}{d\lambda}\mathcal{T}(\lambda)\Big|_{\lambda=\eta/2}$$

The open spin chains: limitations of the solution by ABA

- ⋆ In the diagonal case ($\kappa_\pm = 0$, boundary fields along σ_1^z and σ_N^z only):
  - the state $|\,0\,\rangle$ can still be used as a reference state to construct the eigenstates as Bethe states in the ABA framework [Sklyanin 88]
$$|\,\{\lambda\}\,\rangle = \prod_{k=1}^{n} \mathcal{B}(\lambda_k)|\,0\,\rangle \in \mathcal{H}, \quad \langle\,\{\lambda\}\,| = \langle\,0\,|\prod_{k=1}^{n} \mathcal{C}(\lambda_k) \in \mathcal{H}^*$$
  - $\exists$ generalization of Slavnov's formula for the scalar products of Bethe states $\langle\{\mu\}_{\text{off-shell}}|\{\lambda\}_{\text{on-shell}}\rangle$ [Tsuchiya 98; Wang 02]
  - correlation functions can be computed (but no simple closed formula for the form factors) [Kitanine et al. 07]

- ⋆ it is possible to generalize Bethe ansatz equations to other cases with nevertheless some constraints on the boundary fields [Nepomechie 03] , but
  - problems in the ABA construction of a complete set of Bethe states both in $\mathcal{H}$ and $\mathcal{H}^*$ [Cao et al 03; Yang, Zhang 07; Filali, Kitanine 11]
    $\rightsquigarrow$ scalar products and correlation functions cannot be computed

- ⋆ most general boundaries ? an ABA solution is missing. . .

A complementary approach to ABA: Sklyanin's quantum Separation of Variables (SOV)

Goal: identity a basis of the space of state which "separates the variables" for the transfer matrix spectral problem

Idea: In the QISM framework, use the "operator roots" $\hat{b}_j$ of the operator $B(\lambda)$ from the monodromy matrix to construct this basis [Sklyanin 85,90]

$\rightsquigarrow$ Conditions on $B(\lambda)$: $B(\lambda)$ is diagonalizable with simple spectrum

$\rightsquigarrow$ N commuting "operators roots" $\hat{b}_j$ (with $\mathrm{Spec}(\hat{b}_j) \cap \mathrm{Spec}(\hat{b}_k) = \emptyset$ if $j \neq k$) which can be used to define a basis of the space of states $\mathcal{H}$:

$$| \mathbf{b} \rangle \quad \text{with} \quad \mathbf{b} = (b_1, \dots, b_N) \in \mathrm{Spec}(\hat{b}_1) \times \cdots \times \mathrm{Spec}(\hat{b}_N)$$
$$\hat{b}_n | \mathbf{b} \rangle = b_n | \mathbf{b} \rangle$$

This basis is moreover such that

$$A(\hat{b}_n) | b_1, \dots, b_n, \dots, b_N \rangle = \Delta_+(b_n) | b_1, \dots, b_n + \eta, \dots, b_N \rangle$$
$$D(\hat{b}_n) | b_1, \dots, b_n, \dots, b_N \rangle = \Delta_-(b_n) | b_1, \dots, b_n - \eta, \dots, b_N \rangle$$

$\rightsquigarrow$ In this basis, the multi-dimensional spectral problem for the transfer matrix $t(\lambda) = A(\lambda) + D(\lambda)$ can be reduced to a set of N one-dimensional finite-difference spectral problems

A complementary approach to ABA: Sklyanin's quantum Separation of Variables (SOV)

$\rightsquigarrow$ In this basis, the multi-dimensional spectral problem for the transfer matrix $t(\lambda) = A(\lambda) + D(\lambda)$ can be reduced to a set of N one-dimensional finite-difference spectral problems:

$$t(\lambda) \,|\, \Psi_\tau \,\rangle = \tau(\lambda) \,|\, \Psi_\tau \,\rangle,$$

$$\text{with} \quad |\, \Psi_\tau \,\rangle = \sum_{\mathbf{b}=(b_1,\ldots,b_N)} \psi_\tau(b_1,\ldots,b_N) \,|\, \mathbf{b} \,\rangle,$$

is solved by

$$\psi_\tau(b_1,\ldots,b_N) = \prod_{n=1}^{N} Q_\tau(b_n)$$

where $Q_\tau(b_n)$ and $\tau(b_n)$ are solution of a discrete version of Baxter's T-Q equation, for $n \in \{1,\ldots,N\}$, $b_n \in \mathrm{Spec}(\hat{b}_n)$:

$$\tau(b_n)\, Q_\tau(b_n) = \Delta_+(b_n)\, Q(b_n+\eta) + \Delta_-(b_n)\, Q_\tau(b_n-\eta)$$

Remark: the completeness is given by construction

SOV for the antiperiodic XXZ chain

One can apply this process to the antiperiodic monodromy matrix (with inhomogeneity parameters $\xi_1, \dots, \xi_N$):

$$\bar{T}(\lambda) = \sigma^x \, L_N(\lambda - \xi_N) \dots L_2(\lambda - \xi_2) \, L_1(\lambda - \xi_1)$$
$$= \begin{pmatrix} \bar{A}(\lambda) & \bar{B}(\lambda) \\ \bar{C}(\lambda) & \bar{D}(\lambda) \end{pmatrix} = \begin{pmatrix} C(\lambda) & D(\lambda) \\ A(\lambda) & B(\lambda) \end{pmatrix}$$

- $\bar{B}(\lambda) = D(\lambda)$ is a (trigonometric) polynomial of degree N with N commuting operator roots $\hat{b}_n$
- $\mathrm{Spec}(\hat{b}_n) = \{\xi_n, \xi_n - \eta\} \, (mod \; i\pi)$
  $\rightarrow$ the simplicity condition is fulfilled if $\xi_j \neq \xi_k, \xi_k \pm \eta \, (mod \; i\pi)$ for $j \neq k$

$\rightsquigarrow$ basis $|\,\mathbf{b}\,\rangle$ of $\mathcal{H}$ and $\langle\,\mathbf{b}\,|$ of $\mathcal{H}^*$ which separate the variables for the spectral problem for the antiperiodic transfer matrix $\bar{t}(\lambda) = \bar{A}(\lambda) + \bar{D}(\lambda)$

Remark: Since $\mathbf{b}$ is of the form $(\xi_1 - h_1\eta, \dots, \xi_N - h_N\eta)$ with $\mathbf{h} = (h_1, \dots, h_N) \in \{0,1\}^N$, we shall use from now on the notation $|\,\mathbf{h}\,\rangle$ and $\langle\,\mathbf{h}\,|$ instead of $|\,\mathbf{b}\,\rangle$ and $\langle\,\mathbf{b}\,|$.

SOV for the antiperiodic XXZ chain

$$\bar{T}(\lambda) = \sigma^x \, L_N(\lambda - \xi_N) \dots L_2(\lambda - \xi_2) \, L_1(\lambda - \xi_1) = \begin{pmatrix} \bar{A}(\lambda) & \bar{B}(\lambda) \\ \bar{C}(\lambda) & \bar{D}(\lambda) \end{pmatrix}$$

- $\bar{B}(\lambda) = D(\lambda)$ is a (trigonometric) polynomial of degree N with N commuting operator roots $\hat{b}_n$
- $\mathrm{Spec}(\hat{b}_n) = \{\xi_n, \xi_n - \eta\} \, (mod \; i\pi)$
  $\rightarrow$ the simplicity condition is fulfilled if $\xi_j \neq \xi_k, \xi_k \pm \eta \, (mod \; i\pi)$ for $j \neq k$

$\rightsquigarrow$ basis $|\, \mathbf{b} \,\rangle$ of $\mathcal{H}$ and $\langle\, \mathbf{b} \,|$ of $\mathcal{H}^*$ (denoted by $|\, \mathbf{h} \,\rangle$ and $\langle\, \mathbf{h} \,|$ with $\mathbf{h} = (h_1, \dots, h_N) \in \{0,1\}^N$, $b_n \equiv \xi_n - h_n\eta$) which separate the variables for the spectral problem for the antiperiodic transfer matrix $\bar{t}(\lambda) = \bar{A}(\lambda) + \bar{D}(\lambda)$

$\rightsquigarrow$ eigenvalues $\tau(\lambda)$ of $\bar{t}(\lambda) = \bar{A}(\lambda) + \bar{D}(\lambda)$: (trigonometric) polynomials of degree $N-1$ s. t. there exists $Q_\tau \in \mathrm{Fun}(\cup_{j=1}^N \{\xi_j, \xi_j - \eta\}) \, (mod \; i\pi)$ satisfying

$$\tau(b_n) \, Q_\tau(b_n) = -a(b_n) \, Q_\tau(b_n - \eta) + d(b_n) \, Q_\tau(b_n + \eta),$$

for $b_n \in \{\xi_n, \xi_n - \eta\}$, $1 \leq n \leq N$.

$\rightsquigarrow$ corresponding eigenvectors: $|\, \Psi_\tau \,\rangle = \sum_{\mathbf{h} \in \{0,1\}^N} \prod_{a=1}^{N} Q_\tau(\xi_a - h_a\eta) \, V_{\boldsymbol{\xi}+\mathbf{h}\eta} \, |\, \mathbf{h} \,\rangle$

where $V_{\boldsymbol{\xi}+\mathbf{h}\eta} = \prod_{b<a} \varphi(\xi_a + h_a\eta - \xi_b - h_b\eta)$

Scalar products/form factors in antiperiodic XXZ chain

- The transfer matrix eigenstates are particular cases of "separate states":

$$\langle \alpha | = \sum_{\mathbf{h}\in\{0,1\}^N} \prod_{a=1}^{N} \alpha(\xi_a - h_a\eta)\, V_{\boldsymbol{\xi}-\boldsymbol{h}\eta}\, \langle \mathbf{h} |$$

$$| \beta \rangle = \sum_{\mathbf{h}\in\{0,1\}^N} \prod_{a=1}^{N} \beta(\xi_a - h_a\eta)\, V_{\boldsymbol{\xi}+\boldsymbol{h}\eta}\, | \mathbf{h} \rangle$$

where α, β are arbitrary functions on $\cup_{j=1}^{N}\{\xi_j, \xi_j - \eta\}\,(mod\; i\pi)$

- scalar product for SOV states: $\langle \mathbf{h} \,|\, \mathbf{k} \rangle = \dfrac{\delta_{\mathbf{h},\mathbf{k}}}{V_{\boldsymbol{\xi}-\boldsymbol{h}\eta}}$

where $V_{\boldsymbol{\xi}} = \prod_{k<j} \varphi(\xi_j - \xi_k) = \det_{1\le i,j\le N}[\tilde{\varphi}(\xi_i)^{j-1}]$

$\rightsquigarrow$ determinant representation for the scalar product of left/right separate states (for XXX):

$$\langle \alpha \,|\, \beta \rangle = \det_{1\le i,j\le N} \left[\sum_{h=0}^{1} \alpha(\xi_i - h\eta)\, \beta(\xi_i - h\eta)\, (\xi_i + h\eta)^{j-1} \right]$$

- action of σ_m^α on $|\,\mathbf{h}\,\rangle$ $\rightarrow$ form factors reduce to scalar products of separate states

$\rightsquigarrow$ determinant representations for the finite-size form factors

The XXZ open spin chain by SOV (non diagonal case)

- Similar construction can be performed for the inhomogeneous XXZ open spin chain with (at least) one triangular boundary matrix [Niccoli 13]
- In the XXX case, the most general boundaries can be reduced to this case by means of the $SU(2)$ symmetry [cf. also Frahm et al. 08]
- In the XXZ case, the most general boundaries can be reduced to this case by means of a Vertex-IRF transformation (dynamical gauge transformation) [cf. Baxter 73; Felder & Varchenko 93; Cao et al, 03...]

$$R_{12}(\lambda-\mu)\, S_1(\lambda|\beta)\, S_2(\mu|\beta+\sigma_1^z) = S_2(\mu|\beta)\, S_1(\lambda|\beta+\sigma_2^z)\, R_{12}^{\text{dyn}}(\lambda-\mu|\beta)$$

$$K_-^{\text{dyn}}(\lambda|\beta) = S^{-1}(-\lambda+\eta/2|\beta)\, K_-(\lambda)\, S(\lambda-\eta/2|\beta)$$

with

$$S(\lambda|\beta) = \begin{pmatrix} e^{\lambda-\eta(\beta+\alpha)} & e^{\lambda+\eta(\beta-\alpha)} \\ 1 & 1 \end{pmatrix}$$

$\rightsquigarrow$ new boundary monodromy matrix $\mathcal{U}_-^{\text{dyn}}(\lambda|\beta)$

$$R_{21}^{\text{dyn}}(\lambda-\mu|\beta)\, \mathcal{U}_1^{\text{dyn}}(\lambda|\beta+\sigma_2^z)\, R_{12}^{\text{dyn}}(\lambda+\mu-\eta|\beta)\, \mathcal{U}_2^{\text{dyn}}(\mu|\beta+\sigma_1^z)$$
$$= \mathcal{U}_2^{\text{dyn}}(\mu|\beta+\sigma_1^z)\, R_{21}^{\text{dyn}}(\lambda+\mu-\eta|\beta)\, \mathcal{U}_1^{\text{dyn}}(\lambda|\beta+\sigma_2^z)\, R_{12}^{\text{dyn}}(\lambda-\mu|\beta)$$

$\rightsquigarrow$ spectrum and eigenvectors of
$\mathcal{T}^{\text{dyn}}(\lambda|\beta) = S_{1\dots N}(\{\xi\}|\beta)^{-1}\, \mathcal{T}(\lambda)\, S_{1\dots N}(\{\xi\}|\beta)$

- Similar formulas also hold for the scalar products of separate states

Problems...

All these results (characterization of the transfer matrix spectrum and eigenstates, expressions for the scalar products/form factors...) depend on a non-trivial way on the inhomogeneity parameters of the model

$\rightsquigarrow$ the study of the **homogeneous** ($\rightarrow$ physical model) or **thermodynamic** limits is not easy !

$\rightsquigarrow$ **2 main problems to be solved:**

1. reformulate the discrete characterization (in terms of discrete T-Q equations) of the spectrum in a more convenient way, i.e. in terms of continuous T-Q equations

   $\rightsquigarrow$ Bethe equations (and Bethe-type representation for the eigenstates)

2. transform the determinant representations for the scalar products/form factors into a more convenient form for the consideration of the homogeneous/thermodynamic limit

From discrete to continuous T-Q equations

In the antiperiodic XXX/XXZ case , the SOV characterization of the spectrum (in terms of discrete T-Q eq) can be equivalently reformulated in terms of solutions of a functional T-Q equation:

An entire function $\tau(\lambda)$ is an eigenvalue of the antiperiodic transfer matrix iff there exists a unique function $Q(\lambda) \in \Sigma_Q$ such that

$$\tau(\lambda)\, Q(\lambda) = -a(\lambda)\, Q(\lambda - \eta) + d(\lambda)\, Q(\lambda + \eta).$$

where Σ_Q is the class of functions $Q(\lambda)$ of the form:

- for XXX: $Q(\lambda) = \prod_{j=1}^{R} (\lambda - \lambda_j), \quad R \leq N,$
- for XXZ: $Q(\lambda) = \prod_{j=1}^{N} \sinh\left(\frac{\lambda - \lambda_j}{2}\right),$ [Batchelor et al. 95; Niccoli, VT 15]

with $\lambda_j \in \mathbb{C} \setminus \{\xi_1, \ldots, \xi_N\}$.

$\rightsquigarrow$ complete description of the spectrum in terms of the corresponding Bethe equations for the roots λ_j of $Q(\lambda)$

Remark: In the XXZ case, one has to impose moreover that $\tau(\lambda)$ satisfies the periodicity condition $\tau(\lambda + i\pi) = (-1)^{N-1}\, \tau(\lambda)$

From discrete to continuous T-Q equations

In the (non-diagonal) open XXX/XXZ case , such a reformulation is not known in general. However, it can be shown [Kitanine, Maillet, Niccoli 13] that the SOV characterization of the spectrum (in terms of discrete T-Q eq) can be equivalently reformulated in terms of polynomials (in λ^2 for XXX and in $\cosh 2\lambda$ for XXZ) Q-solutions of a functional T-Q equation with an inhomogeneous term (cf also [Cao et al. 13; Belliard, Crampé 13...]):

An entire function $\tau(\lambda)$ is an eigenvalue of the antiperiodic transfer matrix iff there exists a unique function $Q(\lambda) \in \Sigma_Q$ such that

$$\tau(\lambda)\, Q(\lambda) = \mathbf{A}(\lambda)\, Q(\lambda - \eta) + \mathbf{A}(-\lambda)\, Q(\lambda + \eta) + \mathbf{F}(\lambda),$$

where $\mathbf{A}(\lambda) \equiv \mathbf{A}_{\zeta_\pm,\kappa_\pm}(\lambda)$ and $\mathbf{F}(\lambda) \equiv \mathbf{F}_{\zeta_\pm,\kappa_\pm,\tau_\pm}(\lambda)$ depend on the boundary parameters, with $\mathbf{F}(\xi_n) = \mathbf{F}(\xi_n - \eta) = 0$, $n = 1, \ldots, N$.

$\mathbf{F} = 0$ identically $\iff$ Nepomechie's constraint on the boundary parameters

$\leadsto$ If Nepomechie's constraint on the boundary parameters is satisfied, one recovers a complete characterization of the spectrum in terms of polynomials (in λ^2 for XXX and in $\cosh 2\lambda$ for XXZ) Q-solutions of

$$\tau(\lambda)\, Q(\lambda) = \mathbf{A}(\lambda)\, Q(\lambda - \eta) + \mathbf{A}(-\lambda)\, Q(\lambda + \eta)$$

$+$ the SOV construction also provides the complete set of eigenstates (both in $\mathcal{H}$ and $\mathcal{H}^*$)

Determinant representations for the scalar products and form factors: antiperiodic XXX case [Kitanine, Maillet, Niccoli, VT 15]

For two separate states

$$\langle\, \alpha \,| = \sum_{\mathbf{h}\in\{0,1\}^{\mathsf{N}}} \prod_{a=1}^{\mathsf{N}} \alpha(\xi_a - h_a\eta)\, V_{\boldsymbol{\xi}-\boldsymbol{h}\eta} \,\langle\, \mathbf{h} \,|, \qquad |\, \beta \,\rangle = \sum_{\mathbf{h}\in\{0,1\}^{\mathsf{N}}} \prod_{a=1}^{\mathsf{N}} \beta(\xi_a - h_a\eta)\, V_{\boldsymbol{\xi}+\boldsymbol{h}\eta} \,|\, \mathbf{h} \,\rangle$$

with $\alpha(\lambda) = \prod_{j=1}^{p}(\lambda - \alpha_j)$, $\beta(\lambda) = \prod_{j=1}^{q}(\lambda - \beta_j)$ and $V_{\boldsymbol{\xi}} = \det_{1\le i,j\le N}[\xi_i^{j-1}]$

$$\langle\, \alpha \,|\, \beta \,\rangle = \det{}_{1\le i,j\le N}\left[\sum_{h=0}^{1} \alpha(\xi_i - h\eta)\,\beta(\xi_i - h\eta)\,(\xi_i + h\eta)^{j-1}\right] \qquad (1)$$

Determinant representations for the scalar products and form factors: antiperiodic XXX case [Kitanine, Maillet, Niccoli, VT 15]

For two separate states $\langle\, \alpha \,|$ and $|\, \beta \,\rangle$, $\quad \alpha(\lambda) = \prod_{j=1}^{p} (\lambda - \alpha_j), \quad \beta(\lambda) = \prod_{j=1}^{q} (\lambda - \beta_j)$

$$\langle\, \alpha \,|\, \beta \,\rangle = \det_{1 \le i,j \le N} \left[\sum_{h=0}^{1} \alpha(\xi_i - h\eta)\, \beta(\xi_i - h\eta)\, (\xi_i + h\eta)^{j-1} \right] \qquad (1)$$

- When $p + q = N$, (1) can be transformed, through some algebraic identities, into an Izergin determinant (symmetric in the two sets of variables $\{\xi_j\}$ and $\{\alpha_j\} \cup \{\beta_j\}$), and then into a similar determinant in which the role of the set of variables $\{\xi_j\}$ and $\{\gamma_j\} \equiv \{\alpha_j\} \cup \{\beta_j\}$ are exchanged:

$$\langle\, \alpha \,|\, \beta \,\rangle \propto \det_{1 \le i,j \le p+q} \left[\sum_{h=0}^{1} \prod_{\ell=1}^{N} (\gamma_i + h\eta - \xi_\ell)\, (\gamma_i - h\eta)^{j-1} \right] \qquad (2)$$

- Generalization to $p + q \neq N$ by considering limits of the previous result
- In its turn, (2) can be transformed into a generalized version of Slavnov's determinant (which reduces to the usual Slavnov determinant when $p = q$ and when one of the state is an eigenstate)

$\rightsquigarrow$ One can express the form factors of local operators in a form similar to ABA

Determinant representations for the scalar products: open XXX chain (non-diagonal case) [Kitanine, Maillet, Niccoli, VT 16]

For two separate states $\langle \alpha |, | \beta \rangle$, with $\alpha(\lambda) = \prod_{j=1}^{p}(\lambda^2 - \alpha_j^2)$, $\beta(\lambda) = \prod_{j=1}^{q}(\lambda^2 - \beta_j^2)$

$$\langle \alpha | \beta \rangle \propto \det_{1\le i,j\le N}\left[\sum_{\epsilon=\pm} f_{\bar\zeta_+,\bar\zeta_-}(\epsilon\xi_i)\,\alpha\Big(\xi_i - \epsilon\frac{\eta}{2}\Big)\,\beta\Big(\xi_i - \epsilon\frac{\eta}{2}\Big)\,\Big(\xi_i + \epsilon\frac{\eta}{2}\Big)^{2(j-1)}\right] \quad (3)$$

where $f_{\bar\zeta_+,\bar\zeta_-}(\lambda)$ depends on combinations $\bar\zeta_\pm$ of the $\pm$ boundary parameters

- When $p + q = N$, (3) can be transformed, similarly as in the closed XXX case, into a determinant which is symmetric by exchange of the two sets of variables $\{\xi_j\}$ and $\{\alpha_j\} \cup \{\beta_j\}$ with $\bar\zeta_\pm \to \frac{\eta}{2} - \bar\zeta_\pm$:

$$\langle \alpha | \beta \rangle \propto \mathcal{I}_{\bar\zeta_+,\bar\zeta_-}(\{\xi\},\{\alpha\}\cup\{\beta\}) = \mathcal{I}_{\frac{\eta}{2}-\bar\zeta_+,\frac{\eta}{2}-\bar\zeta_-}(\{\alpha\}\cup\{\beta\},\{\xi\})$$

with $\mathcal{I}_{\xi_+,\xi_-}(\{x\},\{y\})) \propto \det_N\left[\sum_{\epsilon=\pm}\epsilon\frac{(x+\epsilon\xi_+)(x+\epsilon\xi_-)}{x\,[(x+\epsilon\frac{\eta}{2})^2 - y^2)]}\right]$

and then into a similar determinant in which the role of the set of variables $\{\xi_j\}$ and $\{\gamma_j\} \equiv \{\alpha_j\} \cup \{\beta_j\}$ are exchanged:

$$\langle \alpha | \beta \rangle \propto \det_{1\le i,j\le p+q}\left[\sum_{\epsilon=\pm} f_{\frac{\eta}{2}-\bar\zeta_+,\frac{\eta}{2}-\bar\zeta_-}(\epsilon\gamma_i)\prod_{\ell=1}^{N}\Big(\Big(\gamma_i - \epsilon\frac{\eta}{2}\Big)^2 - \xi_\ell^2\Big)\,\Big(\xi_i + \epsilon\frac{\eta}{2}\Big)^{2(j-1)}\right]$$

Determinant representations for the scalar products: open XXX chain (non-diagonal case) [Kitanine, Maillet, Niccoli, VT 16]

For two separate states $\langle\,\alpha\,|,\ |\,\beta\,\rangle$, with $\alpha(\lambda)=\prod_{j=1}^{p}(\lambda^2-\alpha_j^2),\ \ \beta(\lambda)=\prod_{j=1}^{q}(\lambda^2-\beta_j^2)$

$$\langle\,\alpha\,|\,\beta\,\rangle \propto \det_{1\le i,j\le N}\left[\sum_{\epsilon=\pm} f_{\bar\zeta_+,\bar\zeta_-}(\epsilon\xi_i)\,\alpha\Big(\xi_i-\epsilon\frac{\eta}{2}\Big)\,\beta\Big(\xi_i-\epsilon\frac{\eta}{2}\Big)\,\Big(\xi_i+\epsilon\frac{\eta}{2}\Big)^{2(j-1)}\right] \quad (3)$$

where $f_{\bar\zeta_+,\bar\zeta_-}(\lambda)$ depends on combinations $\bar\zeta_\pm$ of the $\pm$ boundary parameters

- When $p+q=N$, (3) can be transformed into a similar determinant in which the role of the set of variables $\{\xi_j\}$ and $\{\gamma_j\}\equiv\{\alpha_j\}\cup\{\beta_j\}$ are exchanged:

$$\langle\,\alpha\,|\,\beta\,\rangle \propto \det_{1\le i,j\le p+q}\left[\sum_{\epsilon=\pm} f_{\frac{\eta}{2}-\bar\zeta_+,\frac{\eta}{2}-\bar\zeta_-}(\epsilon\gamma_i)\prod_{\ell=1}^{N}\left(\Big(\gamma_i-\epsilon\frac{\eta}{2}\Big)^2-\xi_\ell^2\right)\Big(\xi_i+\epsilon\frac{\eta}{2}\Big)^{2(j-1)}\right]$$

- Generalization to $p+q\neq N$ by considering limits of the previous result
- In its turn, this new determinant can be transformed into a generalized version of Slavnov's determinant
- In the case with a constraint, the determinant simplifies if one of the state is an eigenstate

Determinant representations for the scalar products: open XXZ chain (non-diagonal case) [Kitanine, Maillet, Niccoli, VT 18]

$$\langle\, \alpha \,|\, \beta \,\rangle \propto \det_{1\le i,j\le N}\left[\sum_{\epsilon=\pm} f_{\{a\}}(\epsilon\xi_i)\,\alpha\Big(\xi_i-\epsilon\frac{\eta}{2}\Big)\,\beta\Big(\xi_i-\epsilon\frac{\eta}{2}\Big)\cosh^{j-1}(2\xi_i+\epsilon\eta)\right] \quad (4)$$

with $\alpha(\lambda)=\prod_{j=1}^{p}(\cosh 2\lambda-\cosh 2\alpha_j)$, $\beta(\lambda)=\prod_{j=1}^{q}(\cosh 2\lambda-\cosh 2\beta_j)$,
where $f_{\{a\}}(\lambda)$ depends on combinations $\{a\}\equiv\{\alpha_+,\alpha_-,\beta_+,\beta_-\}$ of the $\pm$ boundary parameters $\zeta_\pm,\kappa_\pm$

- When $p+q=N$, (4) can be transformed, using similar identities as in the XXX cases into a new determinant, but this determinant cannot be made completely symmetric by exchange of the two sets of variables $\{\xi_j\}$ and $\{\alpha_j\}\cup\{\beta_j\}$ with $\{a\}\to\{\frac{\eta}{2}-a\}$!

  It is nevertheless possible to exchange the role of the set of variables $\{\xi_j\}$ and $\{\gamma_j\}\equiv\{\alpha_j\}\cup\{\beta_j\}$ in (4) at the price of modifying the last column:

$$\langle\, \alpha \,|\, \beta \,\rangle \propto \det_{1\le i,j\le p+q}\left[\sum_{\epsilon=\pm} f_{\{\frac{\eta}{2}-a\}}(\epsilon\gamma_i)\prod_{\ell=1}^{N}\Big(\cosh(2\gamma_i-\epsilon\eta)-\cosh 2\xi_\ell\Big)\cosh^{j-1}(2\gamma_i+\epsilon\eta) + \delta_{j,N}\, g^{(p+q)}_{\{a\}}(\gamma_i)\right]$$

Determinant representations for the scalar products: open XXZ chain (non-diagonal case) [Kitanine, Maillet, Niccoli, VT 18]

$$\langle\, \alpha \,|\, \beta \,\rangle \propto \det_{1\le i,j\le N}\left[\sum_{\epsilon=\pm} f_{\{a\}}(\epsilon\xi_i)\,\alpha\Big(\xi_i - \epsilon\frac{\eta}{2}\Big)\,\beta\Big(\xi_i - \epsilon\frac{\eta}{2}\Big)\,\cosh^{j-1}(2\xi_i+\epsilon\eta)\right] \quad (4)$$

with $\alpha(\lambda) = \prod_{j=1}^{p}(\cosh 2\lambda - \cosh 2\alpha_j)$, $\beta(\lambda) = \prod_{j=1}^{q}(\cosh 2\lambda - \cosh 2\beta_j)$,
where $f_{\{a\}}(\lambda)$ depends on combinations $\{a\} \equiv \{\alpha_+, \alpha_-, \beta_+, \beta_-\}$ of the $\pm$ boundary parameters $\zeta_\pm, \kappa_\pm$

- When $p+q=N$, (4) can be transformed as

$$\langle\, \alpha \,|\, \beta \,\rangle \propto \det_{1\le i,j\le p+q}\left[\sum_{\epsilon=\pm} f_{\{\frac{\eta}{2}-a\}}(\epsilon\gamma_i)\,\prod_{\ell=1}^{N}\Big(\cosh(2\gamma_i-\epsilon\eta) - \cosh 2\xi_\ell\Big)\,\cosh^{j-1}(2\gamma_i+\epsilon\eta) + \delta_{j,N}\; g^{(p+q)}_{\{a\}}(\gamma_i)\right]$$

- Generalization to $p+q \neq N$ by considering limits of the previous result
- In its turn, this new determinant can be transformed into a generalized version of Slavnov's determinant (much more complicated that in the XXX case !)
- In the case with a constraint, the determinant simplifies drastically if one of the state is an eigenstate thanks to Bethe equations

  $\rightsquigarrow$ usual Slavnov formula if $p=q$!

Generalized Slavnov determinant for open XXZ

Example: the case $p = q$

$$\langle \alpha \,|\, \beta \rangle \propto \det_p \mathcal{S}$$

$$\mathcal{S}_{i,k} = \sum_{\epsilon \in \{+,-\}} f(\epsilon\beta_i)\, X(\beta_i + \epsilon\eta) \Bigg[\frac{f(-\alpha_k)}{\varsigma(\beta_i + \epsilon\frac{\eta}{2}) - \varsigma(\alpha_k + \frac{\eta}{2})} - \frac{f(\alpha_k)\,\varphi(\alpha_k)}{\varsigma(\beta_i + \epsilon\frac{\eta}{2}) - \varsigma(\alpha_k - \frac{\eta}{2})}$$

$$+ \frac{f(-\alpha_k) - f(\alpha_k)\,\varphi(\alpha_k)}{1 + \sum_{\ell=1}^{p} X^g_{f,\ell}} \sum_{j=1}^{p} \frac{X^g_{f,j}}{\varsigma(\beta_i + \epsilon\frac{\eta}{2}) - \varsigma(\alpha_j - \frac{\eta}{2})} \Bigg] + \frac{g(\beta_i)}{X(\beta_i)} \, \frac{f(-\alpha_k) - f(\alpha_k)\,\varphi(\alpha_k)}{1 + \sum_{\ell=1}^{p} X^g_{f,\ell}}.$$

with $\qquad \varsigma(\lambda) = \dfrac{\cosh(2\lambda)}{2}, \quad X(\lambda) = \displaystyle\prod_{\ell=1}^{p} \big[\varsigma(\lambda) - \varsigma(\alpha_\ell)\big],$

and $\qquad X^g_{f,k} = \dfrac{g(\alpha_k)\,\sinh(2\alpha_k - \eta)}{f(-\alpha_k)\,X'(\alpha_k)\,X(\alpha_k - \eta)}, \quad \varphi(\lambda) = \dfrac{\sinh(2\lambda - \eta)}{\sinh(2\lambda + \eta)} \, \dfrac{X(\lambda + \eta)}{X(\lambda - \eta)}.$

The functions f and g depend on the boundary parameters.

Generalized Slavnov determinant for open XXZ

Example: the case $p = q$

$$\langle \alpha \,|\, \beta \rangle \propto \det_p \mathcal{S}$$

$$\mathcal{S}_{i,k} = \sum_{\epsilon\in\{+,-\}} f(\epsilon\beta_i)\, X(\beta_i{+}\epsilon\eta) \left[\frac{f(-\alpha_k)}{\varsigma(\beta_i+\epsilon\frac{\eta}{2}) - \varsigma(\alpha_k+\frac{\eta}{2})} - \frac{f(\alpha_k)\,\varphi(\alpha_k)}{\varsigma(\beta_i+\epsilon\frac{\eta}{2}) - \varsigma(\alpha_k-\frac{\eta}{2})} \right.$$
$$\left. + \frac{f(-\alpha_k) - f(\alpha_k)\,\varphi(\alpha_k)}{1+\sum_{\ell=1}^{p} X^g_{f,\ell}} \sum_{j=1}^{p} \frac{X^g_{f,j}}{\varsigma(\beta_i+\epsilon\frac{\eta}{2}) - \varsigma(\alpha_j-\frac{\eta}{2})} \right] + \frac{g(\beta_i)}{X(\beta_i)}\, \frac{f(-\alpha_k) - f(\alpha_k)\,\varphi(\alpha_k)}{1+\sum_{\ell=1}^{p} X^g_{f,\ell}}.$$

In the case with a constraint, the Bethe equations are

$$f(-\alpha_k) - f(\alpha_k)\,\varphi(\alpha_k) = 0, \;\; k = 1, \ldots p$$

$\rightsquigarrow$ if $|\,\alpha\,\rangle$ is an eigenstate the determinant simplifies into

$$\mathcal{S}_{i,k} = \sum_{\epsilon\in\{+,-\}} f(\epsilon\beta_i)\, X(\beta_i+\epsilon\eta) \left[\frac{f(-\alpha_k)}{\varsigma(\beta_i+\epsilon\frac{\eta}{2}) - \varsigma(\alpha_k+\frac{\eta}{2})} - \frac{f(\alpha_k)\,\varphi(\alpha_k)}{\varsigma(\beta_i+\epsilon\frac{\eta}{2}) - \varsigma(\alpha_k-\frac{\eta}{2})} \right]$$
$$\propto \frac{\partial\tau(\beta_j|\{\alpha\})}{\partial\alpha_k}$$

Further problems

⋆ correlations functions for open XXX/XXZ (with a constraint) ?

⋆ solution of the functional T-Q equation for the general open chain (case without constraint) ?

⋆ **Scalar products for antiperiodic XXZ case ?**
separates states should a priori be associated with functions of the form

$$\alpha(\lambda) = \prod_{j=1}^{p} \sinh\Big(\frac{\lambda - \alpha_j}{2}\Big), \quad \beta(\lambda) = \prod_{j=1}^{q} \sinh\Big(\frac{\lambda - \beta_j}{2}\Big)$$

whereas Sklyanin measure is $V_{\boldsymbol{\xi}} = \prod_{k<j} \sinh(\xi_j - \xi_k)$

$\rightsquigarrow$ the naive generalization of the algebraic identities used in the XXX case does not enable us to transform the determinant for $\langle \alpha | \beta \rangle$