



# Outline

- The  $A_2^{(1)}$  lattice models
  - A family of models: loop, vertex and dimer models
  - Diagrammatic algebras
- Diagrammatic calculus and functional relations
  - Planar identities
  - Wenzl-Jones projectors
  - Fused transfer matrices
  - Fusion hierarchy relations of  $sl_3$  type
  - $Y$ -systems

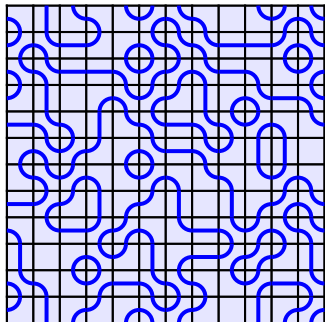
# The $A_2^{(1)}$ loop model

- The elementary face operator: (Nienhuis, Warnaar '93)

$$\boxed{u} = \frac{\sin(\lambda - u)}{\sin \lambda} \left( \boxed{\phantom{u}} + \boxed{\text{diag 1}} \right) + \boxed{\text{diag 2}} + \boxed{\text{diag 3}} + \frac{\sin u}{\sin \lambda} \left( \boxed{\text{diag 4}} + \boxed{\text{diag 5}} + \boxed{\text{diag 6}} \right)$$

- $u$  is the **spectral parameter**
- Loop fugacities:
  - contractible:  $\beta = 2 \cos \lambda = q + q^{-1}$
  - non contractible:  $\alpha$
- **Vacancies are preserved**
- Weights and partition functions:

$$W_\sigma = \alpha^{n_\alpha} \beta^{n_\beta} \prod_f w_f \quad Z = \sum_\sigma W_\sigma$$



A loop configuration  $\sigma$   
on the  $12 \times 12$  torus

## Crossing symmetry and transfer tangles

- No crossing symmetry:  $\boxed{u} \neq \boxed{\lambda-u}$

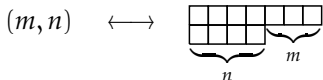
$$\boxed{u} = \frac{\sin(\lambda - u)}{\sin \lambda} \left( \square + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \right) + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} + \frac{\sin u}{\sin \lambda} \left( \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \right)$$

$$\boxed{\lambda-u} = \frac{\sin u}{\sin \lambda} \left( \square + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \right) + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} + \frac{\sin(\lambda - u)}{\sin \lambda} \left( \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} + \begin{array}{|c|} \hline \text{ } \\ \hline \end{array} \right)$$



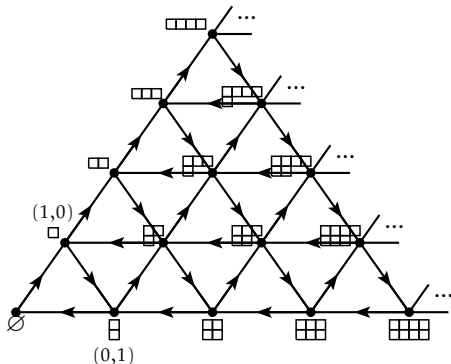
# $sl_3$ representations

- Irreducible representations and Young diagrams:



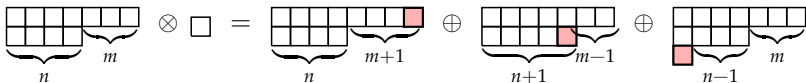
- Tensor products of irreducible representations:

$$(m, n) \otimes (1, 0) = (m + 1, n) \oplus (m - 1, n + 1) \oplus (m, n - 1)$$



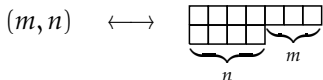
The  $sl_3$  weight lattice

- Equivalently:



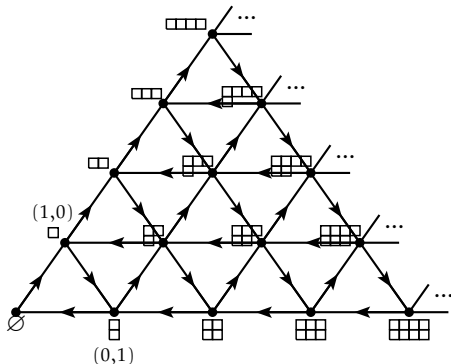
# $sl_3$ representations

- Irreducible representations and Young diagrams:



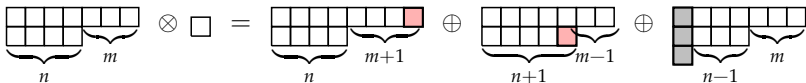
- Tensor products of irreducible representations:

$$(m, n) \otimes (1, 0) = (m + 1, n) \oplus (m - 1, n + 1) \oplus (m, n - 1)$$



The  $sl_3$  weight lattice

- Equivalently:



# Dilute Temperley-Lieb algebra

- The periodic dilute Temperley-Lieb algebra,  $\text{pdTL}_N(\alpha, \beta)$ , is the linear span of **connectivity diagrams**:

$$a_1 = \text{[Diagram 1]} \quad a_2 = \text{[Diagram 2]}$$

Diagram 1: A rectangular box with a light blue background. A horizontal line with a dot at its center passes through the box. Blue wavy lines connect the top and bottom boundaries, forming a pattern that is symmetric about the horizontal line. There are dots on the top and bottom boundaries.

Diagram 2: A rectangular box with a light blue background. A horizontal line with a dot at its center passes through the box. Blue wavy lines connect the top and bottom boundaries, forming a pattern that is not symmetric about the horizontal line. There are dots on the top and bottom boundaries.

- Product of two connectivity diagrams:

$$a_1 a_2 = \text{[Diagram 3]} = \beta \text{[Diagram 4]} = \beta a_3$$

Diagram 3: The product of diagrams  $a_1$  and  $a_2$ . It shows the same horizontal line and dots as  $a_1$ , but the blue wavy lines are more complex, representing the combined paths from both diagrams. There are dots on the top and bottom boundaries.

Diagram 4: Diagram  $a_3$ , which is identical to  $a_1$ . There are dots on the top and bottom boundaries.

- More examples of products for  $N = 6$ :

$$\text{[Diagram 5]} = \alpha \text{[Diagram 6]} \quad \text{[Diagram 7]} = 0$$

Diagram 5: A rectangular box with a light blue background. A horizontal line with a dot at its center passes through the box. Blue wavy lines connect the top and bottom boundaries, forming a pattern that is symmetric about the horizontal line. There are dots on the top and bottom boundaries.

Diagram 6: A rectangular box with a light blue background. A horizontal line with a dot at its center passes through the box. Blue wavy lines connect the top and bottom boundaries, forming a pattern that is not symmetric about the horizontal line. There are dots on the top and bottom boundaries.

Diagram 7: A rectangular box with a light blue background. A horizontal line with a dot at its center passes through the box. Blue wavy lines connect the top and bottom boundaries, forming a pattern that is symmetric about the horizontal line. There are dots on the top and bottom boundaries.

## A smaller algebra

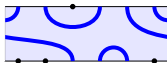
- $\text{pdTL}_{N,v}(\alpha, \beta)$ : subalgebra of  $\text{pdTL}_N(\alpha, \beta)$  with **connectivities that have  $v$  preserved vacancies**
- The algebra  $\mathbf{A}_N(\alpha, \beta)$  is the direct sum of these subalgebras:

$$\mathbf{A}_N(\alpha, \beta) = \bigoplus_{v=0}^N \text{pdTL}_{N,v}(\alpha, \beta)$$

- Examples:



$\in \text{pdTL}_{N,1}(\alpha, \beta) \subset \mathbf{A}_N(\alpha, \beta)$



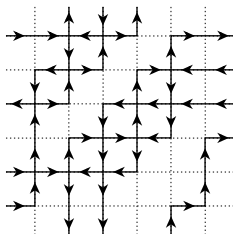
$\notin \mathbf{A}_N(\alpha, \beta)$

- The transfer tangles are elements of  $\mathbf{A}_N(\alpha, \beta)$ :

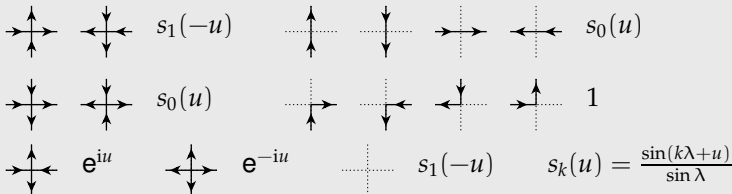
$$T^{(1,0)}(u), T^{(0,1)}(u) \in \mathbf{A}_N(\alpha, \beta)$$

# The $A_2^{(1)}$ vertex model

- A configuration of the 15-vertex model: (Jimbo '86)



■ The 15 admissible vertices and their Boltzmann weights:



# The $A_2^{(1)}$ vertex model

- The vector space is  $(\mathbb{C}^3)^{\otimes N}$ , with the canonical basis:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- The  $\check{R}(u)$  matrix:

$$\check{R}(u) = \begin{pmatrix} s_1(-u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & s_0(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{iu} & 0 & 0 & 0 & s_0(u) & 0 & 0 \\ 0 & s_0(u) & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_1(-u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & s_0(u) & 0 \\ 0 & 0 & s_0(u) & 0 & 0 & 0 & e^{-iu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_0(u) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_1(-u) \end{pmatrix} = \text{diamond}(u)$$

- The local maps between the loop and vertex models:

$$\begin{array}{|c|} \hline i \quad j \\ \hline \text{loop} \\ \hline \end{array} \longrightarrow q^{1/2} \langle \uparrow_i \downarrow_j | + q^{-1/2} \langle \downarrow_i \uparrow_j |$$

$$\begin{array}{|c|} \hline j \\ \hline \text{loop} \\ \hline i \\ \hline \end{array} \longrightarrow |\uparrow_i\rangle \langle \uparrow_j| + |\downarrow_i\rangle \langle \downarrow_j|$$

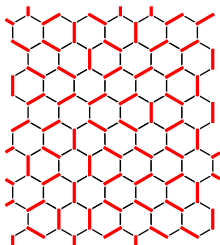
$$\begin{array}{|c|} \hline \text{loop} \\ \hline i \quad j \\ \hline \end{array} \longrightarrow q^{1/2} |\uparrow_i \downarrow_j\rangle + q^{-1/2} |\downarrow_i \uparrow_j\rangle$$

$$\begin{array}{|c|} \hline \text{loop} \\ \hline i \\ \hline \end{array} \longrightarrow |0_i\rangle \quad \begin{array}{|c|} \hline i \\ \hline \text{loop} \\ \hline \end{array} \longrightarrow \langle 0_i|$$



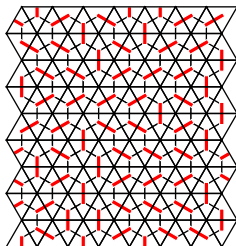
## Dimers on the hexagonal lattice

- Bijection between **dimer matchings** and configurations of the **fully packed loop model**: (Kondev, de Gier, Nienhuis '96)

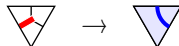
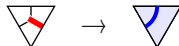
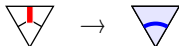


# Dimers on the hexagonal lattice

- Bijection between **dimer matchings** and configurations of the **fully packed loop model**: (Kondev, de Gier, Nienhuis '96)

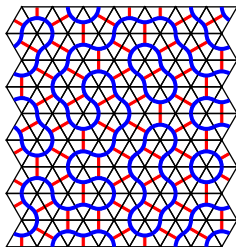


- Local maps:

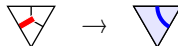
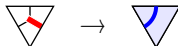
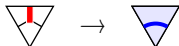


# Dimers on the hexagonal lattice

- Bijection between **dimer matchings** and configurations of the **fully packed loop model**: (Kondev, de Gier, Nienhuis '96)

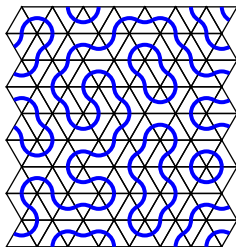


- Local maps:

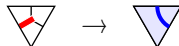
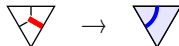
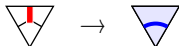


# Dimers on the hexagonal lattice

- Bijection between **dimer matchings** and configurations of the **fully packed loop model**: (Kondev, de Gier, Nienhuis '96)

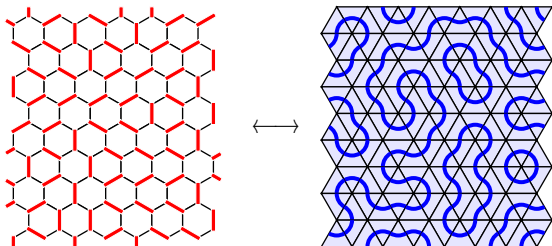


- Local maps:

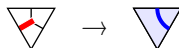
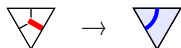
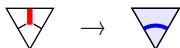


# Dimers on the hexagonal lattice

- Bijection between **dimer matchings** and configurations of the **fully packed loop model**: (Kondev, de Gier, Nienhuis '96)

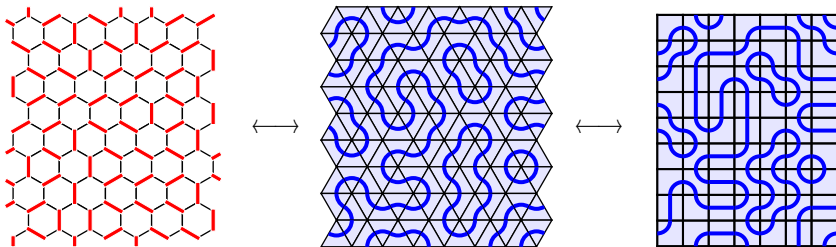


- Local maps:

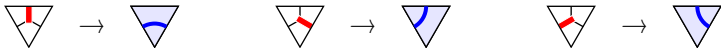


# Dimers on the hexagonal lattice

- Bijection between **dimer matchings** and configurations of the **fully packed loop model**: (Kondev, de Gier, Nienhuis '96)



- Local maps:



- Equivalent to the  $A_2^{(1)}$  loop model at  $\alpha = \beta = 1$  with  $u = \lambda = \pi/3$ :

$$\boxed{\lambda} = \begin{array}{|c|} \hline \text{blue arc} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{blue arc} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{blue vertical line} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{blue horizontal line} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{blue arc} \\ \hline \end{array}$$

# Lattice models and representations

- Family of  $A_2^{(1)}$  lattice models:
  - Loop model
  - Dimer model
  - Vertex model
  - RSOS model

One specific  $A_2^{(1)}$  lattice model  $\longleftrightarrow$  A set of representations of the algebra  $\mathbf{A}_N(\alpha, \beta)$

- To obtain the partition function, one must compute the eigenvalues of the transfer matrices  $T^{(1,0)}(u)$  and  $T^{(0,1)}(u)$
- **Objective:** find relations satisfied by  $T^{(1,0)}(u)$  and  $T^{(0,1)}(u)$
- By doing the calculations in  $\mathbf{A}_N(\alpha, \beta)$ , **we are solving all the  $A_2^{(1)}$  models at once.**



# Inversion identities

- There are two local inversion identities:

$$\begin{aligned}
 \text{Diagram 1} &= s_1(u)s_1(-u) \text{ Diagram 2} & \text{Diagram 3} &= s_0(u)s_3(-u) \text{ Diagram 4}
 \end{aligned}$$

Diagram 1: Two diamonds with parameters  $u$  and  $-u$  and dashed blue lines connecting them.

Diagram 2: A single diamond with parameter  $u$  and dashed blue lines.

Diagram 3: Two diamonds with parameters  $u$  and  $3\lambda - u$  and dashed blue lines connecting them.

Diagram 4: A single diamond with parameter  $u$  and dashed blue lines.

- This is computed as follows:

$$\begin{aligned}
 \text{Diagram 1} &= \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} \\
 &+ \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12} + \text{Diagram 13} + \text{Diagram 14} \\
 &+ \text{Diagram 15} + \text{Diagram 16} + \text{Diagram 17}
 \end{aligned}$$

The diagrams on the right show various configurations of solid blue lines (straight, wavy, or circular) within and between diamonds, representing the expansion of the inversion identity.



# Inversion identities

- There are two local inversion identities:

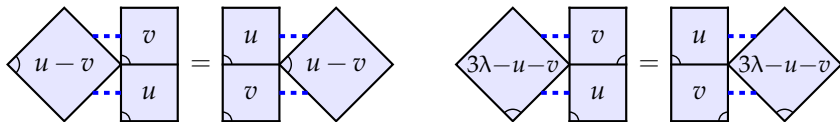
$$\begin{aligned}
 \text{Diagram 1} &= s_1(u)s_1(-u) \text{ Diagram 2} & \text{Diagram 3} &= s_0(u)s_3(-u) \text{ Diagram 4}
 \end{aligned}$$

- This is computed as follows:

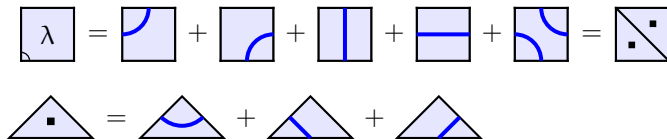
$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 5.1} + \text{Diagram 5.2} + \text{Diagram 5.3} + \text{Diagram 5.4} + \text{Diagram 5.5} \\
 & \qquad \qquad \qquad s_1(-u)s_1(u) \quad 1 \times 1 \quad 1 \times s_0(-u) \quad 1 \times 1 \quad 1 \times s_0(-u) \\
 & + \text{Diagram 5.6} + \text{Diagram 5.7} + \text{Diagram 5.8} + \text{Diagram 5.9} + \text{Diagram 5.10} \\
 & \qquad \qquad \qquad s_0(u) \times 1 \quad s_0(u)s_0(-u) \quad s_0(u) \times 1 \quad s_0(u)s_0(-u) \quad s_1(-u)s_1(u) \\
 & + \text{Diagram 5.11} + \text{Diagram 5.12} + \text{Diagram 5.13} \\
 & \qquad \qquad \qquad s_1(-u)s_0(-u) \quad s_0(u)s_1(u) \quad \beta s_0(u)s_0(-u) \\
 & = \text{Diagram 5.14} + \text{Diagram 5.15} + \text{Diagram 5.16} + \text{Diagram 5.17} + \text{Diagram 5.18} + \text{Diagram 5.19} + \text{Diagram 5.20} \\
 & \qquad \qquad \qquad s_1(u)s_1(-u) \quad s_1(u)s_1(-u) \quad s_1(u)s_1(-u) \quad s_1(u)s_1(-u) \quad 0 \quad 0 \quad 0
 \end{aligned}$$

## More identities

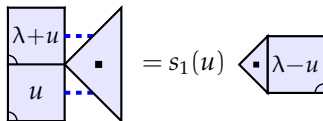
- Two inequivalent **Yang-Baxter equations**:



- Factorisation of the face operator at  $u = \lambda$ :



- A **push-through property**:







# Identities from $sl(3)$ spiders

- Two triangle operators: (reminder:  $\beta = q + q^{-1}$ )

$$\text{Triangle with square and dot} = \text{Triangle with circle} + q \text{ Triangle with arc} + q^{-1} \text{ Triangle with arc} = \beta \text{ Rectangle} + (q + q^{-1}) \text{ Rectangle with lines}$$

- Two identities:

$$\text{Triangle with square and dot} = [2] \text{ Rectangle with lines}$$

$$\text{Hexagon with square and dot} = \text{Diamond with dashed circles} + \text{Diamond with wavy line}$$




where  $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$  and  $\text{Rectangle with wavy line} = q \text{ Rectangle with lines} + q^{-2} \text{ Rectangle}$

- Identical to identities for  $sl(3)$  spiders: (Kuperberg '96)

$$\text{Complex spider} = [2] \text{ Simple spider}$$

$$\text{Square spider} = \text{Spider 1} + \text{Spider 2}$$

# Fusion relation for $T^{2,0}$




- The  $(2, 0)$  projector:  =  -  $\frac{1}{[2]}$  
- The resulting fusion hierarchy relation:

$$T^{1,0}(u)T^{1,0}(u + \lambda) = f_1 T^{0,1}(u) + f_0 T^{2,0}(u)$$

- The diagrammatic derivation:  $(u_k = u + k\lambda)$

$$f_0 T^{2,0}(u) = \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array}$$

# Fusion relation for $T^{2,0}$

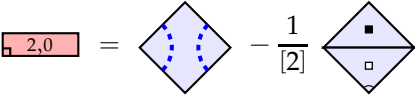
- The  $(2, 0)$  projector:  =  -  $\frac{1}{[2]}$  
- The resulting fusion hierarchy relation:

$$T^{1,0}(u)T^{1,0}(u + \lambda) = f_1 T^{0,1}(u) + f_0 T^{2,0}(u)$$

- The diagrammatic derivation:  $(u_k = u + k\lambda)$

$$f_0 T^{2,0}(u) = \begin{array}{|c|c|c|} \hline \cdot u_1 & \cdot u_1 & \cdot u_1 \\ \hline \cdot u_0 & \cdot u_0 & \cdot u_0 \\ \hline \end{array} \begin{array}{|c|} \hline \cdot \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \cdot u_1 & \cdot u_1 & \cdot u_1 \\ \hline \cdot u_0 & \cdot u_0 & \cdot u_0 \\ \hline \end{array} - \frac{1}{[2]} \begin{array}{|c|c|c|} \hline \cdot u_1 & \cdot u_1 & \cdot u_1 \\ \hline \cdot u_0 & \cdot u_0 & \cdot u_0 \\ \hline \end{array} \begin{array}{|c|} \hline \cdot \\ \hline \end{array}$$

# Fusion relation for $T^{2,0}$

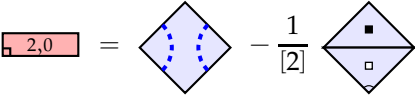
- The  $(2,0)$  projector: 
- The resulting fusion hierarchy relation:

$$T^{1,0}(u)T^{1,0}(u + \lambda) = f_1 T^{0,1}(u) + f_0 T^{2,0}(u)$$

- The diagrammatic derivation:  $(u_k = u + k\lambda)$

$$\begin{aligned}
 f_0 T^{2,0}(u) &= \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline 2,0 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} - \frac{1}{[2]} \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline \blacksquare \\ \hline \square \\ \hline \end{array} \\
 &= T^{1,0}(u)T^{1,0}(u + \lambda) - \frac{1}{[2]} \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array}
 \end{aligned}$$

# Fusion relation for $T^{2,0}$

- The  $(2,0)$  projector: 
- The resulting fusion hierarchy relation:

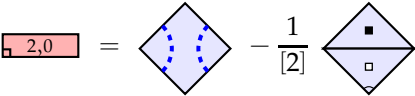
$$T^{1,0}(u)T^{1,0}(u + \lambda) = f_1 T^{0,1}(u) + f_0 T^{2,0}(u)$$

- The diagrammatic derivation:  $(u_k = u + k\lambda)$

$$\begin{aligned}
 f_0 T^{2,0}(u) &= \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline 2,0 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} - \frac{1}{[2]} \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline \blacksquare \\ \hline \square \\ \hline \end{array} \\
 &= T^{1,0}(u)T^{1,0}(u + \lambda) - \frac{1}{[2]} \begin{array}{|c|c|} \hline u_1 & u_1 \\ \hline u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline \blacksquare \\ \hline \square \\ \hline \end{array} \begin{array}{|c|} \hline \lambda - u \\ \hline \end{array} \times s_1(u)
 \end{aligned}$$



# Fusion relation for $T^{2,0}$

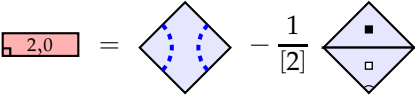
- The  $(2,0)$  projector: 
- The resulting fusion hierarchy relation:

$$T^{1,0}(u)T^{1,0}(u + \lambda) = f_1 T^{0,1}(u) + f_0 T^{2,0}(u)$$

- The diagrammatic derivation:  $(u_k = u + k\lambda)$

$$\begin{aligned}
 f_0 T^{2,0}(u) &= \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline 2,0 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} - \frac{1}{[2]} \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline \blacksquare \\ \hline \square \\ \hline \end{array} \\
 &= T^{1,0}(u)T^{1,0}(u + \lambda) - \frac{1}{[2]} \begin{array}{|c|} \hline \square \\ \hline \blacksquare \\ \hline \end{array} \begin{array}{|c|c|c|} \hline \lambda-u & \lambda-u & \lambda-u \\ \hline \end{array} \times (s_1(u))^N
 \end{aligned}$$

# Fusion relation for $T^{2,0}$




- The  $(2, 0)$  projector: 
- The resulting fusion hierarchy relation:

$$T^{1,0}(u)T^{1,0}(u + \lambda) = f_1 T^{0,1}(u) + f_0 T^{2,0}(u)$$

- The diagrammatic derivation:  $(u_k = u + k\lambda)$

$$\begin{aligned}
 f_0 T^{2,0}(u) &= \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline 2,0 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} - \frac{1}{[2]} \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline \blacksquare \\ \hline \square \\ \hline \end{array} \\
 &= T^{1,0}(u)T^{1,0}(u + \lambda) - \begin{array}{|c|c|c|} \hline \lambda-u & \lambda-u & \lambda-u \\ \hline \end{array} \times (s_1(u))^N
 \end{aligned}$$

## Fusion relation for $T^{2,0}$

- The  $(2, 0)$  projector:  =  -  $\frac{1}{[2]}$  
- The resulting fusion hierarchy relation:

$$T^{1,0}(u)T^{1,0}(u + \lambda) = f_1 T^{0,1}(u) + f_0 T^{2,0}(u)$$

- The diagrammatic derivation:  $(u_k = u + k\lambda)$

$$\begin{aligned} f_0 T^{2,0}(u) &= \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline 2,0 \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} - \frac{1}{[2]} \begin{array}{|c|c|c|} \hline u_1 & u_1 & u_1 \\ \hline u_0 & u_0 & u_0 \\ \hline \end{array} \begin{array}{|c|} \hline \text{diamond} \\ \hline \end{array} \\ &= T^{1,0}(u)T^{1,0}(u + \lambda) - \begin{array}{|c|c|c|} \hline \lambda-u & \lambda-u & \lambda-u \\ \hline \end{array} \times (s_1(u))^N \\ &= T^{1,0}(u)T^{1,0}(u + \lambda) - f_1 T^{0,1}(u) \end{aligned}$$

# The fused transfer tangle $T^{1,1}$

- Diagrammatic definition:

$$T^{1,1}(u) = \frac{1}{f_0} \begin{array}{|c|c|c|c|} \hline -u & -u & \cdots & -u \\ \hline u & u & \cdots & u \\ \hline \end{array} \begin{array}{|c|} \hline \text{red box} \\ \hline \end{array} \quad f_k = (s_k(u))^N$$

- $\begin{array}{|c|} \hline 1,1 \\ \hline \end{array}$  is a projector:  $\begin{array}{|c|} \hline 1,1 \\ \hline \end{array} = \begin{array}{|c|} \hline \text{diamond with dashed lines} \\ \hline \end{array} - \frac{1}{[3]} \begin{array}{|c|} \hline \text{diamond with wavy line} \\ \hline \end{array}$

- Fusion hierarchy relation:

$$T^{1,0}(u)T^{0,1}(u + \lambda) = f_0 T^{1,1}(u) + \sigma f_{-1} f_1 I \quad \sigma = (-1)^N$$

- Similar to the  $sl(3)$  tensor product rule:

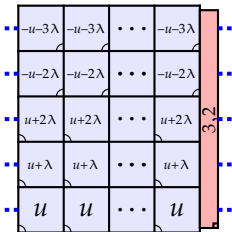
$$\begin{array}{|c|} \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

$$(1, 0) \otimes (0, 1) = (1, 1) \oplus (0, 0)$$

# The general case

- General definition:

$$T^{3,2}(u) = \frac{1}{f_0 f_1 f_2 f_3}$$



$$f_k = (s_k(u))^N$$

- The projectors are defined recursively

■ **The fusion hierarchy relations:**  $(u_k = u + k\lambda)$

$$T^{m,0}(u_0)T^{1,0}(u_m) = f_m T^{m-1,1}(u_0) + f_{m-1} T^{m+1,0}(u_0)$$

$$T^{0,1}(u_0)T^{0,n}(u_1) = \sigma f_{-1} T^{1,n-1}(u_1) + f_0 T^{0,n+1}(u_0)$$

$$T^{m,0}(u_0)T^{0,n}(u_m) = f_{m-1} T^{m,n}(u_0) + \sigma T_0^{m-1,0} T^{0,n-1}(u_{m+1})$$

- Each  $T^{m,n}(u)$  is a polynomial in  $T^{1,0}(u)$  and  $T^{0,1}(u)$

## Closure relations at roots of unity

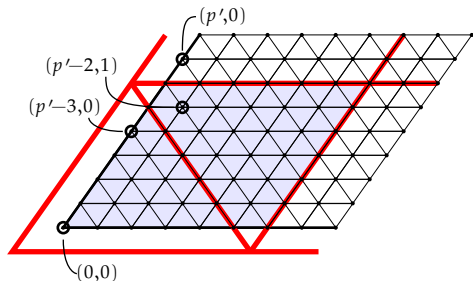
- Roots of unity values:  $\beta = q + q^{-1}$  with  $q^{2p'} = 1$   $p' \in \mathbb{N}$

- **Closure relations** for the fused transfer tangles:

$$T^{p',0}(u) = T^{p'-2,1}(u + \lambda) - \sigma T^{p'-3,0}(u + 2\lambda) + f_{-1}J$$

$$T^{0,p'}(u) = \sigma T^{1,p'-2}(u) - T^{0,p'-3}(u + \lambda) + f_{-1}K$$

- $J$  and  $K$  are tangles that are independent of  $u$
- These are two polynomial equations satisfied by  $T^{1,0}(u)$  and  $T^{0,1}(u)$ , and by their eigenvalues



# Y-systems

- Functions in the Y-system:

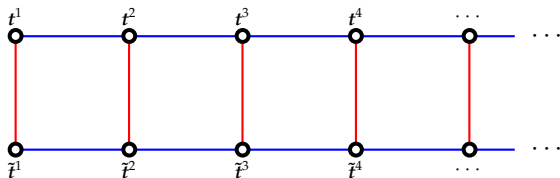
$$t_0^m = \frac{T_0^{m+1,0} T_1^{m-1,0}}{f_m T_0^{0,m}} \quad \tilde{t}_0^n = \sigma^n \frac{T_0^{0,n+1} T_1^{0,n-1}}{f_{-1} T_1^{n,0}} \quad T_k^{m,n} = T^{m,n}(u + k\lambda)$$

$$t_k^m = t^m(u + k\lambda) \quad \tilde{t}_k^m = \tilde{t}^m(u + k\lambda)$$

- Universal Y-system equations:

$$t_0^m t_1^m = \frac{(I + t_0^{m+1})(I + t_1^{m-1})}{I + (\tilde{t}_0^m)^{-1}} \quad \tilde{t}_0^n \tilde{t}_1^n = \frac{(I + \tilde{t}_0^{n+1})(I + \tilde{t}_1^{n-1})}{I + (t_1^n)^{-1}}$$

- Encoded in this Dynkin diagram (for  $q$  generic):





# Outlook

## Overview:

- We derived functional equations satisfied by the transfer matrices of the  $A_2^{(1)}$  models
- These can be rewritten in terms of a  $Y$ -system

## Future work:

- Solve the  $Y$ -system for the eigenvalues
- Extract information on the underlying CFT (with  $W_3$  symmetry)
- Generalize the method to the  $A_2^{(2)}$  models:

# Outlook

## Overview:

- We derived functional equations satisfied by the transfer matrices of the  $A_2^{(1)}$  models
- These can be rewritten in terms of a  $Y$ -system

## Future work:

- Solve the  $Y$ -system for the eigenvalues
- Extract information on the underlying CFT (with  $W_3$  symmetry)
- Generalize the method to the  $A_2^{(2)}$  models:

$$\begin{aligned}
 \boxed{u} &= \rho_1(u) \boxed{\phantom{u}} + \rho_2(u) \boxed{\curvearrowright} + \rho_3(u) \boxed{\curvearrowleft} + \rho_4(u) \boxed{\curvearrowright\curvearrowleft} + \rho_5(u) \boxed{\curvearrowleft\curvearrowright} \\
 &+ \rho_6(u) \boxed{\text{||}} + \rho_7(u) \boxed{\text{—}} + \rho_8(u) \boxed{\curvearrowright\curvearrowright} + \rho_9(u) \boxed{\curvearrowleft\curvearrowleft}
 \end{aligned}$$

# Outlook

## Overview:

- We derived functional equations satisfied by the transfer matrices of the  $A_2^{(1)}$  models
- These can be rewritten in terms of a  $Y$ -system

## Future work:

- Solve the  $Y$ -system for the eigenvalues
- Extract information on the underlying CFT (with  $W_3$  symmetry)
- Generalize the method to the  $A_2^{(2)}$  models:

$$\begin{aligned}
 \boxed{u} &= \rho_1(u) \boxed{\phantom{u}} + \rho_2(u) \boxed{\curvearrowright} + \rho_3(u) \boxed{\curvearrowleft} + \rho_4(u) \boxed{\curvearrowright\curvearrowleft} + \rho_5(u) \boxed{\curvearrowleft\curvearrowright} \\
 &+ \rho_6(u) \boxed{\text{||}} + \rho_7(u) \boxed{\text{—}} + \rho_8(u) \boxed{\curvearrowright\curvearrowright} + \rho_9(u) \boxed{\curvearrowleft\curvearrowleft}
 \end{aligned}$$

Thank you!