Functional relations for the $A_2^{(1)}$ models

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Recent Advances in Quantum Integrable Systems

Annecy-le-Vieux, 13/09/2018

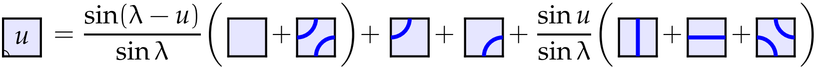
Joint work with Paul A. Pearce and Jørgen Rasmussen

Outline

- The $A_2^{(1)}$ lattice models
  - A family of models: loop, vertex and dimer models
  - Diagrammatic algebras
- Diagrammatic calculus and functional relations
  - Planar identities
  - Wenzl-Jones projectors
  - Fused transfer matrices
  - Fusion hierarchy relations of $s\ell_3$ type
  - Y-systems

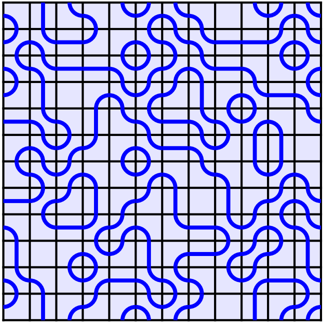
The $A_2^{(1)}$ loop model

- The elementary face operator: (Nienhuis, Warnaar '93)



- u is the **spectral parameter**
- Loop fugacities:
  - contractible: $\beta = 2\cos\lambda = q + q^{-1}$
  - non contractible: α
- **Vacancies are preserved**
- Weights and partition functions:

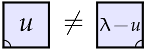
$$W_\sigma = \alpha^{n_\alpha}\beta^{n_\beta}\prod_f w_f \qquad Z = \sum_\sigma W_\sigma$$

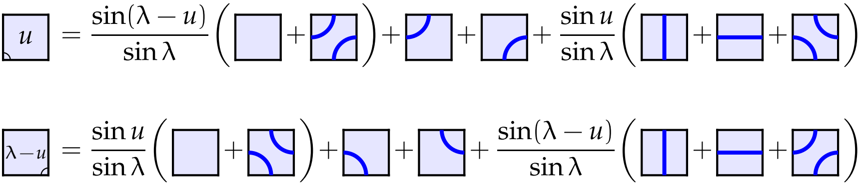


A loop configuration σ on the 12×12 torus

Crossing symmetry and transfer tangles

- No crossing symmetry:





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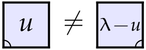
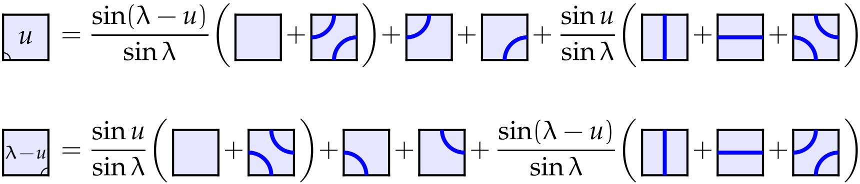
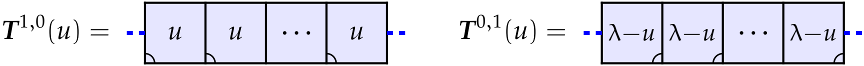
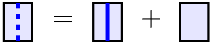




- Two elementary transfer tangles:



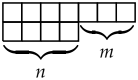
- The identity strand:



- They are assigned labels corresponding to the **two fundamental representations of $s\ell(3)$.**

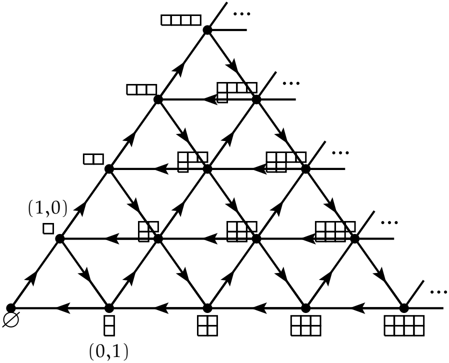
$s\ell_3$ representations

- Irreducible representations and Young diagrams:

$(m, n) \quad \longleftrightarrow$ 

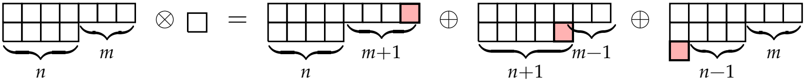
- Tensor products of irreducible representations:

$$(m,n) \otimes (1,0) = (m+1, n) \oplus (m-1, n+1) \oplus (m, n-1)$$



The $s\ell_3$ weight lattice

- Equivalently:



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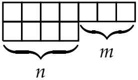

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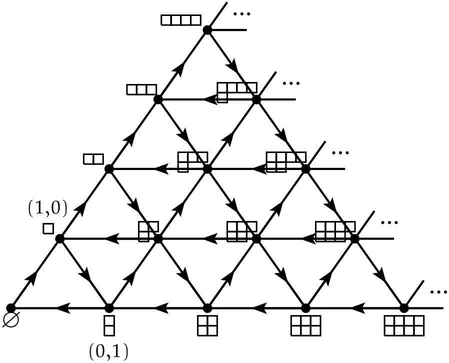

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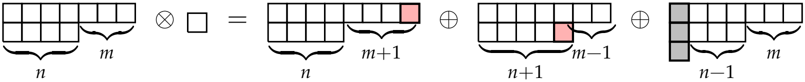
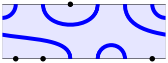
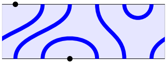
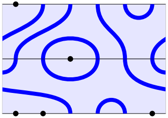
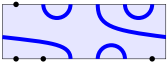


Dilute Temperley-Lieb algebra

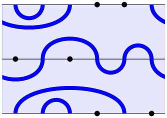
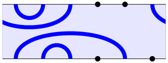
- The periodic dilute Temperley-Lieb algebra, $\mathsf{pdTL}_N(\alpha, \beta)$, is the linear span of **connectivity diagrams:**

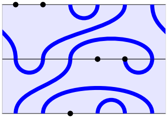
$a_1 =$  $a_2 =$ 

- Product of two connectivity diagrams:

$a_1 a_2 =$  $= \beta$  $= \beta\, a_3$

- More examples of products for $N = 6$:

 $= \alpha$ 

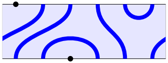
 $= 0$

A smaller algebra

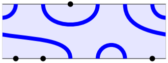
- $\mathsf{pdTL}_{N,v}(\alpha,\beta)$: subalgebra of $\mathsf{pdTL}_N(\alpha,\beta)$ with **connectivities that have v preserved vacancies**
- The algebra $\mathsf{A}_N(\alpha,\beta)$ is the direct sum of these subalgebras:

$$\mathsf{A}_N(\alpha,\beta) = \bigoplus_{v=0}^{N} \mathsf{pdTL}_{N,v}(\alpha,\beta)$$

- Examples:



$\in \mathsf{pdTL}_{N,1}(\alpha,\beta) \subset \mathsf{A}_N(\alpha,\beta)$



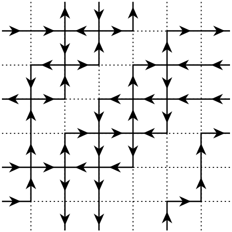
$\notin \mathsf{A}_N(\alpha,\beta)$

- The transfer tangles are elements of $\mathsf{A}_N(\alpha,\beta)$:

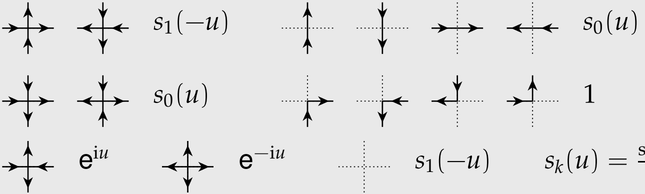
$$\boldsymbol{T}^{(1,0)}(u), \boldsymbol{T}^{(0,1)}(u) \in \mathsf{A}_N(\alpha,\beta)$$

The $A_2^{(1)}$ vertex model

- A configuration of the 15-vertex model: (Jimbo '86)



- The 15 admissible vertices and their Boltzmann weights:



$$s_k(u) = \frac{\sin(k\lambda+u)}{\sin\lambda}$$

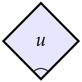
The $A_2^{(1)}$ vertex model

- The vector space is $(\mathbb{C}^3)^{\otimes N}$, with the canonical basis:

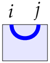
$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- The $\check{R}(u)$ matrix:

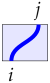
$$\check{R}(u) = \begin{pmatrix} s_1(-u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & s_0(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{iu} & 0 & 0 & 0 & s_0(u) & 0 & 0 \\ 0 & s_0(u) & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_1(-u) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & s_0(u) & 0 \\ 0 & 0 & s_0(u) & 0 & 0 & 0 & \mathrm{e}^{-iu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_0(u) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_1(-u) \end{pmatrix} =$$



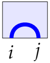
- The local maps between the loop and vertex models:



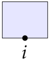
$\longrightarrow q^{1/2}\langle\uparrow_i\downarrow_j| + q^{-1/2}\langle\downarrow_i\uparrow_j|$



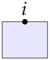
$\longrightarrow |\uparrow_i\rangle\langle\uparrow_j| + |\downarrow_i\rangle\langle\downarrow_j|$



$\longrightarrow q^{1/2}|\uparrow_i\downarrow_j\rangle + q^{-1/2}|\downarrow_i\uparrow_j\rangle$



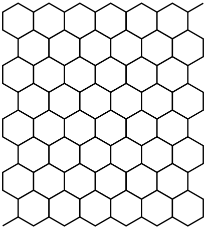
$\longrightarrow |0_i\rangle$



$\longrightarrow \langle 0_i|$

Dimers on the hexagonal lattice

- Bijection between **dimer matchings** and configurations of the **fully packed loop model:** (Kondev, de Gier, Nienhuis '96)



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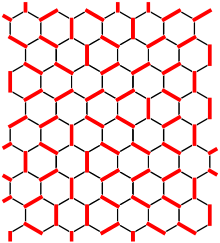

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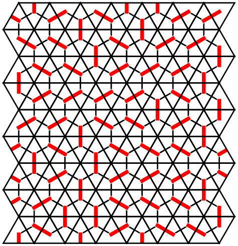
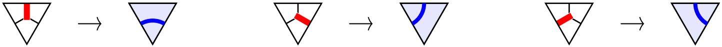


- Local maps:



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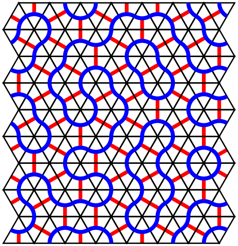

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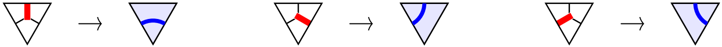

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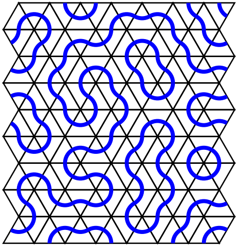

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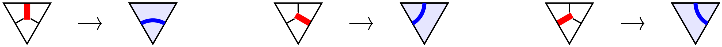

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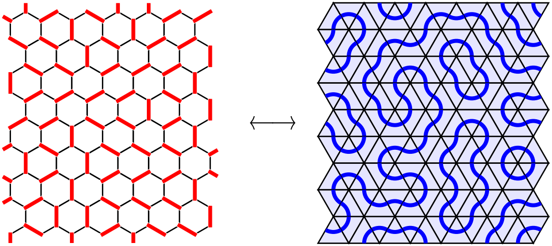

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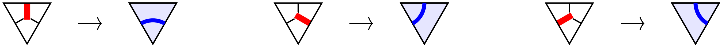

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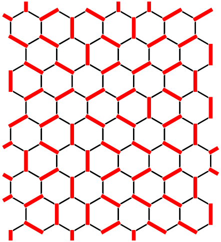
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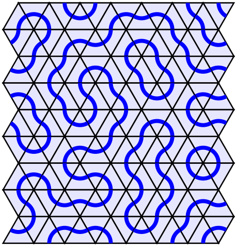
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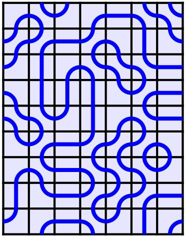

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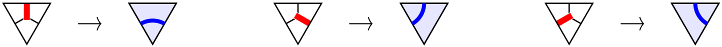
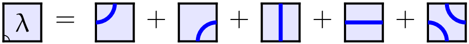


- Equivalent to the $A_2^{(1)}$ loop model at $\alpha = \beta = 1$ with $u = \lambda = \pi/3$:



Lattice models and representations

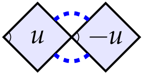
- Family of $A_2^{(1)}$ lattice models:
  - Loop model
  - Dimer model
  - Vertex model
  - RSOS model

One specific $A_2^{(1)}$ lattice model	$\longleftrightarrow$	A set of representations of the algebra $\mathsf{A}_N(\alpha, \beta)$

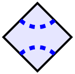
- To obtain the partition function, one must compute the eigenvalues of the transfer matrices $T^{(1,0)}(u)$ and $T^{(0,1)}(u)$
- **Objective:** find relations satisfied by $\boldsymbol{T}^{(1,0)}(u)$ and $\boldsymbol{T}^{(0,1)}(u)$
- By doing the calculations in $\mathsf{A}_N(\alpha, \beta)$, **we are solving all the $A_2^{(1)}$ models at once.**

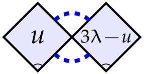
Inversion identities

- There are two local inversion identities:

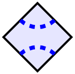


$= s_1(u)s_1(-u)$

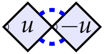




$= s_0(u)s_3(-u)$



- This is computed as follows:



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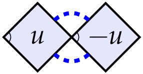
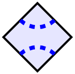
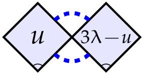


$= s_1(u)s_1(-u)$





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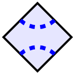

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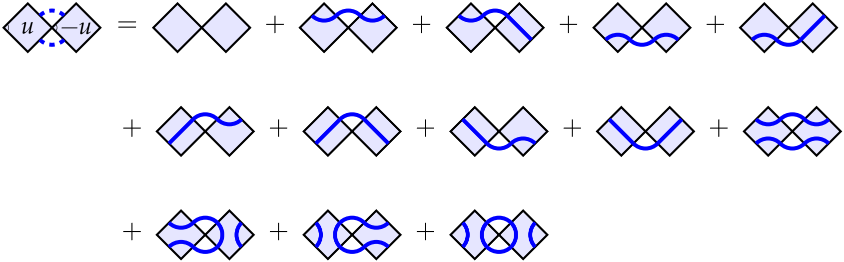

Inversion identities

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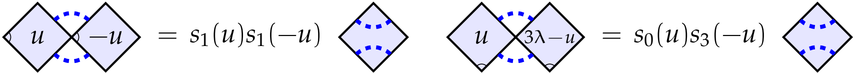
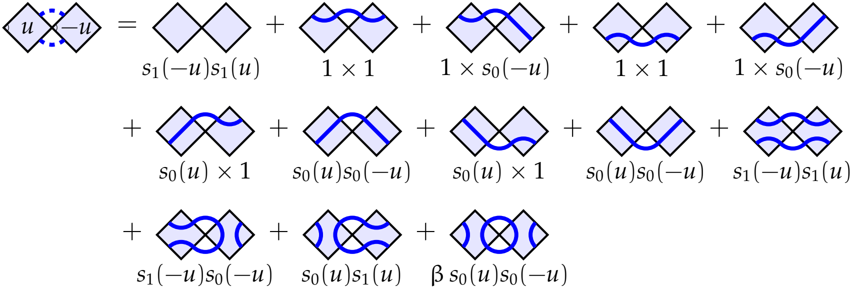


- This is computed as follows:



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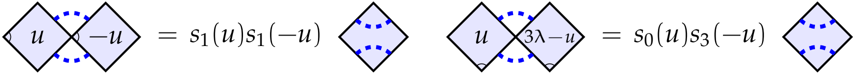

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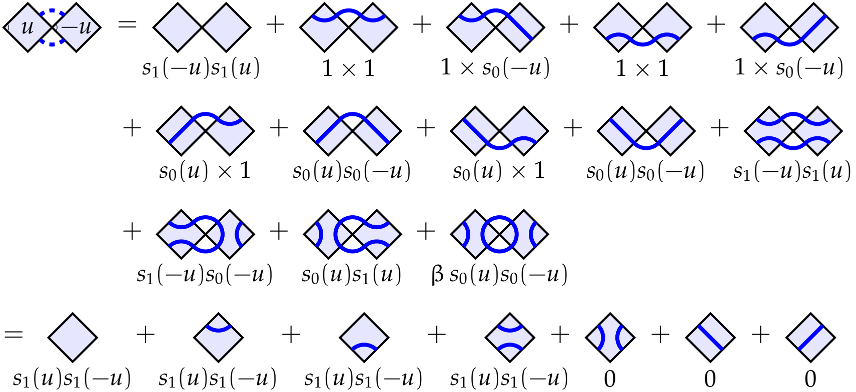
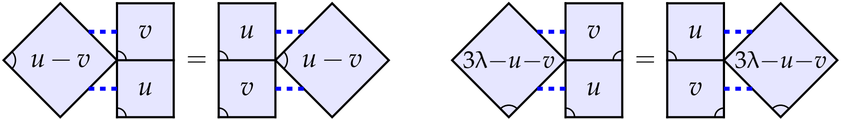
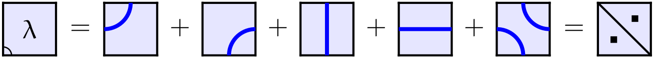


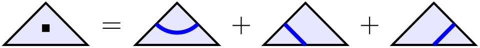
More identities

- Two inequivalent **Yang-Baxter equations:**

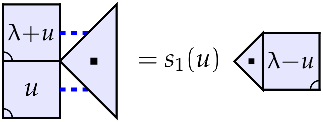


- Factorisation of the face operator at $u = \lambda$:





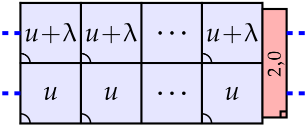
- A **push-through property**:



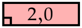
The fused transfer tangle $T^{2,0}$

- Diagrammatic definition:

$$T^{2,0}(u) = \frac{1}{f_0}$$



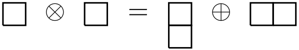
$$f_k = \big(s_k(u)\big)^N$$

- 

is a projector
- Fusion hierarchy relation:

$$T^{1,0}(u)T^{1,0}(u+\lambda) = f_1 T^{0,1}(u) + f_0 T^{2,0}(u)$$

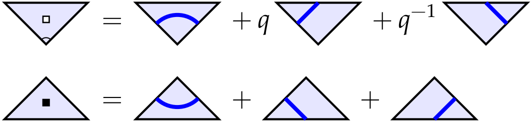
- Similar to the $s\ell(3)$ tensor product rule:



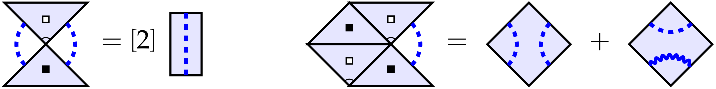
$$(1,0)\otimes(1,0) = (0,1)\oplus(2,0)$$

Identities from $s\ell(3)$ spiders

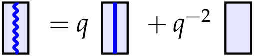
- Two triangle operators: (reminder: $\beta = q + q^{-1}$)



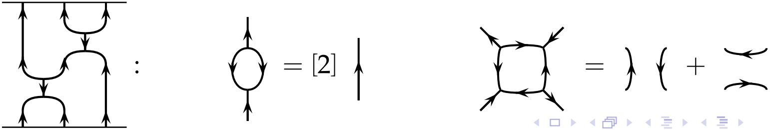
- Two identities:



where $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ and



- Identical to identities for $s\ell(3)$ spiders: (Kuperberg '96)



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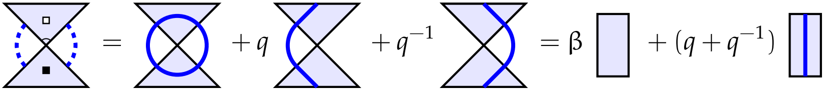

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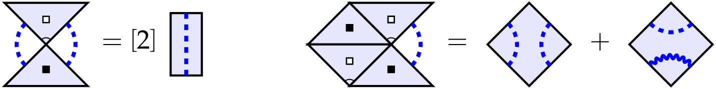

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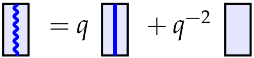

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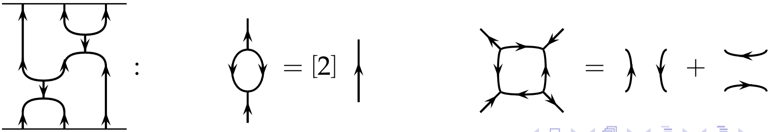
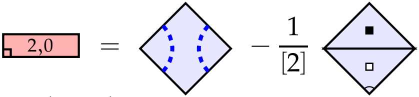


Fusion relation for $\boldsymbol{T}^{2,0}$

- The $(2,0)$ projector:

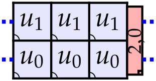


- The resulting fusion hierarchy relation:

$$\boldsymbol{T}^{1,0}(u)\boldsymbol{T}^{1,0}(u+\lambda) = f_1\boldsymbol{T}^{0,1}(u) + f_0\boldsymbol{T}^{2,0}(u)$$

- The diagrammatic derivation: $(u_k = u + k\lambda)$

$f_0\boldsymbol{T}^{2,0}(u) =$



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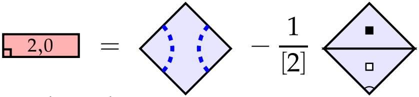

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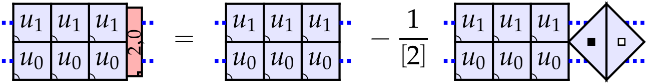

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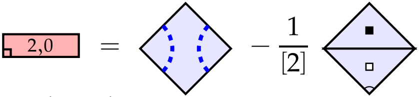

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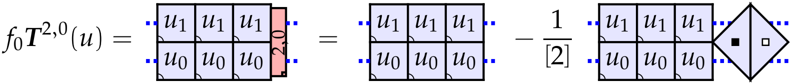

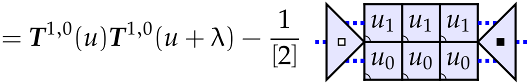

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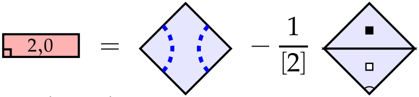

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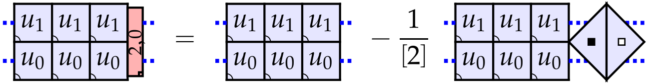
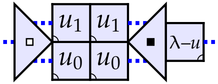


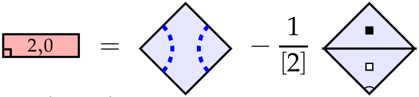
$= \boldsymbol{T}^{1,0}(u)\boldsymbol{T}^{1,0}(u+\lambda) - \frac{1}{[2]}$



$\times\, s_1(u)$

Fusion relation for $T^{2,0}$

- The $(2,0)$ projector:



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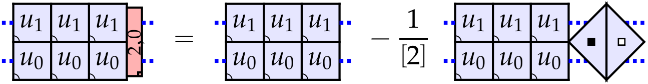

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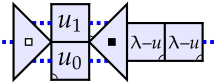

$\times \big(s_1(u)\big)^2$

Fusion relation for $T^{2,0}$

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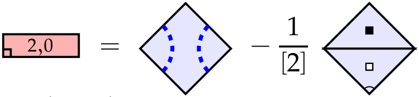

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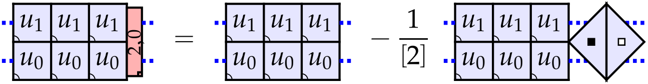

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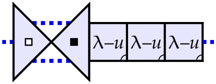
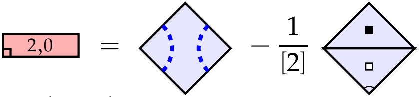


$\times \big(s_1(u)\big)^N$

Fusion relation for $T^{2,0}$

- The $(2,0)$ projector:



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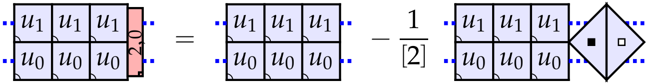
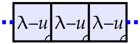


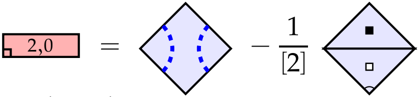
$= T^{1,0}(u)T^{1,0}(u+\lambda) -$



$\times \big(s_1(u)\big)^N$

Fusion relation for $T^{2,0}$

- The $(2,0)$ projector:



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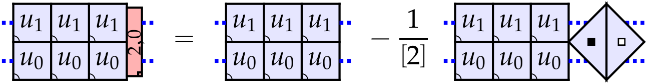

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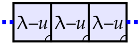
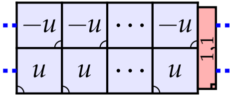


$\times \big(s_1(u)\big)^N$

$$= T^{1,0}(u)T^{1,0}(u+\lambda) - f_1 T^{0,1}(u)$$

The fused transfer tangle $\boldsymbol{T}^{1,1}$

- Diagrammatic definition:

$$\boldsymbol{T}^{1,1}(u) = \frac{1}{f_0}$$


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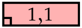
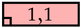
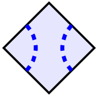
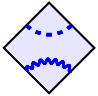
- 

is a projector:

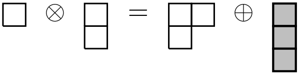


$=$  $-\dfrac{1}{[3]}$ 

- Fusion hierarchy relation:

$$\boldsymbol{T}^{1,0}(u)\boldsymbol{T}^{0,1}(u+\lambda) = f_0\boldsymbol{T}^{1,1}(u) + \sigma f_{-1}f_1\boldsymbol{I} \qquad \sigma = (-1)^N$$

- Similar to the $s\ell(3)$ tensor product rule:

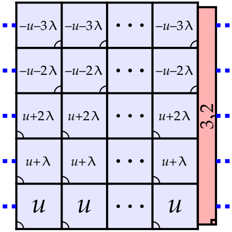


$$(1,0)\otimes(0,1) = (1,1)\oplus(0,0)$$

The general case

- General definition:

$$\boldsymbol{T}^{3,2}(u) = \frac{1}{f_0 f_1 f_2 f_3}$$



$$f_k = \big(s_k(u)\big)^N$$

- The projectors are defined recursively

> ▪ **The fusion hierarchy relations:** $(u_k = u + k\lambda)$
>
> $$\boldsymbol{T}^{m,0}(u_0)\boldsymbol{T}^{1,0}(u_m) = f_m \boldsymbol{T}^{m-1,1}(u_0) + f_{m-1}\boldsymbol{T}^{m+1,0}(u_0)$$
> $$\boldsymbol{T}^{0,1}(u_0)\boldsymbol{T}^{0,n}(u_1) = \sigma f_{-1}\boldsymbol{T}^{1,n-1}(u_1) + f_0\boldsymbol{T}^{0,n+1}(u_0)$$
> $$\boldsymbol{T}^{m,0}(u_0)\boldsymbol{T}^{0,n}(u_m) = f_{m-1}\boldsymbol{T}^{m,n}(u_0) + \sigma\, \boldsymbol{T}_0^{m-1,0}\boldsymbol{T}^{0,n-1}(u_{m+1})$$

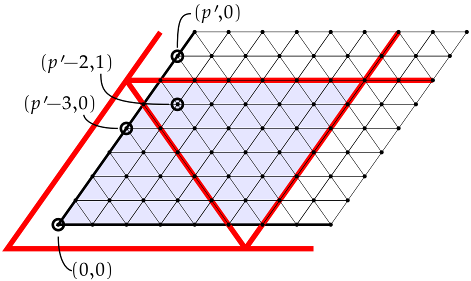
- Each $\boldsymbol{T}^{m,n}(u)$ is a polynomial in $\boldsymbol{T}^{1,0}(u)$ and $\boldsymbol{T}^{0,1}(u)$

Closure relations at roots of unity

- Roots of unity values: $\beta = q + q^{-1}$ with $q^{2p'} = 1$ $p' \in \mathbb{N}$

> ■ **Closure relations** for the fused transfer tangles:
>
> $$\boldsymbol{T}^{p',0}(u) = \boldsymbol{T}^{p'-2,1}(u+\lambda) - \sigma\,\boldsymbol{T}^{p'-3,0}(u+2\lambda) + f_{-1}\boldsymbol{J}$$
> $$\boldsymbol{T}^{0,p'}(u) = \sigma\,\boldsymbol{T}^{1,p'-2}(u) - \boldsymbol{T}^{0,p'-3}(u+\lambda) + f_{-1}\boldsymbol{K}$$

- $\boldsymbol{J}$ and $\boldsymbol{K}$ are tangles that are independent of u
- These are two polynomial equations satisfied by $\boldsymbol{T}^{1,0}(u)$ and $\boldsymbol{T}^{0,1}(u)$, and by their eigenvalues



Y-systems

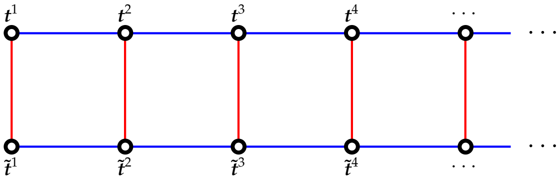
- Functions in the Y-system:

$$t_0^m = \frac{T_0^{m+1,0} T_1^{m-1,0}}{f_m T_0^{0,m}} \qquad \tilde{t}_0^n = \sigma^n \frac{T_0^{0,n+1} T_1^{0,n-1}}{f_{-1} T_1^{n,0}} \qquad \begin{aligned} T_k^{m,n} &= T^{m,n}(u+k\lambda) \\ t_k^m &= t^m(u+k\lambda) \\ \tilde{t}_k^m &= \tilde{t}^m(u+k\lambda) \end{aligned}$$

- Universal Y-system equations:

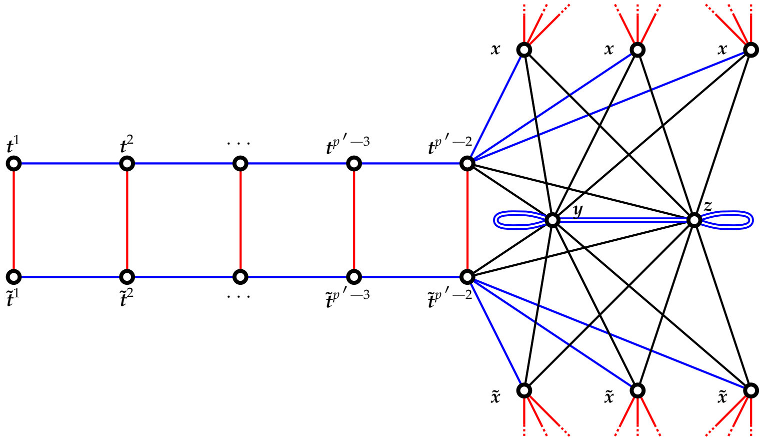
$$t_0^m t_1^m = \frac{(I + t_0^{m+1})(I + t_1^{m-1})}{I + (\tilde{t}_0^m)^{-1}} \qquad \tilde{t}_0^n \tilde{t}_1^n = \frac{(I + \tilde{t}_0^{n+1})(I + \tilde{t}_1^{n-1})}{I + (t_1^n)^{-1}}$$

- Encoded in this Dynkin diagram (for q generic):



Y-system at roots of unity

- For $q^{2p'} = 1$, the Y-system is finite:



- Similar Y-system previously found in complex $su(3)$ Toda theory

  (Saleur, Wehefritz-Kaufmann '00)

Outlook

Overview:

- We derived functional equations satisfied by the transfer matrices of the $A_2^{(1)}$ models
- These can be rewritten in terms of a Y-system

Future work:

- Solve the Y-system for the eigenvalues
- Extract information on the underlying CFT (with W_3 symmetry)
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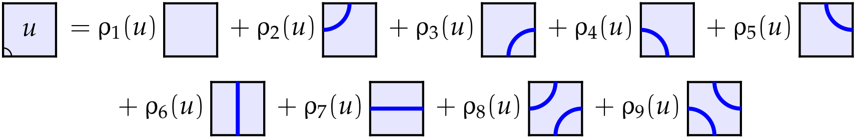

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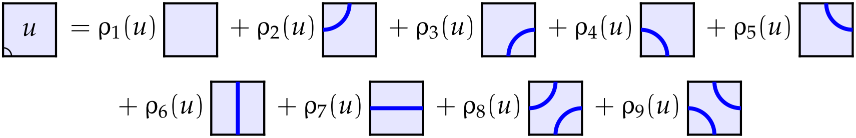

Thank you!