Singularities of dynamic response functions in the massless regime of the XXZ chain

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K. K. Kozlowski *"On the thermodynamic limit of form factor expansions of dynamical correlation functions in the massless regime of the XXZ spin-1/2 chain."* J. Math. Phys. "Ludwig Faddeev memorial volume" (2018).

K. K. Kozlowski *" On singularities of dynamic response functions in the massless regime of the XXZ chain."*, to appear.

K. K. Kozlowski *"Large-distance and long-time asymptotics of two-point dynamical correlation functions in the massless regime of the XXZ chain."*, to appear.

Recent Advances in Quantum Integrable Systems,
LAPTh, Annecy-Le-Vieux

Outline

The XXZ chain

- The XXZ spin-1/2 chain on $\mathfrak{h}_{XXZ} = \otimes_{n=1}^{L}\mathbb{C}^2$, σ^{α} Pauli matrices

$$\mathrm{H} = \sum_{n=1}^{L}\Big\{\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z - h\sigma_n^z\Big\} \quad , \quad \sigma_{n+L}^{\alpha} \equiv \sigma_n^{\alpha}$$

Local operators : $\sigma_n^{\alpha} = \underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{n-1\,\text{times}}\otimes\sigma^{\alpha}\otimes\underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{L-n\,\text{times}}$

- Model solvable by Bethe Ansatz **'31** (Bethe), **'58** (Orbach): $\mathrm{H}\cdot\Upsilon = \widehat{\mathcal{E}}_{\Upsilon}\cdot\Upsilon$

✔ Ground state, Completeness, Norms, Scalar products, Matrix elements of local operators ...

Yang, Yang, Gaudin, McCoy, Wu, Korepin, Slavnov, Kitanine, Maillet, Terras, Mukhin, Tarasov, Varchenko

✔ Spectrum above ground state $\lim\limits_{L\to+\infty}\{\widehat{\mathcal{E}}_{\Upsilon} - \widehat{\mathcal{E}}_{\Omega}\}$

Hulten, Yang, Yang, Des Cloiseau, Gaudin, Ishumira, Shiba, Faddeev, Takhtadjan, Dorlas, Samsonov, Gusev, K.

✔ Large understanding of the static correlators

Izergin, Korepin, Jimbo, Miwa, Miki, Nakayashiki, Kitanine, Maillet, Slavnov, Terras

Dynamic response functions in the XXZ chain

⊛ Time evolution of operators

$$\sigma_m^{\gamma}(t) = \mathrm{e}^{\mathrm{i}Ht} \cdot \sigma_m^{\gamma} \cdot \mathrm{e}^{-\mathrm{i}tH}$$

⊛ Connected dynamical two-point function at zero temperature

$$\Big\langle \big(\sigma_1^{\gamma}(t)\big)^{\dagger} \sigma_{m+1}^{\gamma} \Big\rangle_{\mathrm{c}} = \lim_{L\to+\infty} \Big\{ \big(\Omega, \big(\sigma_1^{\gamma}(t)\big)^{\dagger} \sigma_{m+1}^{\gamma} \Omega\big) - \big|\big(\Omega, \sigma_1^{\gamma}\Omega\big)\big|^2 \Big\}$$

♦ Experiments ⇝ Dynamic Response Functions

$$\mathscr{S}^{(\gamma)}(k,\omega) = \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} \Big\langle \big(\sigma_1^{\gamma}(t)\big)^{\dagger} \sigma_{m+1}^{\gamma} \Big\rangle_{\mathrm{c}} \cdot \mathrm{e}^{\mathrm{i}(\omega t - km)} \frac{\mathrm{d}t}{(2\pi)^2}$$

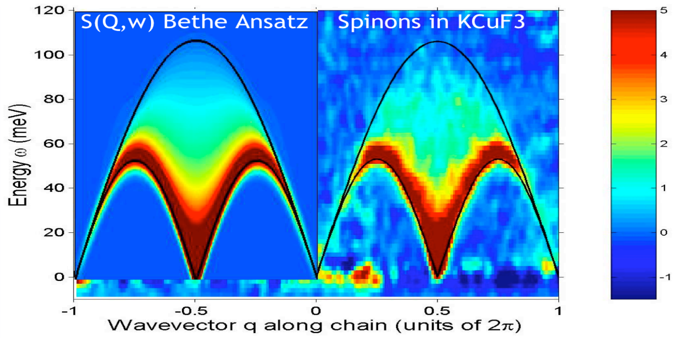
♦ Limited understanding of dynamical correlators at $\Delta \neq 0$

’95 Jimbo, Miwa , **’04** Kitanine, Maillet, Slavnov, Terras : Series representation.

⇝ **Not** in a manageable form to extract physically pertinent information

Expermiental measurements on $\mathrm{KCuF_3}$

- Bragg spectroscopy.
- $\mathrm{KCuF_3}$ $\rightsquigarrow$ XXX chain at $h = 0$.



'05 (Caux, Maillet, Hagemans) **'95** (Tennant, Cowley, Nagler, Tsvelik)

Observations for $\mathscr{S}^{(z)}(k,\omega)$

- Dominant intensity delimited by viking helmet curves
- Large intensity concentrated on lower curves
- Decrease of intensity when approaching top curve

State of the art for the response functions

♦ **Non-linear luttinger liquid phenomenology**

Mahan, Noziére, De Dominicus, Glazman, Kamenev, Khodas, Pustilnik, Imambekov, Cheianov, Affleck, Pereira, White, Caux, Shashi

$$\mathscr{S}^{(z)}(k,\omega) \simeq \mathscr{A}(k)\cdot \Xi\left(\omega-\varepsilon_h(k)\right)\cdot\left[\omega-\varepsilon_h(k)\right]^{\vartheta}$$

- Universal structure of the amplitudes

$$\mathscr{A}(k) = \frac{(2\pi)^2\,\Gamma^{-1}(\mu_R+\mu_L)}{[v(k)+v_F]^{\mu_R}[v_F-v(k)]^{\mu_L}}\cdot\left|\mathcal{F}^{(z)}(k)\right|^2 \quad , \quad \vartheta = \mu_R+\mu_L-1$$

- Bethe Ansatz spectrum $\rightsquigarrow$ closed expressions for $\mu_{R,L}$;
- $\left|\mathcal{F}^{(z)}(k)\right|^2 \rightsquigarrow$ volume-renormalised form factor;

♦ **Integrability based investigation**

⊛ $\mathscr{S}^{(z)}(k,\omega) = \sum_{n\geq 0}\mathcal{S}_{2n}(k,\omega)$ for XXX $h=0$ & (k,ω) behaviour from $n=1,2$
Jimbo, Miwa, Bougourzi, Couture, Kacir, Karbach, Müller, Mütter, Caux, Konno, Sorrel, Weston

⊛ Numerics & Bethe Ansatz
Biegel, Karbach, Müller, Sato, Shiroishi, Takahashi, Caux, Hagemans, Maillet,...

⊛ Formal saddle-point approach to form factor series in NLSM
Kitanine, K., Maillet, Slavnov, Terras

The long-distance and large-time asymptotics

$\hookrightarrow$ Massless regime : $-1 < \Delta \leq 1, \quad |h| < h_{\mathrm{c}}.$

- Universality $\rightsquigarrow$ asymptotics of $\Big\langle \big(\sigma_1^{\gamma}(t)\big)^{\dagger} \sigma_{m+1}^{\gamma} \Big\rangle_{\mathrm{c}}$
  - Exact results:

    $\Delta = 0$: **'82** (McCoy, Perk, Shrock), **'84** (Schrock, Müller), **'85** (McCoy, Tang)
  - Predictions:
    - **'02** (Lukyanov, Terras) $h = 0$, $-1 < \Delta < 1$
    - **'09** (Affleck, Pereira, White) $h_{\mathrm{c}} > |h|$, $-1 < \Delta < 1$
    - **'12** (Imambekov, Schmidt, Glazman) $h_{\mathrm{c}} > |h|$, $-1 < \Delta < 1$

The open problems

Questions

- Can one carry out an *ab inicio*, exact, calculation of $\left\langle \left(\sigma_1^{\gamma}(t)\right)^{\dagger} \sigma_{m+1}^{\gamma} \right\rangle_{c}$, $\mathscr{S}^{(\gamma)}(k,\omega)$?
- Can one access to the predicted universal features for an integrable model?
- Is the non-linear Luttinger picture complete?
- Can one bring the description of this universality to a singularity theory?

Thermodynamic limit of the spectrum of the XXZ chain

⊛ Long history of the $L \rightarrow +\infty$ analysis of the spectrum
Hulten, Yang Yang, Des Cloiseau, Gaudin, Ishumira, Shiba, Faddeev, Takhtadjan, K.

$$\widehat{\mathcal{E}}_{\Upsilon\setminus\Omega} = \mathcal{E}(\mathfrak{R}_{\Upsilon}) + \mathrm{O}(L^{-1})$$

⊛ Relative excitation momentum-energy: $\mathcal{P}(\mathfrak{R}) - \frac{1}{\mathrm{v}} \cdot \mathcal{E}(\mathfrak{R}) = \mathscr{U}(\mathfrak{Y}, \mathrm{v}) + \mathrm{O}(\delta) \qquad \mathrm{v} = \frac{m}{t}$

$$\mathscr{U}(\mathfrak{Y}, \mathrm{v}) = \sum_{r \in \mathfrak{N}} \sum_{a=1}^{n_r} \Big\{ p_r\big(\nu_a^{(r)}\big) - \frac{1}{\mathrm{v}} \varepsilon_r\big(\nu_a^{(r)}\big) \Big\} - \sum_{a=1}^{n_h} \Big\{ p_r(\mu_a) - \frac{1}{\mathrm{v}} \varepsilon_r(\mu_a) \Big\} + \sum_{\upsilon=\pm} \upsilon \ell_{\upsilon} p_F$$

⊛ macroscopic variables of the massive excitations

$$\mathfrak{Y} = \Big\{ \{\mu_a\}_1^{n_h} \,;\, \Big\{ \{\nu_a^{(r)}\}_{a=1}^{n_r} \Big\}_{r \in \mathfrak{N}} \,;\, \{\ell_{\upsilon}\} \Big\} \qquad \text{with} \quad \mathfrak{N} = \mathfrak{N}_{\mathrm{st}} \cup \{1\}$$

The form factor expansion of dynamic correlation functions

- Form factor expansion

$$\left(\Omega, \left(\sigma_1^{\gamma}(t)\right)^{\dagger} \sigma_{m+1}^{\gamma} \Omega\right) = \sum_{\Upsilon} \exp\left\{\mathrm{i} m \widehat{\mathcal{P}}_{\Upsilon\setminus\Omega} - \mathrm{i} t \widehat{\mathcal{E}}_{\Upsilon\setminus\Omega}\right\} \cdot \left|\left(\Upsilon, \sigma_1^{\gamma}\Omega\right)\right|^2$$

- ['17 K.] Thermodynamic limit of the form factor expansion in massless regime

$$\left\langle\left(\sigma_1^{\gamma}(t)\right)^{\dagger} \sigma_{m+1}^{\gamma}\right\rangle = (-1)^{m s_\gamma} \sum_{\boldsymbol{n}\in\mathfrak{S}} \prod_{r\in\mathfrak{N}} \left\{ \int_{\mathscr{C}_{r,\delta}^{n_r}} \frac{\mathrm{d}^{n_r}\nu^{(r)}}{n_r!(2\pi)^{n_r}} \right\} \cdot \int_{\mathscr{C}_{h,\delta}^{n_h}} \frac{\mathrm{d}^{n_h}\mu}{n_h!(2\pi)^{n_h}} \cdot \frac{\mathcal{F}^{(\gamma)}(\mathfrak{Y}) \cdot \mathrm{e}^{\mathrm{i} m \mathscr{U}(\mathfrak{Y}, \mathrm{v})}}{\prod_{\upsilon=\pm} [-\mathrm{i} m_\upsilon]^{\vartheta_\upsilon^2(\mathfrak{Y})}} \Big(1 + \mathfrak{r}(\mathfrak{Y}, m_\upsilon, \delta)\Big) .$$

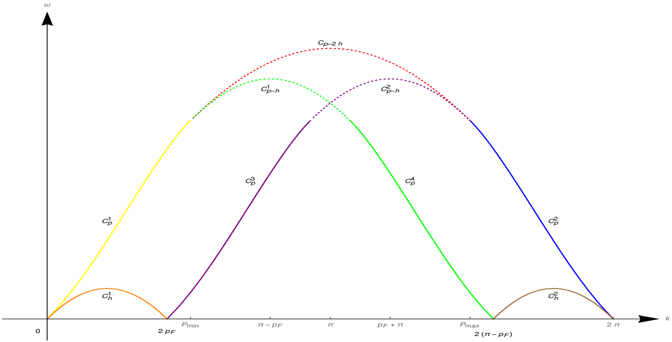
  - Left right moving combinations $m_\upsilon = \upsilon m - \mathrm{v}_F t$, s_γ operator spin
  - remainder $\mathfrak{r}(\mathfrak{Y}, m_\upsilon, \delta) = \mathrm{O}\Big(\delta \ln \delta + \sum_{\upsilon=\pm} \Big\{\delta^2 |m_\upsilon| + \delta |\ln|m_\upsilon|| + \mathrm{e}^{-|m_\upsilon|\delta}\Big\}\Big)$;
  - $\mathfrak{S} = \Big\{ \boldsymbol{n} = \big(n_h, \{n_r\}_{r\in\mathfrak{N}}, \ell_\upsilon\big) \ : \ n_h = \sum_{\upsilon\in\{\pm\}} \ell_\upsilon + \sum_{r\in\mathfrak{N}} r n_r \Big\}$

- First complete expression for a form factor expansion in a massless model.
- Non-perturbative finite-L resummation of low-energy modes $\rightsquigarrow$ constructive IR renormalisation.

The two-particle-hole excitation thresholds

- Form factor series representation for DRF

$$\mathscr{S}^{(\gamma)}(k,\omega) = \sum_{\boldsymbol{n}\in\mathfrak{S}} \mathscr{S}^{(\gamma)}_{\boldsymbol{n}}(k,\omega)$$



The vicinity of a hole threshold

(k,ω) configuration close to the hole excitation line

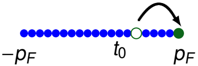
$$(\mathcal{P}_0,\mathcal{E}_0) = (p_F - t_0\,, -\mathfrak{e}_1(t_0)) \qquad \text{with} \qquad t_0 \in \,]-p_F\,;p_F\,[\;.$$

- The *hole* threshold

$$\mathscr{S}_h^{(z)}\big(\mathcal{P}_0,\mathcal{E}_0+\delta\omega\big) = \frac{(2\pi)^2\cdot\Xi\,(\delta\omega)\cdot[\delta\omega]^{\Delta^{(h)}}}{[\mathfrak{v}(t_0)+\mathrm{v}_F]^{\delta_+^{(h)}}\,[\mathrm{v}_F-\mathfrak{v}(t_0)]^{\delta_-^{(h)}}}\cdot\frac{\big|\mathcal{F}_h^{(z)}(t_0)\big|^2}{\Gamma(\delta_+^{(h)}+\delta_-^{(h)})}\Big(1+\mathrm{O}\big([\delta\omega]^{1-0^+}\big)\Big)+\mathscr{S}_{h;\mathrm{reg}}^{(zz)}(\delta\omega)\,.$$

- $\mathfrak{v}(t_0)=\mathfrak{e}_1'(t_0)$: velocity of the hole at t_0 $\qquad$ v_F: velocity excitations on Fremi boundary.
- Edge exponent: $\Delta^{(h)} = \delta_+^{(h)} + \delta_-^{(h)} - 1$
- $\delta_\pm^{(h)}$: microscopic shift of right(+)/left(-) Fermi boundary due to excitation.

$$\big|\mathcal{F}_h^{(z)}(t_0)\big|^2 = \lim_{L\to+\infty}\left\{\Big(\frac{L}{2\pi}\Big)^{\delta_+^{(h)}+\delta_-^{(h)}+1}\frac{\big|\big(\Omega,\sigma_1^z\Upsilon_{t_0}\big)\big|^2}{\|\Omega\|^2\cdot\|\Upsilon_{t_0}\|^2}\right\}$$



- ★ ground state Ω
- ★ excitation Υ_{t_0} $\left\{\begin{array}{lcl}\mathcal{E}_0 & = & -\mathfrak{e}_1(t_0)\\ \mathcal{P}_0 & = & p_F - t_0\end{array}\right.$

Multi particle-hole thresholds

- Particle and holes may have equal velocities $\mathfrak{v}(k)=\mathfrak{v}(t(k))$ $t :]\,K_m\,;K_M\,[\to]-p_F\,;p_F\,[$
  (k,ω) configuration close to an *equal*-velocity particle-hole excitations

$$(\mathcal{P}_0,\mathcal{E}_0)=\Big(n_p k_0-n_h t(k_0)+(n_h-n_p)p_F\,,\,n_p\mathfrak{e}_1(k_0)-n_h\mathfrak{e}_1(t(k_0))\Big) \qquad \text{with} \qquad k_0\in\,]\,K_m\,;K_M\,[\ .$$

- The *particle* threshold

$$\mathscr{S}^{(zz)}_{ph}\big(\mathcal{P}_0,\mathcal{E}_0+\delta\omega\big)=\frac{\big|\mathcal{F}^{(z)}_p(k_0)\big|^2}{\sqrt{n_p-n_h t'(k_0)}}\cdot\frac{(2\pi)^{\frac{n_p+n_h-1}{2}}G(1+n_p)G(1+n_h)}{\Big(\sqrt{-\mathfrak{v}'(k_0)}\Big)^{n_p^2-1}\Big(\sqrt{\mathfrak{v}'(t(k_0))}\Big)^{n_h^2}}\,\frac{[\delta\omega]^{\Delta^{(ph)}}\,\Gamma(-\Delta^{(p)})}{[\mathfrak{v}(k_0)+\mathrm{v}_F]^{\delta_+^{(ph)}}\,[\mathrm{v}_F-\mathfrak{v}(k_0)]^{\delta_-^{(ph)}}}$$

$$\times\Big\{\frac{\Xi(\delta\omega)}{\pi}\sin\Big[\pi\big(\delta_+^{(ph)}+\delta_-^{(ph)}\big)\Big]+\frac{\Xi(-\delta\omega)}{\pi}\sin\Big[\tfrac{\pi}{2}\big(n_p^2+n_h^2-3\big)\Big]\Big\}\cdot\Big(1+\mathrm{O}\Big([\delta\omega]^{1-0^+}\Big)\Big)+\mathscr{S}^{(zz)}_{ph;\mathrm{reg}}(\delta\omega)$$

- Edge exponent: $\Delta^{(ph)}=\delta_+^{(ph)}+\delta_-^{(ph)}+\frac{1}{2}\big(n_p^2+n_h^2-3\big)$

The not-so-long time regime: $|\mathrm{v}| > \mathrm{v}_\infty$

♦ Regime $m, t \to \pm\infty$, $\frac{m}{t} = \mathrm{v}$

$$\Big\langle \big(\sigma_1^\gamma(t)\big)^\dagger \sigma_{m+1}^\gamma \Big\rangle_{\mathrm{c}} = (-1)^{m s_\gamma} \sum_{\ell \in \mathbb{Z}} \mathcal{F}_\ell^{(\gamma)} \cdot \frac{\mathrm{e}^{2\mathrm{i} m \ell p_F}}{\prod_{\upsilon=\pm} [-\mathrm{i} m_\upsilon]^{\vartheta^2_{\upsilon;\ell}}} \cdot \Bigg\{ 1 + \mathrm{O}\Bigg(\sum_{\upsilon=\pm} \bigg| \frac{\ln |m_\upsilon|}{|m_\upsilon|} \bigg| + \mathrm{e}^{-cm} \Bigg) \Bigg\}$$

- $m_\upsilon = \upsilon m - \mathrm{v}_F t$;
- Critical exponents: $\vartheta_{\upsilon;\ell} = \ell \mathcal{Z} - \frac{\upsilon s_\gamma}{2\mathcal{Z}}$;
- Amplitudes $\mathcal{F}_\ell^{(\gamma)}$: properly normalised form factor squared GS to ℓ-Umklapp excitation;
- String excitations (bound states):
  - produce exponentially small contributions in m;
  - are **necessary** to carry out the steepest descent contour deformation
- Not related to space *vs.* time-like regimes ($\mathrm{v}_F > \mathrm{v}^{(\infty)}$);

The long time regime: $|\mathrm{v}| < \mathrm{v}_\infty$

$$\Big\langle \big(\sigma_1^\gamma(t)\big)^\dagger \sigma_{m+1}^\gamma \Big\rangle_{\mathrm{c}} = (-1)^{m s_\gamma} \sum_{\boldsymbol{n} \in \mathfrak{S}_{\mathrm{v}}} C_{\boldsymbol{n}} \cdot \mathcal{F}_{\boldsymbol{n}}^{(\gamma)} \cdot \prod_{\upsilon=\pm} \left\{ \frac{\mathrm{e}^{\mathrm{i}\upsilon \ell_\upsilon m p_F}}{[-\mathrm{i} m_\upsilon]^{\vartheta_{\upsilon;\boldsymbol{n}}^2}} \right\} \cdot \frac{\mathrm{e}^{\mathrm{i}\varphi_{\boldsymbol{n}}(m,t)}}{m^{\vartheta_{sp;\boldsymbol{n}}}} \cdot \left\{ 1 + \mathrm{O}\left(\sum_{\upsilon=\pm} \left| \frac{\ln|m_\upsilon|}{|m_\upsilon|} \right| + \frac{1}{\sqrt{|m|}} \right) \right\}$$

- Critical exponents:

$$\vartheta_{\upsilon;\boldsymbol{n}} = -\varkappa_{\mathrm{v}} n_0 \phi_1(\upsilon q, \omega_0) + \tfrac{1}{2} \mathrm{s}_\gamma \mathcal{Z} - \upsilon \ell_\upsilon - \sum_{r \in \mathfrak{N}} n_r \phi_1(\upsilon q, \omega_r) - \sum_{\upsilon'=\pm} \ell_{\upsilon'} \phi_1(\upsilon q, \upsilon' q) \qquad \text{Fermi endpoints}$$

$$\vartheta_{sp;\boldsymbol{n}} = \tfrac{1}{2}\Big({n_0}^2 + \sum_{r \in \mathfrak{N}} {n_r}^2 \Big) \qquad \text{Saddle-points}$$

- Oscillating phases:

$$\varphi_{\boldsymbol{n}}(m,t) = \varkappa_{\mathrm{v}} n_0 \big[m p_1(\omega_0) - t \varepsilon_1(\omega_0) \big] + \sum_{r \in \mathfrak{N}} n_r \big[m p_r(\omega_r) - t \varepsilon_r(\omega_r) \big]$$

- $\mathcal{F}_{\boldsymbol{n}}^{(\gamma)}$: normalised FF squared GS $\hookrightarrow$ Umklapp & multi-particle saddle point excitation;
- Universal pre-factor

$$C_{\boldsymbol{n}} = (-\varkappa_{\mathrm{v}})^{n_0} \cdot \prod_{r \in \mathfrak{N}} \big(\mathrm{sgn}[p_r'(\omega_r)] \big)^{n_r} \cdot \prod_{r \in \mathfrak{N} \cup \{0\}} \left\{ \frac{G(1+n_r)}{(2\pi)^{\frac{1}{2} n_r}} \left(\frac{\mathrm{i}}{p_r''(\omega_r) - \frac{1}{\mathrm{v}} \varepsilon_r''(\omega_r)} \right)^{\frac{1}{2}{n_r}^2} \right\};$$

- $\varkappa_{\mathrm{v}} = 1$ if $\mathrm{v}_F < |\mathrm{v}| < \mathrm{v}_\infty$ and $\varkappa_{\mathrm{v}} = -1$ if $|\mathrm{v}| < \mathrm{v}_F$;
- $\mathfrak{S}_{\mathrm{v}} = \Big\{ \boldsymbol{n} = \big(\{n_r\}_{r \in \mathfrak{N} \cup \{0\}}; \{\ell_\upsilon\}_{\upsilon=\pm} \big) \; : \; 0 = \varkappa_{\mathrm{v}} n_0 + \sum_{\upsilon \in \{\pm\}} \ell_\upsilon + \sum_{r \in \mathfrak{N}} r n_r \Big\}$

Conclusion and perspectives

Review of the results

- ✔ Rigorous analysis of auxiliary integrals;
- ✔ Confirms predictions of non-linear Luttinger-liquid structure (hole, particle, strings);
- ✔ Exhibits universal structure driven by equal velocity excitations;
- ✔ Phenomenological form of correlators for the Luttinger liquid universality class;
- ✔ Long-distance and large-time asymptotics;
- ✔ universality $\equiv$ singularity structure of form factors & *classical* saddle-point calculation;

Further developments

- ⊛ Solve the dynamical correlation functions at finite T program;

The form factor series

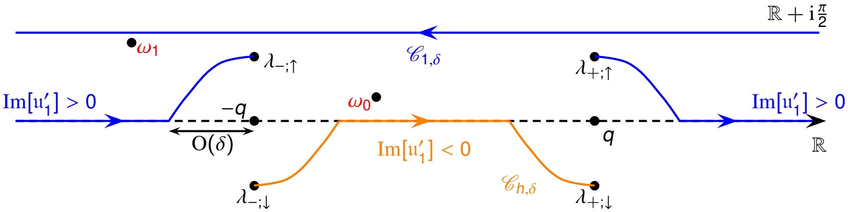
$$\Big\langle \big(\sigma_1^{\gamma}(t)\big)^{\dagger}\sigma_{m+1}^{\gamma}\Big\rangle_{\mathrm{c}} = (-1)^{ms_\gamma}\sum_{\boldsymbol{n}\in\mathfrak{S}}\prod_{r\in\mathfrak{R}}\Bigg\{\frac{1}{n_r!}\prod_{a=1}^{n_r}\int_{\mathscr{C}_{r,\delta}}\frac{\mathrm{d}\nu_a^{(r)}}{2\pi}\,\mathrm{e}^{\mathrm{i}m\mathfrak{u}_r(\nu_a^{(r)};\mathrm{v})}\Bigg\}\cdot\Bigg\{\frac{1}{n_h!}\prod_{a=1}^{n_h}\int_{\mathscr{C}_{h,\delta}}\frac{\mathrm{d}\mu_a}{2\pi}\,\mathrm{e}^{-\mathrm{i}m\mathfrak{u}_1(\mu_a;\mathrm{v})}\Bigg\}\cdot\prod_{\upsilon=\pm}\Bigg\{\frac{\mathrm{e}^{\mathrm{i}\upsilon\ell_\upsilon m p_F}}{[-\mathrm{i}m_\upsilon]^{\vartheta_\upsilon^2(\mathfrak{Y})}}\Bigg\}$$

$$\times\,\mathcal{F}^{(\gamma)}(\mathfrak{Y})\Big(1+\mathfrak{r}(\mathfrak{Y},m_\upsilon,\delta)\Big)\,.$$

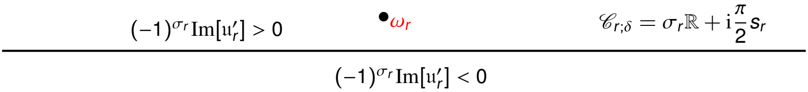
- $\mathfrak{u}_r(\lambda;\mathrm{v}) \;=\; p_r(\lambda)-\frac{1}{\mathrm{v}}\varepsilon_r(\lambda)$
- Saddle-points ω_r: $\mathfrak{u}_r'(\omega_r;\mathrm{v})$ away from integration contours

The integration curves

- Particle and hole contours



- r-string contour:



The integration curves

⇝ Deform contours $\begin{cases} \mathscr{C}_{r,\delta} & \hookrightarrow \quad \text{region where } \operatorname{Im}[\mathfrak{u}'_r(\lambda;\mathrm{v})] > 0 \\ \mathscr{C}_{h,\delta} & \hookrightarrow \quad \text{region where } \operatorname{Im}[\mathfrak{u}'_1(\lambda;\mathrm{v})] < 0 \end{cases}$

BUT

- Contours go to ∞ and $\mathfrak{u}'_r(\infty;\mathrm{v}) \;=\; 0$
- Form factors have jump discontinuities in

$$\nu^{(1)} \in \,]-q\,;-\infty\,[\,\pm\, \mathrm{i}\zeta\,, \qquad \nu^{(r)} \in \,]-q\,;-\infty\,[\,\pm\, \mathrm{i}\zeta\,\tfrac{r\pm 1}{2}\,, \quad r \in \mathfrak{N}_{\mathrm{st}}$$

- Form factors posses kinematical poles $\nu_a^{(r)} - \nu_b^{(p)} = \mathrm{i}\zeta k\,, \quad k \in \mathcal{S}^{(rp)}$

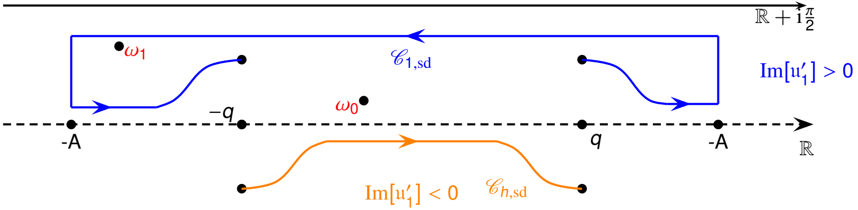
$$\operatorname{Res}\Big(\mathcal{F}^{(\gamma)}(\mathfrak{Y}_{\mathrm{ph}}) \prod_{r\in\mathfrak{N}_{\mathrm{st}}} \prod_{a=1}^{n_r} \prod_{k=2}^{r} \mathrm{d}\nu_{a,k}^{(r)}, \quad \nu_{a,k}^{(r)} = \nu_{a,k-1}^{(r)} - \mathrm{i}\zeta, k = 2,\dots,r \Big) = \prod_{r\in\mathfrak{N}_{\mathrm{st}}} \Big\{ (-\mathrm{i})^{(r-1)n_r} \Big\} \mathcal{F}^{(\gamma)}(\mathfrak{Y})$$

⊛ Rapidities :

$$\mathfrak{Y}_{\mathrm{ph}} \;=\; \Big\{ \{\mu_a\}_1^{n_h}\,;\, \Big\{ \{\nu_a^{(1)}\}_{a=1}^{n_1} \cup \Big\{ \Big\{ \{\nu_{a;k}^{(r)}\}_{k=1}^{r} \Big\}_{a=1}^{n_r} \Big\}_{r\in\mathfrak{N}_{\mathrm{st}}} \Big\}, \{\emptyset\}_{r\in\mathfrak{N}_{\mathrm{st}}}\,;\, \{\ell_\upsilon\} \Big\} \quad , \quad \mathfrak{Y} \;=\; \Big\{ \{\mu_a\}_1^{n_h}\,;\, \Big\{ \{\nu_a^{(r)}\}_{a=1}^{n_r} \Big\}_{r\in\mathfrak{N}}\,;\, \{\ell_\upsilon\} \Big\}$$

The deformed contours

- Particle and hole contours



The deformation of contours for the two-particle sector: $0 < \zeta < \pi/2$

- Total contribution: $$\mathcal{I}^{(2)}_{\mathrm{tot}} = \mathcal{I}_{ph} + \mathcal{I}_2 + \mathcal{I}_{0,1}$$

$$\mathcal{I}_2 = \frac{1}{2}\int_{\mathscr{C}^2_{1,\delta}} \frac{\mathrm{d}\nu_1 \mathrm{d}\nu_2}{(2\pi)^2} J(\nu_1,\nu_2)\,, \qquad \mathcal{I}_{0,1} = \int_{\mathbb{R}} \frac{\mathrm{d}\nu^{(2)}}{2\pi} J_{0,1}(\nu^{(2)}) \qquad \mathcal{I}_{ph} = \int_{\mathscr{C}_{1,\delta}} \frac{\mathrm{d}\nu}{2\pi} \int_{\mathscr{C}_{h,\delta}} \frac{\mathrm{d}\mu}{2\pi} J_{ph}(\nu,\mu)$$

$$\mathcal{I}_2 = \frac{1}{2}\int_{\mathscr{C}_{1,\delta}} \frac{\mathrm{d}\nu_2}{2\pi} \int_{\mathscr{C}_{1,\mathrm{sd}}} \frac{\mathrm{d}\nu_1}{2\pi} J(\nu_1,\nu_2) + \frac{1}{2}\int_{\mathscr{C}_{1,\delta}} \frac{\mathrm{d}\nu_2}{2\pi} \int_{\Gamma_{\mathrm{ext}}} \frac{\mathrm{d}\nu_1}{2\pi} J(\nu_1,\nu_2)$$

- 1st integral $\mathscr{C}_{1,\delta} \hookrightarrow \mathscr{C}_{1,\mathrm{sd}}$ without taking poles
- 2nd integral Γ_{ext} residues $\rightsquigarrow$ poles:

$$\nu_1 = \nu_2 + \mathrm{i}\zeta,\ \nu_2 \in J_A \qquad \text{and} \qquad \nu_1 = \nu_2 - \mathrm{i}\zeta,\ \nu_2 \in J_A + \mathrm{i}\frac{\pi}{2} \qquad J_A = \mathbb{R} \setminus [-A\,;A\,]$$

$$\mathcal{I}_2 = \frac{1}{2}\int_{\mathscr{C}_{1,\mathrm{sd}} \subset \mathscr{C}_{1,\mathrm{sd}}} \frac{\mathrm{d}\nu_1 \mathrm{d}\nu_2}{(2\pi)^2} J(\nu_1,\nu_2) - \frac{1}{2}\int_{J_A + \mathrm{i}\frac{1}{2}\zeta} \frac{\mathrm{d}\nu^{(2)}}{2\pi} J_{0,1}(\nu^{(2)}) - \frac{1}{2}\int_{J_A + \mathrm{i}\frac{1}{2}(\pi-\zeta)} \frac{\mathrm{d}\nu^{(2)}}{2\pi} J_{0,1}(\nu^{(2)})$$

The deformation of contours for the two-particle sector: $0 < \zeta < \pi/2$

♦ Final form

$$\mathcal{I}_{\mathrm{tot}}^{(2)} = \frac{1}{2}\int\limits_{\mathscr{C}_{1,\mathrm{sd}}\subset\mathscr{C}_{1,\mathrm{sd}}} \frac{\mathrm{d}\nu_1\mathrm{d}\nu_2}{(2\pi)^2} J(\nu_1,\nu_2) + \int\limits_{\mathscr{C}_{1,\mathrm{sd}}} \frac{\mathrm{d}\nu}{2\pi} \int\limits_{\mathscr{C}_{h,\mathrm{sd}}} \frac{\mathrm{d}\mu}{2\pi} J_{ph}(\nu,\mu) + \frac{1}{2}\sum_{\sigma\in\{\zeta,\pi-\zeta\}} \int\limits_{-A+\mathrm{i}\frac{1}{2}\sigma}^{-A+\mathrm{i}\frac{1}{2}\sigma} \frac{\mathrm{d}\nu^{(2)}}{2\pi} J_{0,1}(\nu^{(2)})$$

⇝ Direct to evaluate the integrals in $m \to +\infty$

⊛ Similar mechanism for 3-particle sector

CONJECTURE

Cancellation mechanism for contributions from ∞ extends to all particle sectors

The model integral

$$\mathcal{I}(\mathfrak{x}) = \prod_{r=1}^{\ell}\Big\{\int_{\mathscr{I}_r^{n_r}} \mathrm{d}\boldsymbol{p}^{(r)} \prod_{a<b}^{n_r}\big(p_a^{(r)}-p_b^{(r)}\big)^2\Big\}\cdot \mathscr{G}\big(\boldsymbol{p},\mathfrak{z}_+(\boldsymbol{p};\mathfrak{x}),\mathfrak{z}_-(\boldsymbol{p};\mathfrak{x})\big)\cdot\prod_{\upsilon=\pm}\Big\{\Xi\big(\mathfrak{z}_\upsilon(\boldsymbol{p};\mathfrak{x})\big)\cdot\big[\mathfrak{z}_\upsilon(\boldsymbol{p};\mathfrak{x})\big]^{\Delta_\upsilon(\boldsymbol{p})-1}\Big\}$$

- $\mathscr{G}\big(\boldsymbol{p},x,y\big) = \mathcal{G}(\boldsymbol{p}) + \mathrm{O}\big(x^\alpha + y^\alpha\big) \qquad \mathscr{G} = 0$ on $\partial\Big\{\prod_{r=1}^{\ell}\mathscr{I}_r^{n_r}\Big\}\times(\mathbb{R}^+)^2$
- Edge function
$$\mathfrak{z}_\upsilon(\boldsymbol{p};\mathfrak{x}) = \mathcal{E}_0 + \mathfrak{x} - \sum_{r=1}^{\ell}\sum_{a=1}^{n_r}\mathfrak{u}_r(p_a^{(r)}) + \upsilon\mathrm{v}\Big\{\mathcal{P}_0 - \sum_{r=1}^{\ell}\sum_{a=1}^{n_r}p_a^{(r)}\Big\}$$
- $|\mathfrak{u}_r'| > 0$ on $\mathscr{I}_r \qquad \pm\mathrm{v}\notin\mathfrak{u}_r'\big(\mathrm{Int}(\mathscr{I}_r)\big)$

$$\left\{\begin{array}{cccc} & \mathscr{I}_r & = & \mathscr{I}_r^{(\mathrm{in})}\sqcup\mathscr{I}_r^{(\mathrm{out})} \\ t_r\ : & \mathscr{I}_1 & \to & \mathscr{I}_r^{(\mathrm{in})}\end{array}\right. \quad \text{with} \quad \left\{\begin{array}{rcl}\mathfrak{u}_r'\big(\mathrm{Int}(\mathscr{I}_r^{(\mathrm{out})})\big)\cap\mathfrak{u}_1'\big(\mathrm{Int}(\mathscr{I}_1)\big) & = & \emptyset \\ \mathfrak{u}_1'(k) & = & \mathfrak{u}_r'\big(t_r(k)\big)\end{array}\right.$$

- Macroscopic momentum and energy on $\mathscr{I}_1$,

$$\mathcal{P}(k) = \sum_{r=1}^{\ell} n_r\, t_r(k) \quad \text{and} \quad \mathcal{E}(k) = \sum_{r=1}^{\ell} n_r\mathfrak{u}_r\big(t_r(k)\big) \qquad \mathcal{P}'(k)\neq 0 \quad \text{on} \quad \mathrm{Int}(\mathscr{I}_1)$$

Asymptotic behaviour model integral

Theorem '18 K.

a) `The regular case.`

⊛ If $\big(\mathcal{P}_0,\mathcal{E}_0\big) \notin \Big\{\big(\mathcal{P}(k),\mathcal{E}(k)\big) \;:\; k \in \mathscr{I}_1\Big\}$ and $\min\limits_{\substack{b\in\partial\mathscr{I}_1\\ \upsilon=\pm}}\Big|\mathcal{E}_0 - \mathcal{E}(b) + \upsilon \mathrm{v}(\mathcal{P}_0 - \mathcal{P}(b))\Big| > 0$

$\mathfrak{x} \mapsto \mathcal{I}(\mathfrak{x})$ is smooth at $\mathfrak{x} = 0$

b) `The singular case.`

⊛ If $\big(\mathcal{P}_0,\mathcal{E}_0\big) = \big(\mathcal{P}(k_0),\mathcal{E}(k_0)\big)\,,\; k_0 \in \mathrm{Int}(\mathscr{I}_1)$ and $\vartheta \notin \mathbb{N}\,,$

$$\mathcal{I}(\mathfrak{x}) = C\cdot|\mathfrak{x}|^{\vartheta}\cdot\bigg\{\Xi(\mathfrak{x})\frac{\sin[\pi\nu_+]}{\pi} + \Xi(-\mathfrak{x})\frac{\sin[\pi\nu_-]}{\pi}\bigg\} + \mathfrak{r}(\mathfrak{x}) + \mathrm{O}\Big(|\mathfrak{x}|^{\vartheta+\alpha}\Big)$$

$$C = \mathcal{G}\big(\boldsymbol{t}(k_0)\big)\cdot\frac{\Gamma(\delta_+)\Gamma(\delta_-)\Gamma(-\vartheta)\cdot(2\mathrm{v})^{\delta_++\delta_--1}}{\sqrt{|\mathcal{P}'(k_0)|}\cdot\prod\limits_{\upsilon=\pm}\big|\mathrm{v}-\upsilon\mathfrak{u}_1'(k_0)\big|^{\delta_\upsilon}}\cdot\prod_{r=1}^{\ell}\left\{\frac{G(2+n_r)\cdot(2\pi)^{\frac{n_r-\delta_{r,1}}{2}}}{\big|\mathfrak{u}_r''(t_r(k_0))\big|^{\frac{1}{2}(n_r^2-\delta_{r,1})}}\right\} \qquad \text{with} \quad \delta_\upsilon = \Delta_\upsilon(\boldsymbol{t}(k_0))$$

$$\vartheta = \frac{1}{2}\sum_{r=1}^{\ell}n_r^2 - \frac{3}{2} + \delta_+ + \delta_-\,, \qquad \nu_\pm = \frac{1}{2}\sum_{\substack{r=1:\\ \varepsilon_r=\mp1}}^{\ell}n_r^2 - \frac{1\mp\mathfrak{s}}{4} + \sum_{\substack{\upsilon=\pm:\\ \pm[\mathrm{v}-\upsilon\mathfrak{u}_1'(k_0)]>0}}\delta_\upsilon \quad \text{and} \quad \begin{cases} \mathfrak{s} = -\mathrm{sgn}\Big[\frac{\mathcal{P}'(k_0)}{\mathfrak{u}_1''(k_0)}\Big] \\ \varepsilon_r = -\mathrm{sgn}\Big[\mathfrak{u}_r''(t_r(k_0))\Big] \end{cases}$$

$$\boldsymbol{t}(k_0) = \Big(\boldsymbol{t}_1(k_0),\dots,\boldsymbol{t}_\ell(k_0)\Big)\in\mathbb{R}^{\overline{\boldsymbol{n}}_\ell} \qquad \text{with} \qquad \boldsymbol{t}_r(k_0) = \Big(t_r(k_0),\dots,t_r(k_0)\Big)\in\mathbb{R}^{n_r}$$