# Four-point functions in the Fortuin-Kasteleyn cluster model

Jesper L. Jacobsen [1,2,3]

[1]'Ecole Normale Supérieure, Paris (Laboratoire de Physique Théorique)

[2]Sorbonne Université, Paris (Faculté des Sciences et Ingénierie)

[3]Commisariat à l'Energie Atomique, Saclay (Institut de Physique Théorique)

RAQIS'18, 13 September 2018

Collaborator: Hubert Saleur (CEA-Saclay)

Also mentions work with: Romain Couvreur (ENS), Youjin Deng (Hefai), Xiaojun Tan (Hefai), Romain Vasseur (Berkeley)

# Introduction

## General setup

- Statistical model initially defined on a 2D lattice (e.g. square)
- Local short-range interactions (e.g. nearest neighbour)
- Adjustable parameter $T$ (temperature / interaction strength)

# Introduction

## General setup

- Statistical model initially defined on a 2D lattice (e.g. square)
- Local short-range interactions (e.g. nearest neighbour)
- Adjustable parameter $T$ (temperature / interaction strength)

## Critical phenomena

- Take continuum limit (lattice mesh $\epsilon \to 0$)
- Scale invariance for some critical $T = T_c$
- Actually conformal invariance

# Introduction

## General setup

- Statistical model initially defined on a 2D lattice (e.g. square)
- Local short-range interactions (e.g. nearest neighbour)
- Adjustable parameter $T$ (temperature / interaction strength)

## Critical phenomena

- Take continuum limit (lattice mesh $\epsilon \to 0$)
- Scale invariance for some critical $T = T_{\mathrm{c}}$
- Actually conformal invariance

## Integrability

- No Yang-Baxter related integrability in this talk. However:
  - The model is integrable (XXZ chain).
  - Algebraic structures and form-factor-like expansions will appear.

# $Q$-state Potts model

## Definition

- Spin $\sigma_i = 1, 2, \ldots, Q$ defined on each vertex $i \in V$
- Interaction $\delta_{\sigma_i,\sigma_j}$ along each edge $(ij) \in E$
- Interaction strength $K = J/T$ (with coupling constant $J$)
- Partition function

$$Z = \sum_{\{\sigma\}} \prod_{(ij)\in E} \mathrm{e}^{K\delta_{\sigma_i,\sigma_j}}$$

- At $K_\mathrm{c}$ on the square lattice, related to integrable XXZ chain.

# Connection to random geometry

## Random geometry

- Non-local observables: curves and clusters
- Examples: Percolation, self-avoiding walk, Ising spin clusters
- At $T = T_c$: self-similarity and fractal structures

# Connection to random geometry

## Random geometry

- Non-local observables: curves and clusters
- Examples: Percolation, self-avoiding walk, Ising spin clusters
- At $T = T_c$: self-similarity and fractal structures

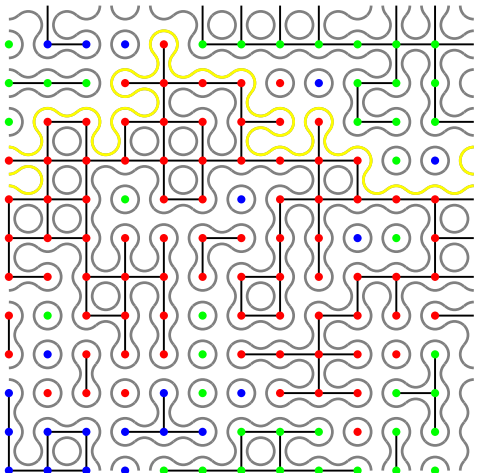
## Fortuin-Kasteleyn trick

- Recall $Z = \sum_{\{\sigma\}} \prod_{(ij)\in E} \mathrm{e}^{K\delta_{\sigma_i,\sigma_j}}$
- Set $\mathrm{e}^{K\delta_{\sigma_i,\sigma_j}} = 1 + (\mathrm{e}^K - 1)\delta_{\sigma_i,\sigma_j}$
- Expand $\prod_{(ij)\in E}$ and perform $\sum_{\{\sigma\}}$ to obtain

$$Z = \sum_{A\subseteq E} Q^{k(A)} (\mathrm{e}^K - 1)^{|A|}$$

- $k(A)$ is the number of connected components (clusters) in $A \subseteq E$

# Illustration



- Colours: Potts spins
- Black: FK clusters, weight $Q$
- Gray: Surrounding loops (on medial graph), weight $n = \sqrt{Q}$

## Special cases at $\mathrm{e}^{K_c} = 1 + \sqrt{Q}$

- $Q \to 1$: Critical percolation (at $p_c = \frac{1}{2}$)
- $Q \to 0$: Uniform spanning trees (alias dense polymers)

# Correlation functions: standard CFT results

## Two and three-point functions

- Functional form fixed by global conformal invariance [Polyakov 1970]
- Denoting $r_{ij} \equiv |r_i - r_j|$:

$$\langle \varphi_1(r_1)\varphi_2(r_2)\rangle = \frac{C_{12}\delta_{\Delta_1,\Delta_2}}{r_{12}^{2\Delta_1}}$$

$$\langle \varphi_1(r_1)\varphi_2(r_2)\varphi_3(r_3)\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3}\, r_{23}^{\Delta_2+\Delta_3-\Delta_1}\, r_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

# Correlation functions: standard CFT results

## Two and three-point functions

- Functional form fixed by global conformal invariance [Polyakov 1970]
- Denoting $r_{ij} \equiv |r_i - r_j|$:

$$\langle \varphi_1(r_1)\varphi_2(r_2)\rangle = \frac{C_{12}\delta_{\Delta_1,\Delta_2}}{r_{12}^{2\Delta_1}}$$

$$\langle \varphi_1(r_1)\varphi_2(r_2)\varphi_3(r_3)\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3}\, r_{23}^{\Delta_2+\Delta_3-\Delta_1}\, r_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

## Four-point functions

- Arbitrary dependence on conformal invariants (anharmonic ratios):

$$\langle \varphi_1(r_1)\varphi_2(r_2)\varphi_3(r_3)\varphi_4(r_4)\rangle = f\left(\frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{23}r_{14}}\right) \prod_{i<j}^{4} r_{ij}^{(\frac{1}{3}\sum_k \Delta_k)-\Delta_i-\Delta_j}$$

# Four-point functions in two-dimensional CFT

Radial quantisation and study of conserved current (stress tensor) leads to the Virasoro algebra [Belavin-Polyakov-Zamolodchikov 1984]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$

where $c$ is the central charge, $L_n = \frac{1}{2\pi i} \oint \mathrm{d}z\, z^{n+1} T(z)$ are mode operators, and $z = x + iy$, $\bar{z} = x - iy$ are complex coordinates.

# Four-point functions in two-dimensional CFT

Radial quantisation and study of conserved current (stress tensor) leads to the Virasoro algebra [Belavin-Polyakov-Zamolodchikov 1984]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$

where $c$ is the central charge, $L_n = \frac{1}{2\pi i}\oint \mathrm{d}z\, z^{n+1}\, T(z)$ are mode operators, and $z = x + iy$, $\bar{z} = x - iy$ are complex coordinates.

## Highest weight representations

- Primary operators $\varphi$ satisfy $L_0\varphi = \Delta\varphi$ and $L_n\varphi = 0$ for $n > 0$.
- Descendents: Linear combinations of $L_{-n_1} \cdots L_{-n_k}$ with $\sum_i n_i = N$.
- Correlation function involving a descendent: Linked to that of its corresponding primary, through action by differential operator.

# The usual situation — and two major challenges

## Diagonalisability of $L_0$

- "Usually" the dilatation operator $\mathcal{D} \equiv L_0 + \bar{L}_0$ is diagonalisable.
  - Critical exponents $\Delta + \bar{\Delta}$ are its eigenvalues.
  - Fields $\varphi$ normalised by setting $C_{12} = 1$ in two-point functions.
- But if $Q \in 4\cos^2(\pi\mathbb{Q})$ this may be untrue, and $\mathcal{D}$ has Jordan cells!
  - Corr. functions have power law *and* logarithmic dependence on $r_{ij}$.

# The usual situation — and two major challenges

## Diagonalisability of $L_0$

- "Usually" the dilatation operator $\mathcal{D} \equiv L_0 + \bar{L}_0$ is diagonalisable.
  - Critical exponents $\Delta + \bar{\Delta}$ are its eigenvalues.
  - Fields $\varphi$ normalised by setting $C_{12} = 1$ in two-point functions.
- But if $Q \in 4\cos^2(\pi\mathbb{Q})$ this may be untrue, and $\mathcal{D}$ has Jordan cells!
  - Corr. functions have power law *and* logarithmic dependence on $r_{ij}$.

## Computability of four-point functions

- For these $Q$, "usually" the highest-weight representations have null vectors $\chi$: a descendent which is itself primary.
  - Since $\chi$ has norm zero, one takes the quotient $\chi = 0$.
  - Leads to diff. eq. for four-point functions, solvable if $N$ small.
  - In this way one can obtain e.g. Cardy's crossing formulae and Schramm's left-passage probability.
- But connectivity-related operators in FK cluster model do not have null vectors! So their correlators are difficult to obtain.

# Logarithms and non-unitarity [Cardy 1999]

## Standard unitary CFT

- Expand local density $\Phi(r)$ on sum of scaling operators $\varphi(r)$

$$\langle \Phi(r)\Phi(0)\rangle \sim \sum_{ij} \frac{A_{ij}}{r^{\Delta_i+\Delta_j}}$$

- $A_{ij} \propto \delta_{ij}$ by conformal symmetry [Polyakov 1970]
- $A_{ii} \geq 0$ by reflection positivity (unitarity)
- Hence only power laws appear

# Logarithms and non-unitarity [Cardy 1999]

## Standard unitary CFT

- Expand local density $\Phi(r)$ on sum of scaling operators $\varphi(r)$

$$\langle \Phi(r)\Phi(0)\rangle \sim \sum_{ij} \frac{A_{ij}}{r^{\Delta_i+\Delta_j}}$$

- $A_{ij} \propto \delta_{ij}$ by conformal symmetry [Polyakov 1970]
- $A_{ii} \geq 0$ by reflection positivity (unitarity)
- Hence only power laws appear

## The non-unitary case

- Cancellations and signs may occur
- Suppose $A_{ii} \sim -A_{jj} \to \infty$ with $A_{ii}(\Delta_i - \Delta_j)$ finite
- Then leading term is $r^{-2\Delta_i} \log r$

# Symmetry classification of operators in FK model

- $N$-spin operators irreducible under $S_Q$ and $S_N$ symmetries

# Symmetry classification of operators in FK model

- $N$-spin operators irreducible under $S_Q$ and $S_N$ symmetries

## Operators acting on one spin

- Most general one-spin operator: $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i) = \sum_{a=1}^{Q} \mathcal{O}_a \delta_{a,\sigma_i}$

$$\underbrace{\delta_{a,\sigma_i}}_{\text{reducible}} = \underbrace{\frac{1}{Q}}_{\text{invariant}} + \underbrace{\left(\delta_{a,\sigma_i} - \frac{1}{Q}\right)}_{\varphi_a(\sigma_i)}$$

- Dimensions of representations: $(Q) = (1) \oplus (Q-1)$

## Operators acting symmetrically on two spins

- $Q \times Q$ matrices $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i, \sigma_j) = \sum_{a=1}^{Q} \sum_{b=1}^{Q} \mathcal{O}_{ab} \delta_{a,\sigma_i} \delta_{b,\sigma_j}$
- The $Q$ operators with $\sigma_i = \sigma_j$ decompose as before: $(1) \oplus (Q-1)$
- Other $\frac{Q(Q-1)}{2}$ operators with $\sigma_i \neq \sigma_j$: $(1) + (Q-1) + \left(\frac{Q(Q-3)}{2}\right)$

## Operators acting symmetrically on two spins

- $Q \times Q$ matrices $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i, \sigma_j) = \sum_{a=1}^{Q} \sum_{b=1}^{Q} \mathcal{O}_{ab} \delta_{a,\sigma_i} \delta_{b,\sigma_j}$
- The $Q$ operators with $\sigma_i = \sigma_j$ decompose as before: $(1) \oplus (Q-1)$
- Other $\frac{Q(Q-1)}{2}$ operators with $\sigma_i \neq \sigma_j$: $(1) + (Q-1) + \left(\frac{Q(Q-3)}{2}\right)$

## Case $\sigma_i \neq \sigma_j$ from representation theory

$$\varepsilon = \delta_{\sigma_i \neq \sigma_j} = 1 - \delta_{\sigma_i, \sigma_j}$$

$$\phi_a = \delta_{\sigma_i \neq \sigma_j} \left( \varphi_a(\sigma_i) + \varphi_a(\sigma_j) \right)$$

$$\psi_{ab} = \delta_{\sigma_i,a} \delta_{\sigma_j,b} + \delta_{\sigma_i,b} \delta_{\sigma_j,a} - \frac{1}{Q-2} \left( \phi_a + \phi_b \right) - \frac{2}{Q(Q-1)} \varepsilon$$

## Operators acting symmetrically on two spins

- $Q \times Q$ matrices $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i, \sigma_j) = \sum_{a=1}^{Q} \sum_{b=1}^{Q} \mathcal{O}_{ab} \delta_{a,\sigma_i} \delta_{b,\sigma_j}$
- The $Q$ operators with $\sigma_i = \sigma_j$ decompose as before: $(1) \oplus (Q-1)$
- Other $\frac{Q(Q-1)}{2}$ operators with $\sigma_i \neq \sigma_j$: $(1) + (Q-1) + \left(\frac{Q(Q-3)}{2}\right)$

## Case $\sigma_i \neq \sigma_j$ from representation theory

$$\varepsilon = \delta_{\sigma_i \neq \sigma_j} = 1 - \delta_{\sigma_i, \sigma_j}$$

$$\phi_a = \delta_{\sigma_i \neq \sigma_j} \left( \varphi_a(\sigma_i) + \varphi_a(\sigma_j) \right)$$

$$\psi_{ab} = \delta_{\sigma_i,a} \delta_{\sigma_j,b} + \delta_{\sigma_i,b} \delta_{\sigma_j,a} - \frac{1}{Q-2} (\phi_a + \phi_b) - \frac{2}{Q(Q-1)} \varepsilon$$

- Scalar $\varepsilon$ (energy)
- Vector $\varphi_a$ (order parameter)
- Tensor $\psi_{ab}$ (two-cluster operator)
- Pole at $Q = 1$ means that $\varepsilon$ and $\psi_{ab}$ must mix into a Jordan cell

# Geometrical interpretation in terms of FK clusters

## One-spin results

$$\langle \varphi_a(r)\varphi_b(0)\rangle = \frac{1}{Q}\left(\delta_{a,b} - \frac{1}{Q}\right)\mathbb{P}\left(\right) .$$

# Geometrical interpretation in terms of FK clusters

## One-spin results

$$\langle \varphi_a(r)\varphi_b(0)\rangle = \frac{1}{Q}\left(\delta_{a,b} - \frac{1}{Q}\right)\mathbb{P}\left(\right) .$$

- In general we do not know exactly (even in $d = 2$) the probability $\mathbb{P}\left(\right)$ that the two spins belong to the same FK cluster.
- But its large-distance asymptotics is predicted from CFT.

## Two-spin results

$$\langle \varepsilon(r)\varepsilon(0)\rangle = \left(\frac{Q-1}{Q}\right)^2 \left(\mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right) + \mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right)\right) + \frac{Q-1}{Q}\mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right),$$

$$\langle \phi_a(r)\phi_b(0)\rangle = \frac{Q-2}{Q^2}\left(\delta_{a,b} - \frac{1}{Q}\right)\left(\frac{Q-2}{Q}\mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right) + 2\mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right)\right),$$

$$\langle \psi_{ab}(r)\psi_{cd}(0)\rangle = \frac{2}{Q^2}\left(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{1}{Q-2}(\delta_{ac} + \delta_{bd} + \delta_{ad} + \delta_{bc}) + \frac{2}{(Q-2)(Q-1)}\right)\mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right).$$

# Percolation limit $Q \to 1$

## Avoiding the $Q \to 1$ divergence

- The "scalar" part of $\langle \psi_{ab}(r)\psi_{cd}(0)\rangle$ diverges
- Therefore $\Delta_\psi = \Delta_\varepsilon$ (in any dimension!)
- So we can cure the divergence by mixing the two operators:

$$\tilde{\psi}_{ab}(r) = \psi_{ab}(r) + \frac{2}{Q(Q-1)}\varepsilon(r).$$

- Thus emerges the Jordan cell which contains logarithmic correlators.

# Isolate the logarithm

Define:

$\mathbb{P}_2 = \mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \bullet\end{smallmatrix}\right)$, $\mathbb{P}_1 = \mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \bullet\end{smallmatrix}\right)$, $\mathbb{P}_0 = \mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet \\ \bullet & \bullet\end{smallmatrix}\right)$, and $\mathbb{P}_{\neq} \equiv \mathbb{P}(\sigma_i \neq \sigma_{i+1})$

**Construct an observable that behaves purely logarithmically**

$$F(r) \equiv \frac{\mathbb{P}_0(r) + \mathbb{P}_1(r) - \mathbb{P}_{\neq}^2}{\mathbb{P}_2(r)} \sim \underbrace{\left(2 \times \lim_{Q \to 1} \frac{\Delta_\psi - \Delta_\varepsilon}{Q-1}\right)}_{\text{universal}} \log r,$$

# Isolate the logarithm

Define:

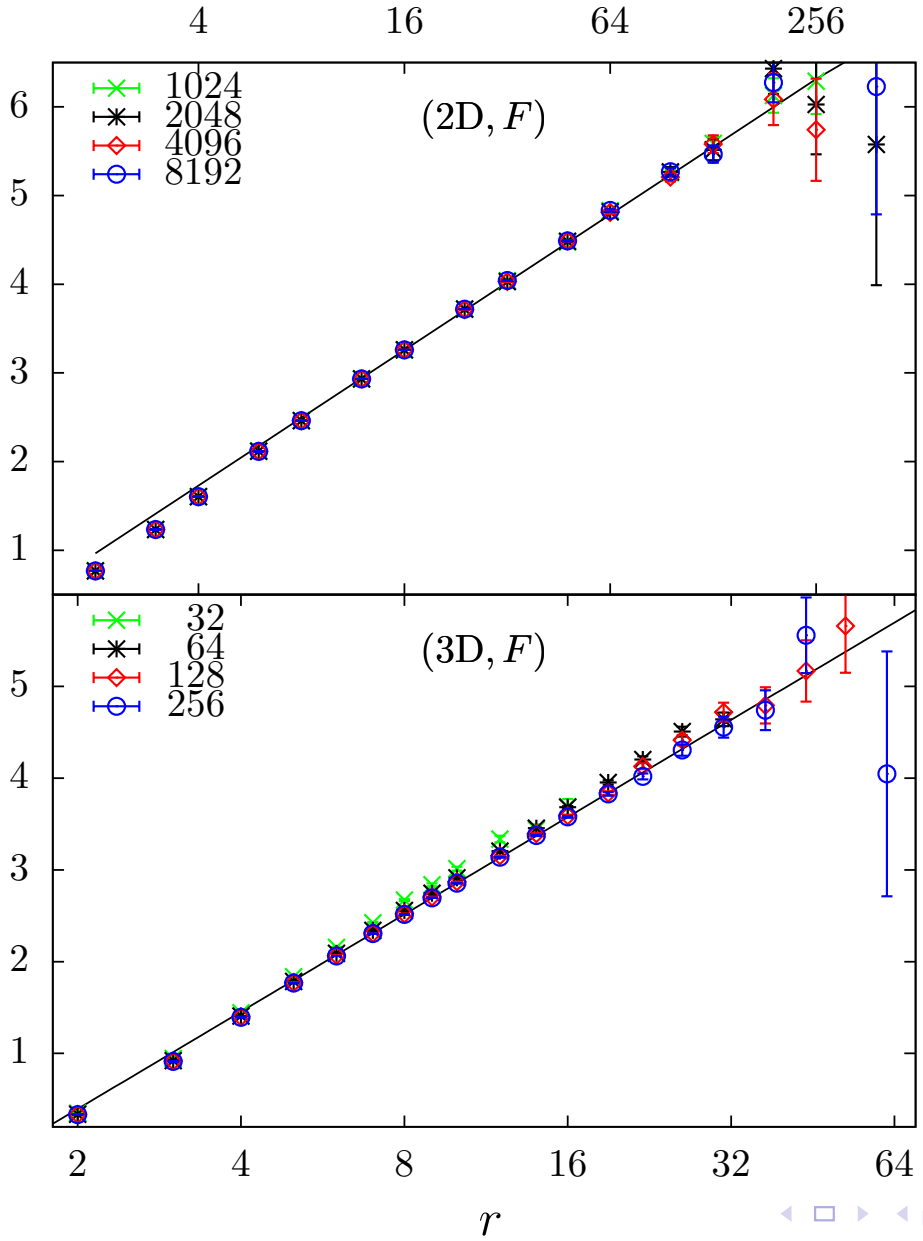
$\mathbb{P}_2 = \mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right)$, $\mathbb{P}_1 = \mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right)$, $\mathbb{P}_0 = \mathbb{P}\left(\begin{smallmatrix}\bullet & \bullet\\ \bullet & \bullet\end{smallmatrix}\right)$, and $\mathbb{P}_{\neq} \equiv \mathbb{P}(\sigma_i \neq \sigma_{i+1})$

**Construct an observable that behaves purely logarithmically**

$$F(r) \equiv \frac{\mathbb{P}_0(r) + \mathbb{P}_1(r) - \mathbb{P}_{\neq}^2}{\mathbb{P}_2(r)} \sim \underbrace{\left(2 \times \lim_{Q\to 1} \frac{\Delta_\psi - \Delta_\varepsilon}{Q-1}\right)}_{\text{universal}} \log r,$$

- In two dimensions, the universal pre-factor is $\frac{2\sqrt{3}}{\pi}$.

# Numerical check



# Back to four-point functions

## General setup

- 4 primaries $\Phi_i(z_i, \bar{z}_i)$ of scaling dimension $h_i + \bar{h}_i$ and spin $h_i - \bar{h}_i$
- Send 3 points to $0, 1, \infty$ and study dependence on $z = \frac{z_{12} z_{34}}{z_{13} z_{24}}$:

$$G(z, \bar{z}) \equiv \lim_{\Lambda \to \infty} \Lambda^{2h_3} \bar{\Lambda}^{2\bar{h}_3} \left\langle \Phi_1(z, \bar{z}) \Phi_2(0, 0) \Phi_3(\Lambda, \bar{\Lambda}) \Phi_4(1, 1) \right\rangle$$

# Back to four-point functions

## General setup

- 4 primaries $\Phi_i(z_i, \bar{z}_i)$ of scaling dimension $h_i + \bar{h}_i$ and spin $h_i - \bar{h}_i$
- Send 3 points to $0, 1, \infty$ and study dependence on $z = \frac{z_{12} z_{34}}{z_{13} z_{24}}$:

$$G(z, \bar{z}) \equiv \lim_{\Lambda \to \infty} \Lambda^{2h_3} \bar{\Lambda}^{2\bar{h}_3} \left\langle \Phi_1(z, \bar{z}) \Phi_2(0,0) \Phi_3(\Lambda, \bar{\Lambda}) \Phi_4(1,1) \right\rangle$$

## Conformal bootstrap approach

$$G(z, \bar{z}) = \sum_{(\Delta, \bar{\Delta}) \in \mathcal{S}} C_{\Phi_1, \Phi_2, \Phi_{\Delta, \bar{\Delta}}} C_{\Phi_{\Delta, \bar{\Delta}}, \Phi_3, \Phi_4} \mathcal{F}_{\Delta}^{(s)}(z) \bar{\mathcal{F}}_{\bar{\Delta}}^{(s)}(\bar{z})$$

- Three different channels must give same result, order by order, in: $s$-channel ($z_1 \to z_2$), $t$-channel ($z_1 \to z_4$), $u$-channel ($z_1 \to z_3$)
- E.g., $s$-channel corresponds to $z \to 0$ (i.e. $z_1 \sim z_2$ and $z_3 \sim z_4$)

$$G(z,\bar{z}) = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} C_{\Phi_1,\Phi_2,\Phi_{\Delta,\bar{\Delta}}} C_{\Phi_{\Delta,\bar{\Delta}},\Phi_3,\Phi_4} \mathcal{F}_{\Delta}^{(s)}(z)\bar{\mathcal{F}}_{\bar{\Delta}}^{(s)}(\bar{z})$$

## Diagramatically

$$G(z,\bar{z}) = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} C_{\Phi_1,\Phi_2,\Phi_{\Delta,\bar{\Delta}}} C_{\Phi_{\Delta,\bar{\Delta}},\Phi_3,\Phi_4} \mathcal{F}_{\Delta}^{(s)}(z)\bar{\mathcal{F}}_{\bar{\Delta}}^{(s)}(\bar{z})$$

## Diagramatically

$$G(z,\bar{z}) = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} C_{\Phi_1,\Phi_2,\Phi_{\Delta,\bar{\Delta}}} C_{\Phi_{\Delta,\bar{\Delta}},\Phi_3,\Phi_4} \mathcal{F}_{\Delta}^{(s)}(z)\bar{\mathcal{F}}_{\bar{\Delta}}^{(s)}(\bar{z})$$

## Conformal blocks

$$\mathcal{F}_{\Delta}^{(s)}(z) = z^{\Delta-h_1-h_2}\left[1+O(z,\bar{z})\right]$$

- For given $(c,\Delta)$ use Zamolodchikov's recursion formula.

$$G(z,\bar{z}) = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} C_{\Phi_1,\Phi_2,\Phi_{\Delta,\bar{\Delta}}} C_{\Phi_{\Delta,\bar{\Delta}},\Phi_3,\Phi_4} \mathcal{F}_{\Delta}^{(s)}(z)\bar{\mathcal{F}}_{\bar{\Delta}}^{(s)}(\bar{z})$$

## Conformal blocks

$$\mathcal{F}_{\Delta}^{(s)}(z) = z^{\Delta-h_1-h_2}\left[1+O(z,\bar{z})\right]$$

- For given $(c,\Delta)$ use Zamolodchikov's recursion formula.

## Structure constants

- $C_{\Phi_1,\Phi_2,\Phi_{\Delta,\bar{\Delta}}}$ same as in the three-point functions
- Not known in general for our problem.
- Some cases linked to DOZZ formula of time-like Liouville theory [Delfino-Viti 2011], [Ikhlef-Jacobsen-Saleur 2015]

$$G(z,\bar{z}) = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} C_{\Phi_1,\Phi_2,\Phi_{\Delta,\bar{\Delta}}} C_{\Phi_{\Delta,\bar{\Delta}},\Phi_3,\Phi_4} \mathcal{F}_{\Delta}^{(s)}(z)\bar{\mathcal{F}}_{\bar{\Delta}}^{(s)}(\bar{z})$$

## Conformal blocks

$$\mathcal{F}_{\Delta}^{(s)}(z) = z^{\Delta-h_1-h_2}\left[1+O(z,\bar{z})\right]$$

- For given $(c,\Delta)$ use Zamolodchikov's recursion formula.

## Structure constants

- $C_{\Phi_1,\Phi_2,\Phi_{\Delta,\bar{\Delta}}}$ same as in the three-point functions
- Not known in general for our problem.
- Some cases linked to DOZZ formula of time-like Liouville theory [Delfino-Viti 2011], [Ikhlef-Jacobsen-Saleur 2015]

## The $s$-channel spectrum problem

- For a given 4-point function, what is the spectrum $\mathcal{S}$?

# The $s$-channel spectrum problem

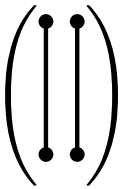
- A conjecture for $\mathcal{S}$ was made in [Ribault-Santachiara-Picco 2016].
- We believe it is incomplete and correct it [JJ-Saleur 2018].
- Remaining conformal bootstrap programme: work in progress.

# The correlation functions

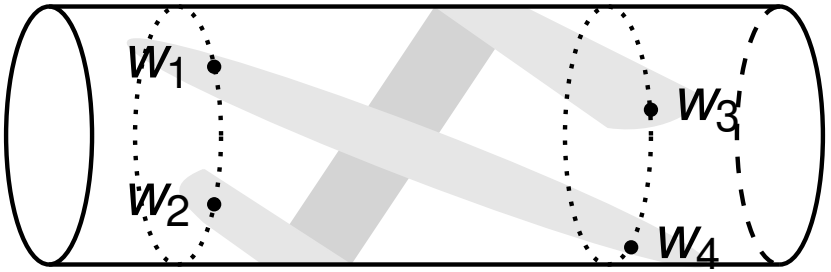
- We are interested in the 15 probabilities $P_{aaaa}, P_{abab}, \ldots, P_{abcd}$
  - Indices refer to $z_1, z_2, z_3, z_4$ (with $z_1 \sim z_2$ and $z_3 \sim z_4$ for $s$-channel).
  - Equal indices mean: points belong to the same FK cluster.

# The correlation functions

- We are interested in the 15 probabilities $P_{aaaa}, P_{abab}, \ldots, P_{abcd}$
  - Indices refer to $z_1, z_2, z_3, z_4$ (with $z_1 \sim z_2$ and $z_3 \sim z_4$ for $s$-channel).
  - Equal indices mean: points belong to the same FK cluster.

E.g. $P_{abab} = \mathbb{P}_2 = \mathbb{P}$  in previous notation. On the cylinder:


$P_{abab}$




$P_{abba}$

## A (slightly) simpler problem

- Consider Potts-model order parameter correlators

$$G_{a_1,a_2,a_3,a_4} = \left\langle \prod_{i=1}^{4} (Q\delta_{\sigma_i,a_i} - 1) \right\rangle$$

- $\{G_{aaaa}, G_{aabb}, G_{abba}, G_{abab}\}$ linearly related to $\{P_{aaaa}, P_{aabb}, P_{abba}, P_{abab}\}$.
- The system is invertible, except for $Q = 0, 1, 2, 3$.

## To keep things (relatively) simple

- We are particularly interested in $P_{abab} - P_{abba}$.
- Exactly computable for $Q = 0, 2, 4$ so three useful checks.

## To keep things (relatively) simple

- We are particularly interested in $P_{abab} - P_{abba}$.
- Exactly computable for $Q = 0, 2, 4$ so three useful checks.

## Parameterisation for $\sqrt{Q} = 2\cos\left(\frac{\pi}{m+1}\right)$ and $m \in [1, \infty)$

$$c = 1 - \frac{6}{m(m+1)} \qquad h_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)}$$

## To keep things (relatively) simple

- We are particularly interested in $P_{abab} - P_{abba}$.
- Exactly computable for $Q = 0, 2, 4$ so three useful checks.

## Parameterisation for $\sqrt{Q} = 2\cos\left(\frac{\pi}{m+1}\right)$ and $m \in [1, \infty)$

$$c = 1 - \frac{6}{m(m+1)} \qquad h_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)}$$

## The conjectures for $\mathcal{S} = \{h_{r,s}\}$

- $(r, s) \in (\mathbb{Z} + \frac{1}{2}, 2\mathbb{Z})$ by [Ribault-Santachiara-Picco 2016].
- $(r, s) \in (\mathbb{Z} + \frac{p}{N}, n)$ with $n > 0$ even, and $\frac{np}{N}$ odd, according to us.
- Technical ingredients: CFT, affine TL algebra, transfer matrix.

# Determining $\mathcal{S}$ numerically: First method

## Geometrical setup

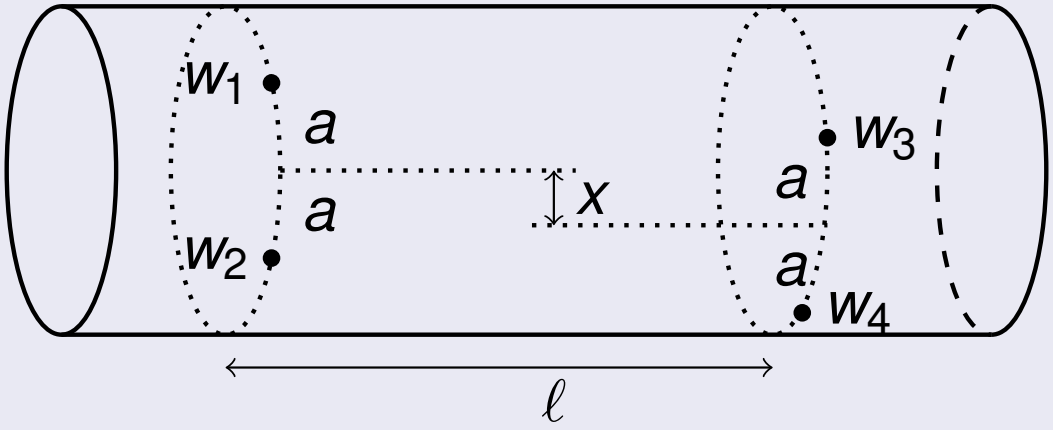
- Conformally transform from the plane to the cylinder: $w = \frac{L}{2\pi} \ln z$.
- Place $w_1, w_2 = \pm ia$ on one "slice" and $w_3, w_4 = \pm ia + \ell$ on another.
- Take $2a \sim \frac{L}{2}$, $L$ as large as possible, and $\ell \gg L$.

# Determining $\mathcal{S}$ numerically: First method

## Geometrical setup

- Conformally transform from the plane to the cylinder: $w = \frac{L}{2\pi} \ln z$.
- Place $w_1, w_2 = \pm ia$ on one "slice" and $w_3, w_4 = \pm ia + \ell$ on another.
- Take $2a \sim \frac{L}{2}$, $L$ as large as possible, and $\ell \gg L$.

## Cylinder geometry



## Transfer matrix method

- Fix $L$ and $2a$.
- Compute $P_{a_1 a_2 a_3 a_4}$ exactly ($\sim$ 4000 digits) for many ($\sim$ 500) $\ell \gg L$.
- Obtain full spectrum $\{\lambda_i\}$ of the transfer matrix.

$$P_{a_1 a_2 a_3 a_4} = \sum_i (A_i + B_i \ell + C_i \ell^2 + \cdots) \left(\frac{\lambda_i}{\lambda_0}\right)^\ell$$

- Invert to get simple amplitudes $A_i$, or Jordan cells $\{A_i, B_i, C_i, \ldots\}$.

# Determining $\mathcal{S}$ numerically: Second method

## Setup and restrictions

- Applies only to order-parameter correlators $G_{a_1 a_2 a_3 a_4}$.
- Change to repr. where operator $\Sigma_{a_i}(\sigma_i) \equiv Q\delta_{\sigma_i, a_i}$ is well defined.
- Applies only to simple amplitudes $A_i$.

# Determining $\mathcal{S}$ numerically: Second method

## Setup and restrictions

- Applies only to order-parameter correlators $G_{a_1 a_2 a_3 a_4}$.
- Change to repr. where operator $\Sigma_{a_i}(\sigma_i) \equiv Q\delta_{\sigma_i, a_i}$ is well defined.
- Applies only to simple amplitudes $A_i$.

## Scalar product method

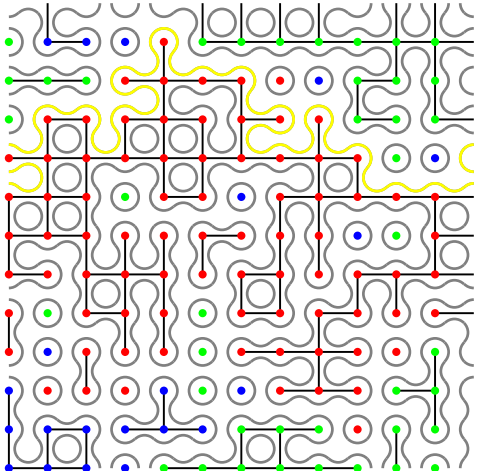
- Obtain left and right eigenstates: $\langle i|$ and $|i\rangle$

$$A_i = \frac{\langle 0|\Sigma_{a_3}\Sigma_{a_4}|i\rangle \; \langle i|\Sigma_{a_1}\Sigma_{a_2}|0\rangle}{\langle 0|0\rangle \; \langle i|i\rangle}$$

- Extensive checks that 1st and 2nd method give the same results.

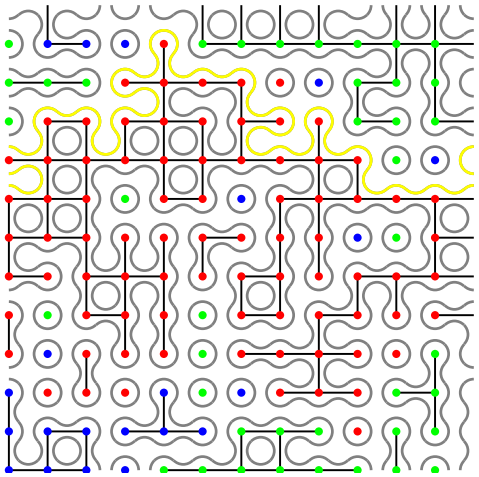
# The affine Temperley-Lieb algebra $\mathrm{TL}_N^a(n)$

- Describes "time evolution" of gray loops (each of weight $n = \sqrt{Q} = \mathfrak{q} + \mathfrak{q}^{-1}$) on the cylinder.



# The affine Temperley-Lieb algebra $\mathrm{TL}_N^a(n)$

- Describes "time evolution" of gray loops (each of weight $n = \sqrt{Q} = \mathfrak{q} + \mathfrak{q}^{-1}$) on the cylinder.



## Defining algebraic relations

- Monoid $e_i$ (with $i \in \mathbb{Z}_L$) and translator $u$.

$$\begin{aligned} e_i^2 &= n e_i & u e_i u^{-1} &= e_{i+1} \\ e_i e_{i\pm 1} e_i &= e_i & u^2 e_{L-1} &= e_1 e_2 \ldots e_{L-1} \end{aligned}$$

- Note that this algebra is infinite-dimensional (for finite $N$).

## Standard modules

- Finite-dimensional modules are classified [Martin-Saleur, Graham-Lehrer]
- $\mathcal{W}_{j,z^2}$ corresponds to $2j$ through-lines, with phase $z$ per winding.
- $u^N$ is central in $\mathrm{TL}_N^a(n)$.
- For $j = 0$, weight $n_{\mathrm{NC}} = z + z^{-1}$ per non-contractible loop.
- For $j > 0$, we have $u^N = z^{2j}$ in $\mathcal{W}_{j,z^2}$. Natural to set $z^{2j} = 1$.

## Standard modules

- Finite-dimensional modules are classified [Martin-Saleur, Graham-Lehrer]
- $\mathcal{W}_{j,z^2}$ corresponds to $2j$ through-lines, with phase $z$ per winding.
- $u^N$ is central in $\mathrm{TL}_N^a(n)$.
- For $j = 0$, weight $n_{\mathrm{NC}} = z + z^{-1}$ per non-contractible loop.
- For $j > 0$, we have $u^N = z^{2j}$ in $\mathcal{W}_{j,z^2}$. Natural to set $z^{2j} = 1$.

## Method 1 relies on $\mathcal{W}_{j,z^2}$

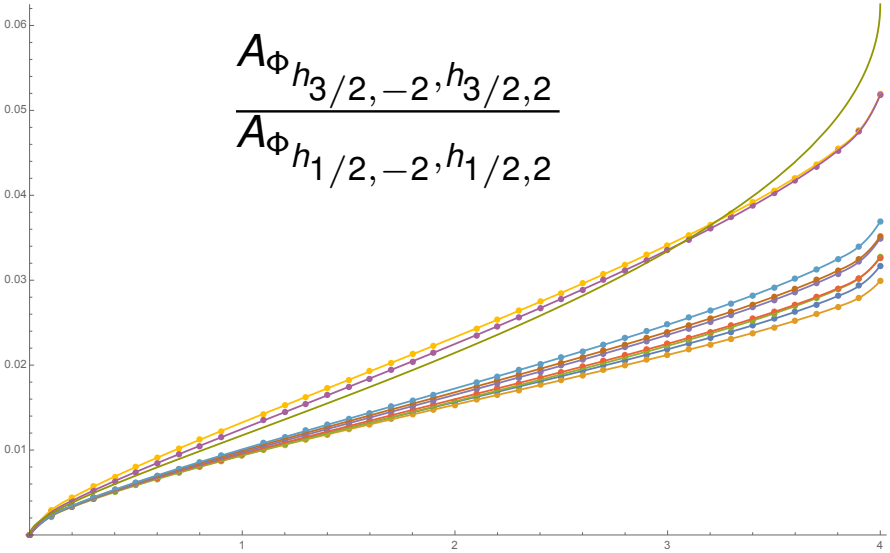
- Choice of standard module depends on # of through-clusters:
  - 0 clusters: $\overline{\mathcal{W}}_{0,\mathrm{q}^2}$ to get $n_{\mathrm{NC}} = \sqrt{Q}$.
  - 1 cluster: $\mathcal{W}_{0,-1}$ to get $n_{\mathrm{NC}} = 0$.
  - $j > 1$ clusters: $\mathcal{W}_{j,\mathrm{e}^{2i\pi p/M}}$ with $M|j$.
- The transfer matrix used in method 2 has this same spectrum, but restricted to $j$ even! (Probably clue for determining $\mathcal{S}$.)
- The CFT limit of these objects are known [Di Francesco-Saleur-Zuber]

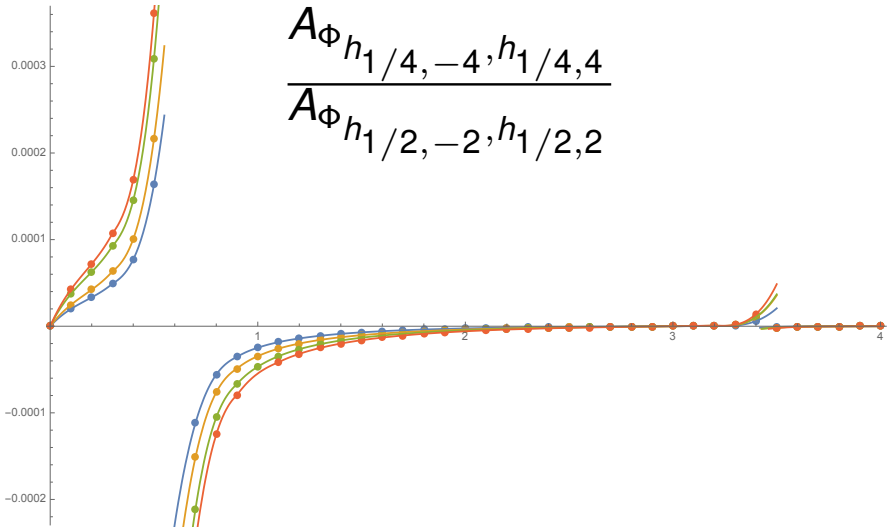
# Sample results: Amplitude ratios in $P_{aabb} - P_{abba}$

$$
\begin{aligned}
P_{aabb} - P_{abba} \quad \propto \quad & (z\bar{z})^{-2h_{1/2,0}} \Big( A_{\Phi_{h_{1/2,-2},h_{1/2,2}}} z^{h_{1/2,-2}} \bar{z}^{h_{1/2,2}} \\
+ \quad & A_{\Phi_{h_{3/2,-2},h_{3/2,2}}} z^{h_{3/2,-2}} \bar{z}^{h_{3/2,2}} + A_{\Phi_{h_{1/4,-4},h_{1/4,4}}} z^{h_{1/4,-4}} \bar{z}^{h_{1/4,4}} + \cdots \Big)
\end{aligned}
$$

# Sample results: Amplitude ratios in $P_{aabb} - P_{abba}$

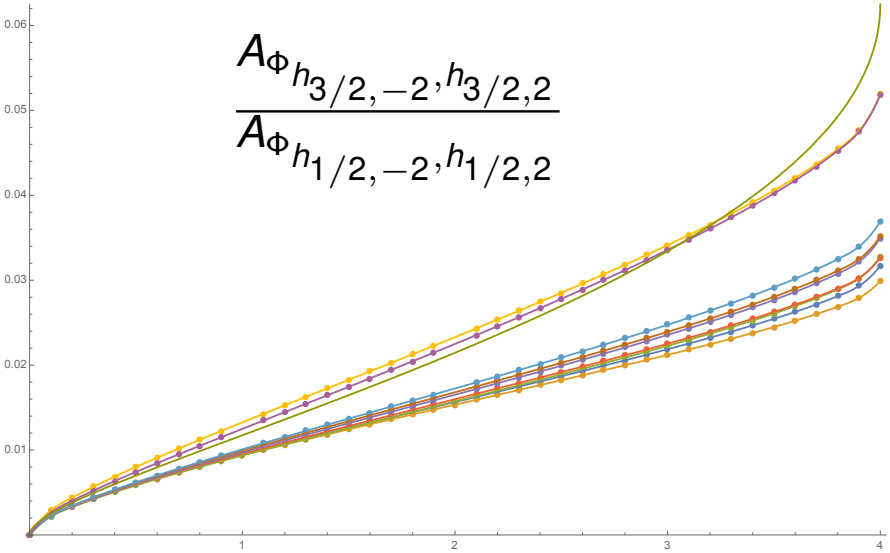
$$P_{aabb} - P_{abba} \propto (z\bar{z})^{-2h_{1/2,0}} \Big( A_{\Phi_{h_{1/2,-2},h_{1/2,2}}} z^{h_{1/2,-2}} \bar{z}^{h_{1/2,2}} + A_{\Phi_{h_{3/2,-2},h_{3/2,2}}} z^{h_{3/2,-2}} \bar{z}^{h_{3/2,2}} + A_{\Phi_{h_{1/4,-4},h_{1/4,4}}} z^{h_{1/4,-4}} \bar{z}^{h_{1/4,4}} + \cdots \Big)$$

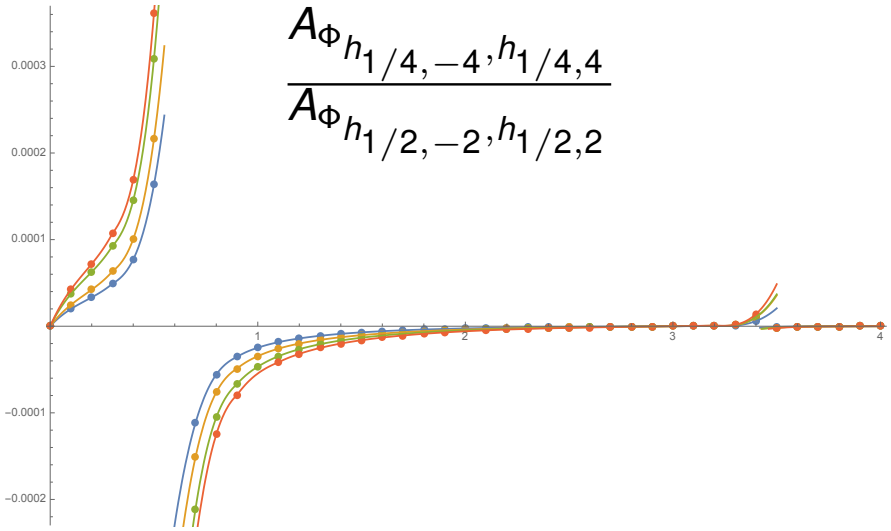


# Sample results: Amplitude ratios in $P_{aabb} - P_{abba}$

$$P_{aabb} - P_{abba} \propto (z\bar{z})^{-2h_{1/2,0}} \Big( A_{\Phi_{h_{1/2,-2},h_{1/2,2}}} z^{h_{1/2,-2}} \bar{z}^{h_{1/2,2}} + A_{\Phi_{h_{3/2,-2},h_{3/2,2}}} z^{h_{3/2,-2}} \bar{z}^{h_{3/2,2}} + A_{\Phi_{h_{1/4,-4},h_{1/4,4}}} z^{h_{1/4,-4}} \bar{z}^{h_{1/4,4}} + \cdots \Big)$$




- 1st case: Reasonable agreement with [Ribault et al] for $L \to \infty$.
- 2nd case: Not in their conjectured spectrum $(r, s) \in (\mathbb{Z} + \frac{1}{2}, 2\mathbb{Z})$ Divergences at $Q = 4\cos^2\left(\frac{3\pi}{8}\right), 4\cos^2\left(\frac{\pi}{8}\right)$ due to Jordan cells.

# Conclusion and summary

- The 2D Potts model in the Fortuin-Kasteleyn cluster formulation breaks two basic paradigms of CFT:
  1. Decomposability of irreducible representations.
  2. Degenerate highest-weight representations ($\exists$ null vectors).

# Conclusion and summary

- The 2D Potts model in the Fortuin-Kasteleyn cluster formulation breaks two basic paradigms of CFT:
  1. Decomposability of irreducible representations.
  2. Degenerate highest-weight representations ($\exists$ null vectors).
- This leads to remarkable consequences:
  1. Dilatation operator may be non-diagonalisable (Jordan cells).
     Correlation functions thus contain logarithms.
     Happens at particular $Q$ of physical relevance (e.g. $Q = 1$).
  2. Four-point functions exhibit rich spectrum in the $s$-channel.

# Conclusion and summary

- The 2D Potts model in the Fortuin-Kasteleyn cluster formulation breaks two basic paradigms of CFT:
  1. Decomposability of irreducible representations.
  2. Degenerate highest-weight representations ($\exists$ null vectors).
- This leads to remarkable consequences:
  1. Dilatation operator may be non-diagonalisable (Jordan cells).
     Correlation functions thus contain logarithms.
     Happens at particular $Q$ of physical relevance (e.g. $Q = 1$).
  2. Four-point functions exhibit rich spectrum in the $s$-channel.
- Challenges:
  1. Clarify relation to time-like Liouville theory and compute all structure constants of three-point functions.
  2. Solve the conformal bootstrap for four-point functions and compute analytically all universal amplitude ratios.

# Conclusion and summary

- The 2D Potts model in the Fortuin-Kasteleyn cluster formulation breaks two basic paradigms of CFT:
  1. Decomposability of irreducible representations.
  2. Degenerate highest-weight representations ($\exists$ null vectors).
- This leads to remarkable consequences:
  1. Dilatation operator may be non-diagonalisable (Jordan cells).
     Correlation functions thus contain logarithms.
     Happens at particular $Q$ of physical relevance (e.g. $Q = 1$).
  2. Four-point functions exhibit rich spectrum in the $s$-channel.
- Challenges:
  1. Clarify relation to time-like Liouville theory and compute all structure constants of three-point functions.
  2. Solve the conformal bootstrap for four-point functions and compute analytically all universal amplitude ratios.
- Quantum integrability might give results even for finite $a$ and $\ell$...