Four-point functions in the Fortuin-Kasteleyn cluster model

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RAQIS’18, 13 September 2018

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**Introduction**

**General setup**
- Statistical model initially defined on a 2D lattice (e.g. square)
- Local short-range interactions (e.g. nearest neighbour)
- Adjustable parameter $T$ (temperature / interaction strength)

**Critical phenomena**
- Take continuum limit ($\epsilon \to 0$)
- Scale invariance for some critical $T = T_c$
- Actually conformal invariance

**Integrability**
- No Yang-Baxter related integrability in this talk. However:
  - The model is integrable (XXZ chain).
  - Algebraic structures and form-factor-like expansions will appear.
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Q-state Potts model

Definition

- Spin $\sigma_i = 1, 2, \ldots, Q$ defined on each vertex $i \in V$
- Interaction $\delta_{\sigma_i, \sigma_j}$ along each edge $(ij) \in E$
- Interaction strength $K = J / T$ (with coupling constant $J$)
- Partition function

$$Z = \sum_{\{\sigma\}} \prod_{(ij) \in E} e^{K\delta_{\sigma_i, \sigma_j}}$$

- At $K_c$ on the square lattice, related to integrable XXZ chain.
Random geometry

- Non-local observables: curves and clusters
- Examples: Percolation, self-avoiding walk, Ising spin clusters
- At $T = T_c$: self-similarity and fractal structures
Connection to random geometry

Random geometry

- Non-local observables: curves and clusters
- Examples: Percolation, self-avoiding walk, Ising spin clusters
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Fortuin-Kasteleyn trick

- Recall $Z = \sum_{\{\sigma\}} \prod_{(ij) \in E} e^{K\delta_{\sigma_i,\sigma_j}}$
- Set $e^{K\delta_{\sigma_i,\sigma_j}} = 1 + (e^K - 1)\delta_{\sigma_i,\sigma_j}$
- Expand $\prod_{(ij) \in E}$ and perform $\sum_{\{\sigma\}}$ to obtain

$$Z = \sum_{A \subseteq E} Q^{k(A)}(e^K - 1)^{|A|}$$

- $k(A)$ is the number of connected components (clusters) in $A \subseteq E$
- Colours: Potts spins
- Black: FK clusters, weight $Q$
- Gray: Surrounding loops (on medial graph), weight $n = \sqrt{Q}$

**Special cases at $e^{K_c} = 1 + \sqrt{Q}$**

- $Q \rightarrow 1$: Critical percolation (at $p_c = \frac{1}{2}$)
- $Q \rightarrow 0$: Uniform spanning trees (alias dense polymers)
Correlation functions: standard CFT results

Two and three-point functions

- Functional form fixed by global conformal invariance [Polyakov 1970]
- Denoting \( r_{ij} \equiv |r_i - r_j| \):

\[
\langle \varphi_1(r_1) \varphi_2(r_2) \rangle = \frac{C_{12} \delta_{\Delta_1, \Delta_2}}{r_{12}^{2\Delta_1}}
\]

\[
\langle \varphi_1(r_1) \varphi_2(r_2) \varphi_3(r_3) \rangle = \frac{C_{123}}{r_{12}^{\Delta_1 + \Delta_2 - \Delta_3} r_{23}^{\Delta_2 + \Delta_3 - \Delta_1} r_{31}^{\Delta_3 + \Delta_1 - \Delta_2}}
\]
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Two and three-point functions

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\]

Four-point functions

- Arbitrary dependence on conformal invariants (anharmonic ratios):

\[
\langle \varphi_1(r_1)\varphi_2(r_2)\varphi_3(r_3)\varphi_4(r_4) \rangle = f \left( \frac{r_{12}r_{34}}{r_{13}r_{24}}, \frac{r_{12}r_{34}}{r_{23}r_{14}} \right) \prod_{i<j} r_{ij}^{\left( \frac{1}{3} \sum_k \Delta_k \right) - \Delta_i - \Delta_j}
\]
Radial quantisation and study of conserved current (stress tensor) leads to the Virasoro algebra \cite{Belavin-Polyakov-Zamolodchikov 1984}

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0}
\]

where \(c\) is the central charge, \(L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z)\) are mode operators, and \(z = x + iy, \tilde{z} = x - iy\) are complex coordinates.
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**Highest weight representations**

- Primary operators \( \varphi \) satisfy \( L_0 \varphi = \Delta \varphi \) and \( L_n \varphi = 0 \) for \( n > 0 \).
- Descendents: Linear combinations of \( L_{-n_1} \cdots L_{-n_k} \) with \( \sum_i n_i = N \).
- Correlation function involving a descendent: Linked to that of its corresponding primary, through action by differential operator.
The usual situation — and **two major challenges**

### Diagonalisability of $L_0$

- “Usually” the dilatation operator $\mathcal{D} \equiv L_0 + \bar{L}_0$ is diagonalisable.
  - Critical exponents $\Delta + \bar{\Delta}$ are its eigenvalues.
  - Fields $\varphi$ normalised by setting $C_{12} = 1$ in two-point functions.
- But if $Q \in 4 \cos^2(\pi Q)$ this may be **untrue**, and $\mathcal{D}$ has Jordan cells!
- Corr. functions have power law *and* logarithmic dependence on $r_{ij}$.

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The usual situation — and two major challenges

**Diagonalisability of $L_0$**

- “Usually” the dilatation operator $D \equiv L_0 + \bar{L}_0$ is diagonalisable.
- Critical exponents $\Delta + \bar{\Delta}$ are its eigenvalues.
- Fields $\varphi$ normalised by setting $C_{12} = 1$ in two-point functions.
- But if $Q \in 4 \cos^2(\pi Q)$ this may be **untrue**, and $D$ has Jordan cells!
- Corr. functions have power law *and* logarithmic dependence on $r_{ij}$.

** Computability of four-point functions**

- For these $Q$, “usually” the highest-weight representations have **null vectors** $\chi$: a descendent which is itself primary.
  - Since $\chi$ has norm zero, one takes the quotient $\chi = 0$.
  - Leads to diff. eq. for four-point functions, solvable if $N$ small.
  - In this way one can obtain e.g. Cardy’s crossing formulae and Schramm’s left-passage probability.

- But connectivity-related operators in FK cluster model **do not** have null vectors! So their correlators are **difficult to obtain**.
Logarithms and non-unitarity [Cardy 1999]

Standard unitary CFT

- Expand local density $\Phi(r)$ on sum of scaling operators $\varphi(r)$

$$\langle \Phi(r)\Phi(0) \rangle \sim \sum_{ij} \frac{A_{ij}}{r^{\Delta_i+\Delta_j}}$$

- $A_{ij} \propto \delta_{ij}$ by conformal symmetry [Polyakov 1970]
- $A_{ii} \geq 0$ by reflection positivity (unitarity)
- Hence only power laws appear

The non-unitary case

Cancellations and signs may occur

Suppose $A_{ii} \sim -A_{jj} \to \infty$ with $A_{ii}(\Delta_i - \Delta_j)$ finite

Then leading term is $r^{-2\Delta_i} \log r$
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Symmetry classification of operators in FK model

- $N$-spin operators irreducible under $S_Q$ and $S_N$ symmetries
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Operators acting on one spin

- Most general one-spin operator: $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i) = \sum_{a=1}^{Q} O_a \delta_{a,\sigma_i}$

$$
\delta_{a,\sigma_i} = \frac{1}{Q} \left[ \varphi_{a}(\sigma_i) + \left( \delta_{a,\sigma_i} - \frac{1}{Q} \right) \right]
$$

- Dimensions of representations: $(Q) = (1) \oplus (Q - 1)$
Operators acting symmetrically on two spins

- $Q \times Q$ matrices $\mathcal{O}(r_i) \equiv \mathcal{O}(\sigma_i, \sigma_j) = \sum_{a=1}^{Q} \sum_{b=1}^{Q} \mathcal{O}_{ab} \delta_{a,\sigma_i} \delta_{b,\sigma_j}$
- The $Q$ operators with $\sigma_i = \sigma_j$ decompose as before: $(1) \oplus (Q - 1)$
- Other $\frac{Q(Q-1)}{2}$ operators with $\sigma_i \neq \sigma_j$: $(1) + (Q - 1) + \left( \frac{Q(Q-3)}{2} \right)$
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Case $\sigma_i \neq \sigma_j$ from representation theory

- $\varepsilon = \delta_{\sigma_i \neq \sigma_j} = 1 - \delta_{\sigma_i, \sigma_j}$
- $\phi_a = \delta_{\sigma_i \neq \sigma_j} (\varphi_a(\sigma_i) + \varphi_a(\sigma_j))$
- $\psi_{ab} = \delta_{\sigma_i, a} \delta_{\sigma_j, b} + \delta_{\sigma_i, b} \delta_{\sigma_j, a} - \frac{1}{Q - 2} (\phi_a + \phi_b) - \frac{2}{Q(Q-1)} \varepsilon$
Operators acting symmetrically on two spins

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Case \( \sigma_i \neq \sigma_j \) from representation theory

\[
\begin{align*}
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\psi_{ab} &= \delta_{\sigma_i,a} \delta_{\sigma_j,b} + \delta_{\sigma_i,b} \delta_{\sigma_j,a} - \frac{1}{Q-2} (\phi_a + \phi_b) - \frac{2}{Q(Q-1)} \varepsilon
\end{align*}
\]

- Scalar \( \varepsilon \) (energy)
- Vector \( \varphi_a \) (order parameter)
- Tensor \( \psi_{ab} \) (two-cluster operator)
- Pole at \( Q = 1 \) means that \( \varepsilon \) and \( \psi_{ab} \) must mix into a Jordan cell
Geometrical interpretation in terms of FK clusters

One-spin results

\[ \langle \varphi_a(r) \varphi_b(0) \rangle = \frac{1}{Q} \left( \delta_{a,b} - \frac{1}{Q} \right) \mathbb{P} \left( \begin{array}{c} \cdot \\ \cdot \end{array} \right). \]
One-spin results

\[ \langle \varphi_a(r) \varphi_b(0) \rangle = \frac{1}{Q} \left( \delta_{a,b} - \frac{1}{Q} \right) P \left( \begin{array}{c} \bullet \\ \bullet \end{array} \right). \]

- In general we do not know exactly (even in \( d = 2 \)) the probability \( P \left( \begin{array}{c} \bullet \\ \bullet \end{array} \right) \) that the two spins belong to the same FK cluster.
- But its **large-distance asymptotics** is predicted from CFT.
Two-spin results

\[ \langle \varepsilon(r) \varepsilon(0) \rangle = \left( \frac{Q-1}{Q} \right)^2 \left( \mathbb{P} \left( \begin{array}{c} \cdot \cdot \\ \end{array} \right) + \mathbb{P} \left( \begin{array}{c} \cdot \\ \end{array} \right) \right) + \frac{Q-1}{Q} \mathbb{P} \left( \begin{array}{c} \cdot \\ \cdot \end{array} \right), \]

\[ \langle \phi_a(r) \phi_b(0) \rangle = \frac{Q-2}{Q^2} \left( \delta_{a,b} - \frac{1}{Q} \right) \left( \frac{Q-2}{Q} \mathbb{P} \left( \begin{array}{c} \cdot \\ \end{array} \right) + 2 \mathbb{P} \left( \begin{array}{c} \cdot \\ \cdot \end{array} \right) \right), \]

\[ \langle \psi_{ab}(r) \psi_{cd}(0) \rangle = \frac{2}{Q^2} \left( \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc} - \frac{1}{Q-2} (\delta_{ac} + \delta_{bd} + \delta_{ad} + \delta_{bc}) \right. \]

\[ \left. + \frac{2}{(Q-2)(Q-1)} \mathbb{P} \left( \begin{array}{c} \cdot \\ \cdot \end{array} \right) \right. \]
Avoiding the $Q \to 1$ divergence

- The “scalar” part of $\langle \psi_{ab}(r) \psi_{cd}(0) \rangle$ diverges.
- Therefore $\Delta \psi = \Delta \varepsilon$ (in any dimension!)
- So we can cure the divergence by mixing the two operators:
  \[
  \tilde{\psi}_{ab}(r) = \psi_{ab}(r) + \frac{2}{Q(Q-1)} \varepsilon(r).
  \]
- Thus emerges the Jordan cell which contains logarithmic correlators.
Define:
\[ P_2 = P \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right), \quad P_1 = P \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right), \quad P_0 = P \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right), \text{ and } P \neq \equiv P(\sigma_i \neq \sigma_{i+1}) \]

Construct an observable that behaves purely logarithmically

\[ F(r) \equiv \frac{P_0(r) + P_1(r) - P_2}{P_2(r)} \sim \left( 2 \times \lim_{Q \to 1} \frac{\Delta_\psi - \Delta_\varepsilon}{Q - 1} \right) \log r, \]

universal
Isolate the logarithm

Define:
\[ P_2 = \mathcal{P} \left( \begin{array}{c} \cdot \\ \cdot \end{array} \right), \quad P_1 = \mathcal{P} \left( \begin{array}{c} \cdot \\ \cdot \end{array} \right), \quad P_0 = \mathcal{P} \left( \begin{array}{c} \cdot \\ \cdot \end{array} \right), \quad \text{and} \quad P_\neq \equiv \mathcal{P}(\sigma_i \neq \sigma_{i+1}) \]

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In two dimensions, the universal pre-factor is \( \frac{2\sqrt{3}}{\pi} \).
Numerical check

\begin{align*}
\begin{array}{c}
\text{(2D, } F) \\
\text{(3D, } F) \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
\text{1024}\quad 2048\quad 4096\quad 8192 \quad 32\quad 64\quad 128\quad 256 \\
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\end{array}
\end{align*}
Back to four-point functions

General setup

- 4 primaries $\Phi_i(z_i, \bar{z}_i)$ of scaling dimension $h_i + \bar{h}_i$ and spin $h_i - \bar{h}_i$
- Send 3 points to $0, 1, \infty$ and study dependence on $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$

$$G(z, \bar{z}) \equiv \lim_{\Lambda \to \infty} \Lambda^{2h_3} \bar{\Lambda}^{2\bar{h}_3} \left\langle \Phi_1(z, \bar{z})\Phi_2(0, 0)\Phi_3(\Lambda, \bar{\Lambda})\Phi_4(1, 1) \right\rangle$$
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Conformal bootstrap approach

$$G(z, \bar{z}) = \sum_{(\Delta, \bar{\Delta}) \in S} C_{\Phi_1, \Phi_2, \Phi_{\Delta, \bar{\Delta}}} C_{\Phi_{\Delta, \bar{\Delta}}, \Phi_3, \Phi_4} F^{(s)}_{\Delta}(z) \bar{F}^{(s)}_{\bar{\Delta}}(\bar{z})$$

- Three different channels must give same result, order by order, in: s-channel ($z_1 \to z_2$), t-channel ($z_1 \to z_4$), u-channel ($z_1 \to z_3$)
- E.g., s-channel corresponds to $z \to 0$ (i.e. $z_1 \sim z_2$ and $z_3 \sim z_4$)
\[ G(z, \bar{z}) = \sum_{(\Delta, \bar{\Delta}) \in S} C_{\Phi_1, \Phi_2, \phi_\Delta, \bar{\phi}_{\bar{\Delta}}} C_{\Phi_{\Delta, \bar{\Delta}}, \phi_3, \phi_4} F_{\Delta}^{(s)}(z) \bar{F}_{\bar{\Delta}}^{(s)}(\bar{z}) \]

Diagramatically

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\[ G(z, \bar{z}) = \sum_{(\Delta, \bar{\Delta}) \in S} C_{\Phi_1, \Phi_2, \Phi_{\Delta}, \bar{\Delta}} C_{\Phi_{\Delta}, \bar{\Phi}_{\bar{\Delta}}, \Phi_3, \Phi_4} \mathcal{F}^{(s)}_{\Delta}(z) \bar{\mathcal{F}}^{(s)}_{\bar{\Delta}}(\bar{z}) \]
\[ G(z, \bar{z}) = \sum_{(\Delta, \bar{\Delta}) \in S} C_{\Phi_1, \Phi_2, \Phi_\Delta, \bar{\Phi}} C_{\Phi_\Delta, \bar{\Phi}, \Phi_3, \Phi_4} \mathcal{F}^{(s)}_\Delta(z) \mathcal{F}^{(s)}_{\bar{\Delta}}(\bar{z}) \]

**Conformal blocks**

\[ \mathcal{F}^{(s)}_\Delta(z) = z^{\Delta - h_1 - h_2} [1 + O(z, \bar{z})] \]

- For given \((c, \Delta)\) use Zamolodchikov’s recursion formula.
\[ G(z, \bar{z}) = \sum_{(\Delta, \bar{\Delta}) \in S} C_{\Phi_1, \Phi_2, \Phi_{\Delta, \bar{\Delta}}} C_{\Phi_{\Delta, \bar{\Delta}}, \Phi_3, \Phi_4} \mathcal{F}_\Delta^{(s)}(z) \bar{\mathcal{F}}_{\bar{\Delta}}^{(s)}(\bar{z}) \]

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**Structure constants**

- \(C_{\Phi_1, \Phi_2, \Phi_{\Delta, \bar{\Delta}}}\) same as in the three-point functions
- Not known in general for our problem.
- Some cases linked to DOZZ formula of time-like Liouville theory [Delfino-Viti 2011], [Ikhlef-Jacobsen-Saleur 2015]
\[ G(z, \bar{z}) = \sum_{(\Delta, \bar{\Delta}) \in S} C_{\Phi_1, \Phi_2, \Phi_\Delta, \bar{\Delta}} C_{\Phi_\Delta, \bar{\Delta}, \Phi_3, \Phi_4} \mathcal{F}_\Delta^{(s)}(z) \mathcal{F}_{\bar{\Delta}}^{(s)}(\bar{z}) \]

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  \([\text{Delfino-Viti 2011}], [\text{Ikhlef-Jacobsen-Saleur 2015}]\)

**The s-channel spectrum problem**

- For a given 4-point function, what is the spectrum \(S\)?
The \textit{s-channel} spectrum problem

- A conjecture for $S$ was made in \cite{Ribault-Santachiara-Picco 2016}.
- We believe it is incomplete and correct it \cite{JJ-Saleur 2018}.
- Remaining conformal bootstrap programme: work in progress.
The correlation functions

- We are interested in the 15 probabilities $P_{aaaa}, P_{abab}, \ldots, P_{abcd}$.
- Indices refer to $z_1, z_2, z_3, z_4$ (with $z_1 \sim z_2$ and $z_3 \sim z_4$ for s-channel).
- Equal indices mean: points belong to the same FK cluster.
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  - Equal indices mean: points belong to the same FK cluster.

E.g. $P_{abab} = P_2 = \mathbb{P} \left( \begin{array}{c} \hline \hline \end{array} \right)$ in previous notation. On the cylinder:

- $P_{abab}$
- $P_{abba}$
A (slightly) simpler problem

- Consider Potts-model order parameter correlators

\[ G_{a_1, a_2, a_3, a_4} = \left\langle \prod_{i=1}^{4} (Q \delta_{\sigma_i, a_i} - 1) \right\rangle \]

- \( \{ G_{aaaa}, G_{aabb}, G_{abba}, G_{abab} \} \) linearly related to
  \( \{ P_{aaaa}, P_{aabb}, P_{abba}, P_{abab} \} \).

- The system is invertible, except for \( Q = 0, 1, 2, 3 \).
To keep things (relatively) simple

- We are particularly interested in $P_{abab} - P_{abba}$.
- Exactly computable for $Q = 0, 2, 4$ so three useful checks.
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Parameterisation for $\sqrt{Q} = 2 \cos\left(\frac{\pi}{m+1}\right)$ and $m \in [1, \infty)$

$$c = 1 - \frac{6}{m(m+1)} \quad h_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)}$$
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The conjectures for $S = \{h_{r,s}\}$

- $(r, s) \in (\mathbb{Z} + \frac{1}{2}, 2\mathbb{Z})$ by [Ribault-Santachiara-Picco 2016].
- $(r, s) \in (\mathbb{Z} + \frac{p}{N}, n)$ with $n > 0$ even, and $\frac{np}{N}$ odd, according to us.
Determining $S$ numerically: First method

Geometrical setup

- Conformally transform from the plane to the cylinder: $w = \frac{L}{2\pi} \ln z$.
- Place $w_1, w_2 = \pm ia$ on one “slice” and $w_3, w_4 = \pm ia + \ell$ on another.
- Take $2a \sim \frac{L}{2}$, $L$ as large as possible, and $\ell \gg L$. 
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**Cylinder geometry**
Transfer matrix method

- Fix $L$ and $2a$.
- Compute $P_{a_1 a_2 a_3 a_4}$ exactly ($\sim 4000$ digits) for many ($\sim 500$) $\ell \gg L$.
- Obtain full spectrum $\{\lambda_i\}$ of the transfer matrix.

$$P_{a_1 a_2 a_3 a_4} = \sum_i (A_i + B_i \ell + C_i \ell^2 + \cdots) \left(\frac{\lambda_i}{\lambda_0}\right)^\ell$$

- Invert to get simple amplitudes $A_i$, or Jordan cells $\{A_i, B_i, C_i, \ldots\}$. 
Determining $S$ numerically: Second method

**Setup and restrictions**

- Applies only to order-parameter correlators $G_{a_1a_2a_3a_4}$.
- Change to repr. where operator $\Sigma a_i(\sigma_i) \equiv Q\delta_{\sigma_i,a_i}$ is well defined.
- Applies only to simple amplitudes $A_i$. 

Scalar product method

Obtain left and right eigenstates: $\langle i| \text{ and } |i\rangle$

$A_i = \langle 0| \Sigma a_3 \Sigma a_4 |i\rangle \langle i| \Sigma a_1 \Sigma a_2 |0\rangle \langle 0| 0 \rangle \langle i| i \rangle$

Extensive checks that 1st and 2nd method give the same results.
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The affine Temperley-Lieb algebra $\mathcal{TL}_N^a(n)$

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Defining algebraic relations

- Monoid $e_i$ (with $i \in \mathbb{Z}_L$) and translator $u$.

$$
e_i^2 = ne_i \quad ue_iu^{-1} = e_{i+1}
$$

$$
e_i e_{i \pm 1} e_i = e_i \quad u^2 e_{L-1} = e_1 e_2 \cdots e_{L-1}
$$

- Note that this algebra is infinite-dimensional (for finite $N$).
Standard modules

- Finite-dimensional modules are classified [Martin-Saleur, Graham-Lehrer]
- $\mathcal{W}_{j,z^2}$ corresponds to $2j$ through-lines, with phase $z$ per winding.
- $u^N$ is central in $\text{TL}_N^a(n)$.
- For $j = 0$, weight $n_{NC} = z + z^{-1}$ per non-contractible loop.
- For $j > 0$, we have $u^N = z^{2j}$ in $\mathcal{W}_{j,z^2}$. Natural to set $z^{2j} = 1$. 

Method 1 relies on $\mathcal{W}_{j,z^2}$

Choice of standard module depends on number of through-clusters:

- 0 clusters: $\mathcal{W}_0$, $q^2$ to get $n_{NC} = \sqrt{Q}$.
- 1 cluster: $\mathcal{W}_0$, $-1$ to get $n_{NC} = 0$.
- $j > 1$ clusters: $\mathcal{W}_j$, $e^{2i\pi p/M}$ with $M\mid j$.

The transfer matrix used in method 2 has this same spectrum, but restricted to $j$ even! (Probably clue for determining $S$.)

The CFT limit of these objects are known [Di Francesco-Saleur-Zuber]
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Sample results: Amplitude ratios in $P_{aabb} - P_{abba}$

$$P_{aabb} - P_{abba} \propto (z\bar{z})^{-2h_{1/2},0} \left( A_{\Phi_{h_1/2, -2, h_1/2, 2}} z^{h_1/2, -2} \bar{z}^{h_1/2, 2} + A_{\Phi_{h_3/2, -2, h_3/2, 2}} z^{h_3/2, -2} \bar{z}^{h_3/2, 2} + A_{\Phi_{h_1/4, -4, h_1/4, 4}} z^{h_1/4, -4} \bar{z}^{h_1/4, 4} + \ldots \right)$$
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- 1st case: Reasonable agreement with [Ribault et al] for $L \rightarrow \infty$.
- 2nd case: Not in their conjectured spectrum $(r, s) \in (\mathbb{Z} + \frac{1}{2}, 2\mathbb{Z})$
Divergences at $Q = 4 \cos^2 \left( \frac{3\pi}{8} \right), 4 \cos^2 \left( \frac{\pi}{8} \right)$ due to Jordan cells.
Conclusion and summary

- The 2D Potts model in the Fortuin-Kasteleyn cluster formulation breaks two basic paradigms of CFT:
  1. Decomposability of irreducible representations.
  2. Degenerate highest-weight representations ($\exists$ null vectors).

  This leads to remarkable consequences:
  1. Dilatation operator may be non-diagonalisable (Jordan cells).
     Correlation functions thus contain logarithms. Happens at particular $Q$ of physical relevance (e.g. $Q = 1$).
  2. Four-point functions exhibit rich spectrum in the $s$-channel.

Challenges:
  1. Clarify relation to time-like Liouville theory and compute all structure constants of three-point functions.
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Quantum integrability might give results even for finite $a$ and $\ell$...
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