Exact regimes of collapsed and extra two-string solutions in the two down-spin sector of the spin-1/2 massive XXZ spin chain

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(arXiv:1807.08885)

2018.9.11

Talk plan

We analyze the Bethe ansatz equations of the anti-ferromagnetic massive XXZ spin chain in two down-spin sector.

- We show that an extra two string solution appears, which is not predicted by the string hypothesis.
- We obtain the condition of the collapse. The collapse means some of two-string solutinos predicted by the string-hypothesis become real solutions.

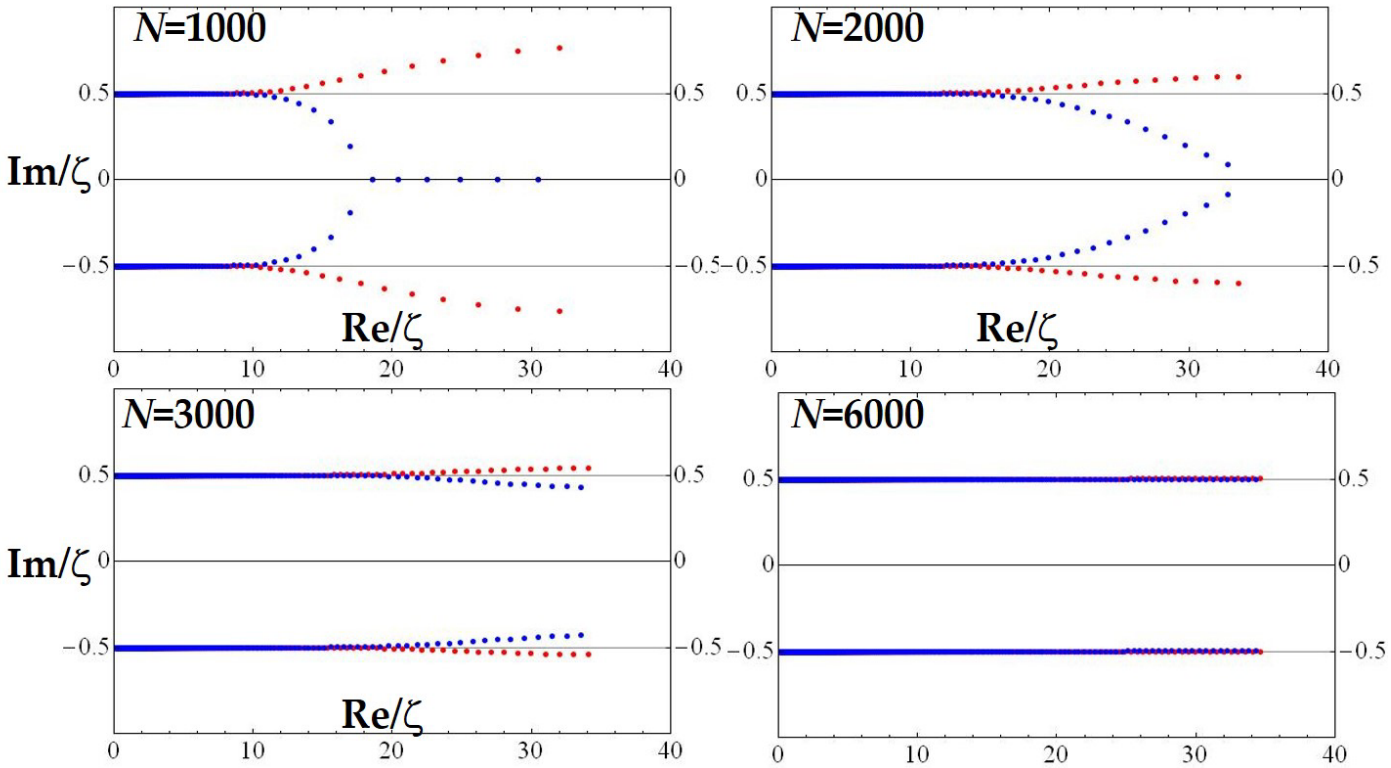


Figure: $\Delta = 1.001$ and $\Delta = \cosh\zeta$

Spin-1/2 XXZ chain Hamiltonians

The Hamiltonian of the XXZ spin chain under the P.B.C is given by

$$H = \sum_{n=1}^{N} \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z \right]$$

where $\sigma_j^a (a = X, Y, Z)$ are the Pauli matices defined on the jth site ,Δ denotes the anisotropic parameter, and N is the site number.

- $|\Delta| > 1$: massive regime
- $-1 \leq \Delta \leq 1$: massless regime
- $\Delta = 1$: anti-ferromagnetic XXX spin chain

Bethe ansatz equations(BAE)[1]

In M down-spin sector with rapidities $\lambda_1, \lambda_2, \cdots, \lambda_M$, the Bethe ansatz equations(BAE) of the spin-1/2 XXZ Heisenberg spin chain in the massive regime are given by

$$\left(\frac{\sin(\lambda_j + i\zeta/2)}{\sin(\lambda_j - i\zeta/2)}\right)^N = \prod_{k\neq j, k=1}^{M} \frac{\sin(\lambda_j - \lambda_k + i\zeta)}{\sin(\lambda_j - \lambda_k - i\zeta)} \quad (j = 1, 2, \cdots, M) \tag{1}$$

where $\Delta = \cosh\zeta$.

[1] H.Bethe(1931)

Arctangent function-part 1

A factor of RHS of the BAE(1)

$$\frac{\sin(\lambda_j - \lambda_k + i\zeta)}{\sin(\lambda_j - \lambda_k - i\zeta)} = \frac{\tan(\lambda_j - \lambda_k)/\tanh(\zeta) + i}{\tan(\lambda_j - \lambda_k)/\tanh(\zeta) - i}$$
$$= (-1)\exp\left[-2i\tan^{-1}\left(\frac{\tan(\lambda_j - \lambda_k)}{\tanh(\zeta)}\right)\right]. \tag{2}$$

In the right hand side of (2) the difference $\lambda_j - \lambda_k$ may be greater than $\pi/2$. In order to make the phase factor continuous at $\lambda_j - \lambda_k = \pi/2$ Takahashi introduced the following function[2] for real variable λ,

$$\theta_n(\lambda;\zeta) = 2\tan^{-1}\left(\frac{\tan\lambda}{\tanh(n\zeta/2)}\right) + 2\pi\left[\frac{2\lambda+\pi}{2\pi}\right] \tag{3}$$

where $[x]$ is the greatest integer that is not larger than x. We remark $-\pi/2 < Re(\tan^{-1}(x)) < \pi/2$ for $-\infty < x < \infty$. This formulation is not analytically continuous at $\lambda = \pi/2$. The logarithmic form of the BAE is

$$\theta_1(\lambda_j;\zeta) = \frac{2\pi}{N} I_j + \frac{1}{N}\sum_{k\neq j, k=1}^{M} \theta_2(\lambda_j - \lambda_k;\zeta). \tag{4}$$

This formulation is not analytically continuous at $\lambda = \pi/2$.

[2] M.Takahashi (1999) Thermodynamics of One-Dimensional Solvable Models

Arctangent function-part 2

In order to derive complex solutions we analytically continue the functions $\theta_n(\lambda)$ through the two branches of the logarithmic function.

$$\theta_n(\lambda;\zeta) = \frac{1}{i}\left(\log^{(+)}\left(\frac{\tan\lambda}{\tanh(n\zeta/2)}\right) - \log^{(s)}\left(\frac{\tan\lambda}{\tanh(n\zeta/2)}\right)\right). \tag{5}$$

where $\alpha, \beta \in \mathbb{R}$, $n \in \mathbb{Z}$,

$$\log^{(s)}(\alpha + i\beta) = i(\theta^{(s)}(\alpha + i\beta) + 2\pi n) + \frac{1}{2}\log(\alpha^2 + \beta^2) \tag{6}$$

$$\log^{(+)}(\alpha + i\beta) = i(\theta^{(+)}(\alpha + i\beta) + 2\pi n) + \frac{1}{2}\log(\alpha^2 + \beta^2) \tag{7}$$

and

$$\theta^{(s)}(\alpha + i\beta) = \begin{cases} \tan^{-1}(\beta/\alpha) + \pi H(-\alpha) sgn(\beta_+) & (\text{for } \alpha \neq 0) \\ sgn(\beta_+)\pi/2 & (\text{for } \alpha = 0) \end{cases}$$

$$\theta^{(+)}(\alpha + i\beta) = \begin{cases} \tan^{-1}(\beta/\alpha) + \pi H(-\alpha) + 2\pi H(\alpha) H(-\beta) & (\text{for } \alpha \neq 0) \\ sgn(\beta_+)\pi/2 + 2\pi H(-\beta) & (\text{for } \alpha = 0) \end{cases}$$

Here the step function $H(x)$ and the sign functions $sgn(x_+), sgn(x_-)$ are given by

$$H(x) = \begin{cases} 1 & (\text{for } \ x > 0) \\ 0 & (\text{otherwise}) \end{cases}, sgn(x_+) = 1 - 2H(-x), \quad sgn(x_-) = 2H(x) - 1.$$

Arctangent function-part 3

In this talk, we use the following definition.

$$\theta_n(\lambda;\zeta) = \frac{1}{i}\left(\log^{(+)}\left(\frac{\tan\lambda}{\tanh(n\zeta/2)}\right) - \log^{(s)}\left(\frac{\tan\lambda}{\tanh(n\zeta/2)}\right)\right).$$

Therefore, the logarithmic form of the BAE is

$$\theta_1(\lambda_j;\zeta) = \frac{2\pi}{N}J_j + \frac{1}{N}\sum_{k\neq j,k=1}^{M}\theta_2(\lambda_j-\lambda_k;\zeta).$$

When we use this definition, the arctangent function $2\widehat{\tan^{-1}}(\alpha+i\beta)$ is

$$\begin{aligned}
2\widehat{\tan^{-1}}(\alpha+i\beta) =& \frac{1}{i}\Big(\log^{(+)}(1+i(\alpha+i\beta)) - \log^{(s)}(1-i(\alpha+i\beta))\Big) \\
=& \tan^{-1}\left(\frac{\alpha}{1-\beta}\right) + \pi H(\beta-1)sgn(\alpha_+) + 2\pi H(1-\beta)H(-\alpha) \\
&+ \tan^{-1}\left(\frac{\alpha}{1+\beta}\right) - \pi H(-\beta-1)sgn(\alpha_-) + \frac{1}{2i}\log\left(\frac{\alpha^2+(\beta-1)^2}{\alpha^2+(\beta+1)^2}\right).
\end{aligned} \tag{8}$$

This formulation is analytically continuous at $\lambda = \pi/2$ for $\lambda \in \mathbb{C}$.

The logarithmic form of BAE in the massive regime of XXZ spin chain

The logarithmic form of Bethe ansatz equations are given by

$$2\widehat{\tan^{-1}}\left(\frac{\tan\lambda_i}{\tanh(\zeta/2)}\right) = \frac{2\pi}{N}J_i + \frac{1}{N}\sum_{k=1}^{M} 2\widehat{\tan^{-1}}\left(\frac{\tan(\lambda_i-\lambda_k)}{\tanh\zeta}\right) \tag{9}$$

$$J_i \equiv \frac{1}{2}(N-M-1) \quad (\mathrm{mod}1) \ \text{ for } i = 1, 2, \cdots, M$$

where $\Delta = \cosh\zeta$. We call J_i Bethe quantum numbers.

Massive regime in two down-spin sector

The logarithmic form of Bethe ansatz equations in two down-spin sector

$$2\widehat{\tan^{-1}}\left(\frac{\tan\lambda_1}{\tanh\zeta/2}\right) = \frac{2\pi}{N}J_1 + \frac{2}{N}\widehat{\tan^{-1}}\left(\frac{\tan(\lambda_1-\lambda_2)}{\tanh\zeta}\right) \tag{10}$$

$$2\widehat{\tan^{-1}}\left(\frac{\tan\lambda_2}{\tanh\zeta/2}\right) = \frac{2\pi}{N}J_2 + \frac{2}{N}\widehat{\tan^{-1}}\left(\frac{\tan(\lambda_2-\lambda_1)}{\tanh\zeta}\right) \tag{11}$$

We call J_1, J_2 the Bethe quantum numbers.

Self-conjugacy [3]

$$\{\lambda_1^*, \lambda_2^*, \cdots, \lambda_M^*\} = \{\lambda_1, \lambda_2, \cdots, \lambda_M\}. \tag{12}$$

2-string solution

We assume the form of a two-string solution as

$$\lambda_1 = x + \frac{i}{2}\zeta + i\delta, \quad \lambda_2 = x - \frac{i}{2}\zeta - i\delta \ \ (x, \delta \in \mathbb{R}) \tag{13}$$

where x: string center, δ: string deviation. We remark $\delta > -\zeta/2$.

[3] A.A.Vladimirov (1986)

The Bethe ansatz equation(10) is equivalent to

$$\frac{2\pi}{N}J_1 = \underbrace{\tan^{-1}\left(\frac{a}{1-b}\right) + \tan^{-1}\left(\frac{a}{1+b}\right) + \pi\left(H(b-1) + 2H(1-b)H(-a) - \frac{H(\delta)}{N}\right)}_{\textit{real part}}$$
$$+ \underbrace{\frac{1}{2i}\log\left\{\left(\frac{a^2+(b-1)^2}{a^2+(b+1)^2}\right)\left(\frac{(\tanh(\zeta+2\delta)/\tanh(\zeta)+1)^2}{(\tanh(\zeta+2\delta)/\tanh(\zeta)-1)^2}\right)^{1/N}\right\}}_{\textit{imaginary part}}. \tag{14}$$

$$a := \frac{\tan x(1-w^2t^2)}{t(1+(\tan^2 x)w^2t^2)} \quad b := \frac{(1+\tan^2 x)w}{\left(1+(\tan^2 x)w^2t^2\right)} \tag{15}$$

where $w = \tanh(\zeta/2+\delta)\big/\tanh(\zeta/2)$, $t = \tanh(\zeta/2)$ and $H(y) = \begin{cases} 1 & \text{for } y > 0 \\ 0 & \text{otherwise} \end{cases}$.

We remark that w depend on δ.

Counting function

We define the counting function as

$$Z_1(\delta(w), x(w), \zeta) \equiv \frac{1}{2\pi}\tan^{-1}\left(\frac{a}{1-b}\right) + \frac{1}{2\pi}\left(\frac{a}{1+b}\right) + \frac{1}{2}\left(H(b-1) + 2H(1-b)H(-a) - \frac{H(\delta)}{N}\right)$$
$$Z_2(\delta(w), x(w), \zeta) \equiv \frac{1}{2\pi}\tan^{-1}\left(\frac{a}{1-b}\right) + \frac{1}{2\pi}\left(\frac{a}{1+b}\right) + \frac{1}{2}\left(H(b-1)sgn(a_-) + \frac{H(\delta)}{N}\right).$$

The real part of the Bethe ansatz equation

$$Z_1(\delta(w), x(w), \zeta) := \frac{J_1}{N} \quad Z_2(\delta(w), x(w), \zeta) := \frac{J_2}{N}. \tag{16}$$

We obtain the relation between $Z_1(w)$ and $Z_2(w)$:

$$Z_2(w) - Z_1(w) = \frac{1}{N} H(\delta) - H(-a). \tag{17}$$

If we find the Bethe quantum numbers J_1, J_2 which the solution of the BAE exists corrsponding to, we obtain the solutions of BAE corresponding to the Bethe quantum numbers numerically.

The imaginary part of the Bethe ansatz equation

$$\left(\frac{a^2 + (b-1)^2}{a^2 + (b+1)^2}\right) = \left(\frac{(\tanh(\zeta + 2\delta)/\tanh(\zeta) + 1)^2}{(\tanh(\zeta + 2\delta)/\tanh(\zeta) - 1)^2}\right)^{1/N}. \tag{18}$$

We regard the imaginary part(18) of the Bethe ansatz equation(10) as a constraint on x and λ. With using this constraint, we analyze the real part.

The imaginary part of the Bethe ansatz equation(10) is equivalent to the quadratic equation of $X = \tan^2 x$ as follows.

$$A(w)X^2 + B(w)X + C(w) = 0, \tag{19}$$

where

$$A(w) = w^2(1+wt^2)^2\left\{\left(\frac{-(1-w)(1-wt^2)}{1+w^2t^2}\right)^2\right\}^{1/N} - w^2(1-wt^2)^2\left\{\left(\frac{(1+w)(1+wt^2)}{1+w^2t^2}\right)^2\right\}^{1/N}.$$

$$B(w) = \left\{\frac{(1-w^2t^2)^2}{t^2} + 2w(1+w)(1+wt^2)\right\}\left\{\left(\frac{-(1-w)(1-wt^2)}{1+w^2t^2}\right)^2\right\}^{1/N}$$
$$- \left\{\frac{(1-w^2t^2)^2}{t^2} - 2w(1-w)(1-wt^2)\right\}\left\{\left(\frac{(1+w)(1+wt^2)}{1+w^2t^2}\right)^2\right\}^{1/N}.$$

$$C(w) = (1+w)^2\left\{\left(\frac{-(1-w)(1-wt^2)}{1+w^2t^2}\right)^2\right\}^{1/N} - (1-w)^2\left\{\left(\frac{(1+w)(1+wt^2)}{1+w^2t^2}\right)^2\right\}^{1/N}.$$

where $w = \tanh(\zeta/2+\delta)\big/\tanh(\zeta/2)$, $t = \tanh(\zeta/2)$.
Solutions of eq(19) is given by

$$X_+ = \frac{1}{2A(w)}\left(-B(w) + \sqrt{B(w)^2 - 4A(w)C(w)}\right). \tag{20}$$

$$X_- = \frac{1}{2A(w)}\left(-B(w) - \sqrt{B(w)^2 - 4A(w)C(w)}\right). \tag{21}$$

We consider the interval of w satisfying $(\tan^2 x =)X(w) > 0$.

Proposition

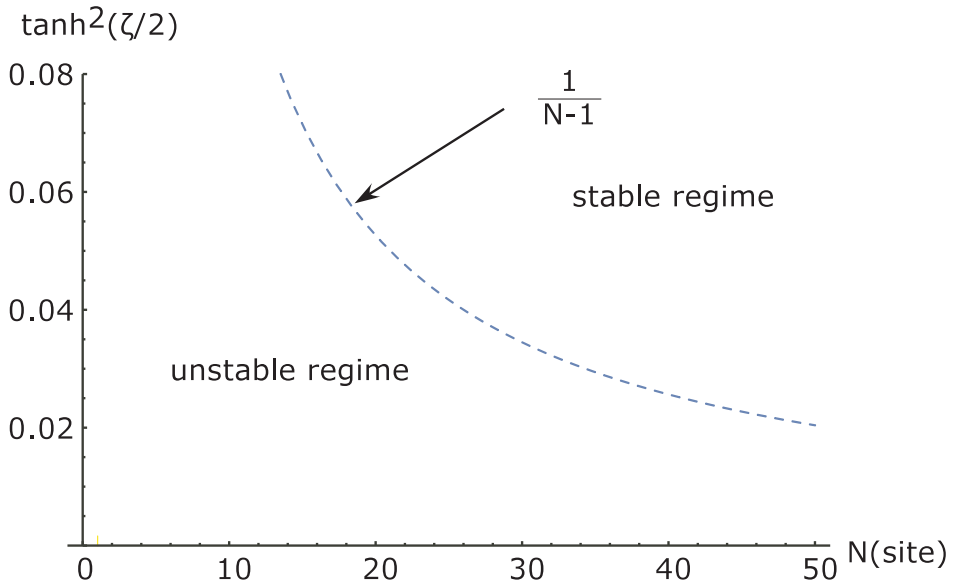
There doesn't exist w satisfying $X_+(w) > 0$.
$X_-(w) > 0$ if and only if we have $A(w) < 0$.

Therefore,the domain of definition for the counting function $Z(w)$ is equivalent to the interval of w satisfying $A(w) < 0$.

Stable and unstable regime

- stable regime: $A(0) > 0$ case (i.e. $X(0) < 0$)
- unstable regime: $A(0) \leq 0$ case (i.e. $X(0) \geq 0$)

$$A(0) = 0 \Leftrightarrow \tanh^2(\zeta/2) = \frac{1}{N-1}. \tag{22}$$



Monotonicity of the counting function

We define w_1, w_2, w_3 as

$$w_1 \equiv \min\{w|A(w) \geq 0\}, \quad w_2 \equiv 1, \quad w_3 \equiv \max\{w|A(w) \geq 0\} \tag{23}$$

We partially prove the monotonicity of the counting function $Z(w)$ in $w \in [w_1, w_2]$ and $w \in [w_2, w_3]$.

- $w \in [w_1, w_2]$ in stable regime, $Z(w)$ decreases monotonically
- $w \in [w_2, w_3]$ in stable regime, $Z(w)$ increases monotonically
- $w \in [w_2, w_3]$ in unstable regime, $Z(w)$ increases monotonically
- $w \in [w_1, w_2]$ in unstable regime, the monotonicity of the counting function $Z(w)$ is conjecture but numerically checked.

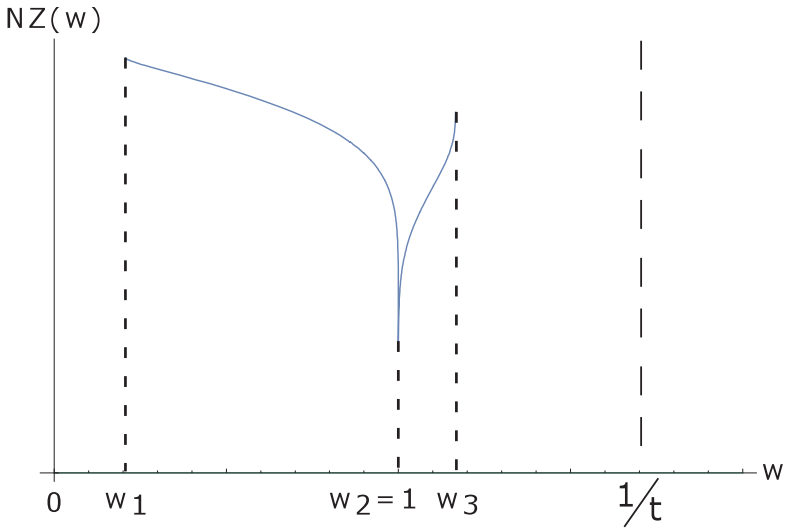


Figure: $NZ(w)$ in stable regime

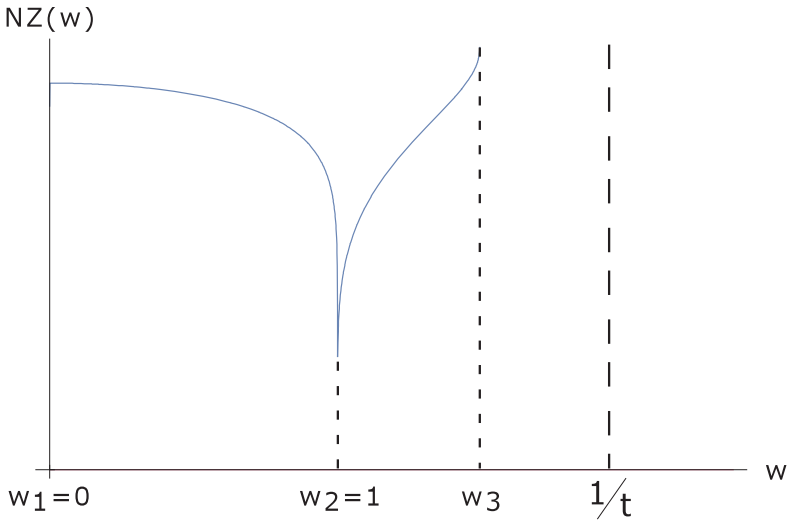


Figure: $NZ(w)$ in unstable regime

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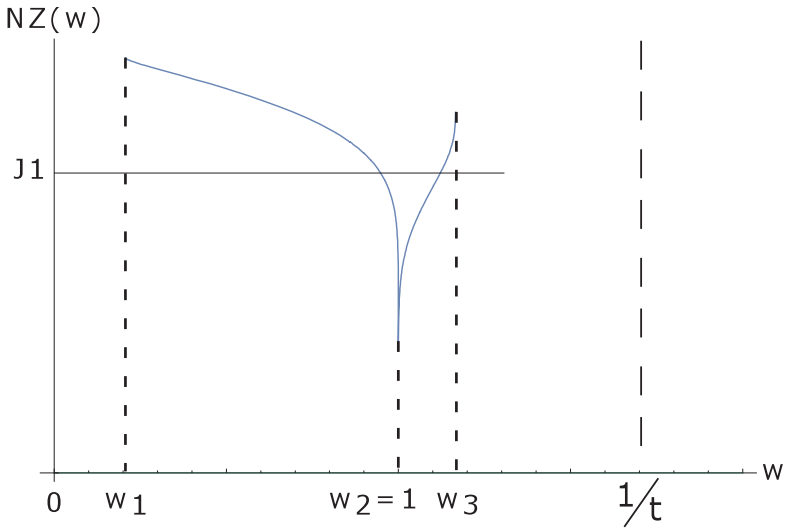

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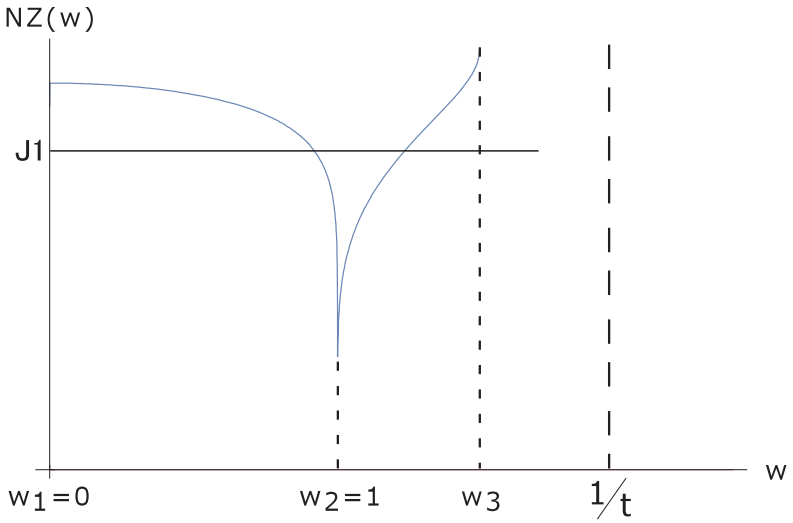


Figure: $NZ(w)$ in unstable regime

narrow pair/wide pair

If $w < 1$, we call the solution of BAE a narrow pair, because δ is negative.

If $w > 1$, we call the solution of BAE a wide pair, because δ is positive.

We remark $w = \tanh(\zeta/2 + \delta)\big/\tanh(\zeta/2)$, $t = \tanh(\zeta/2)$

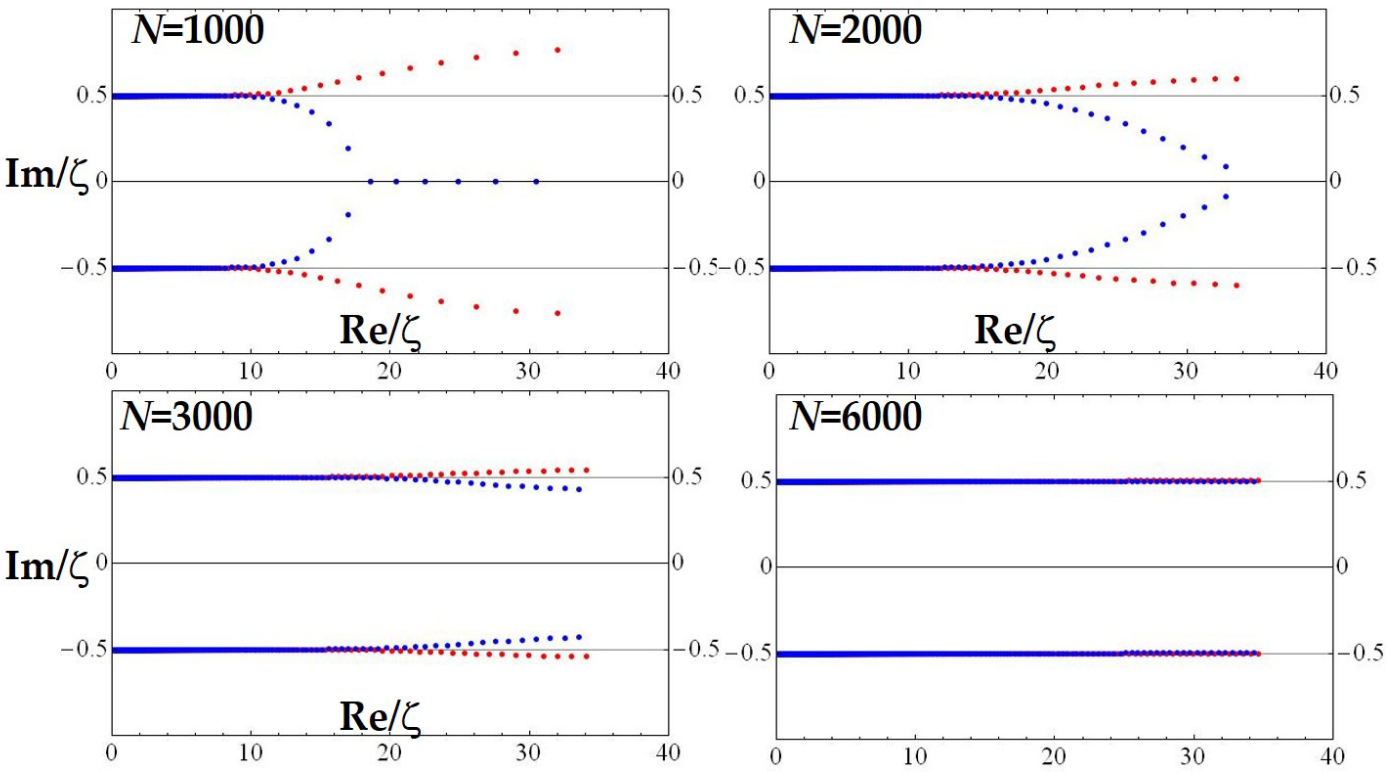
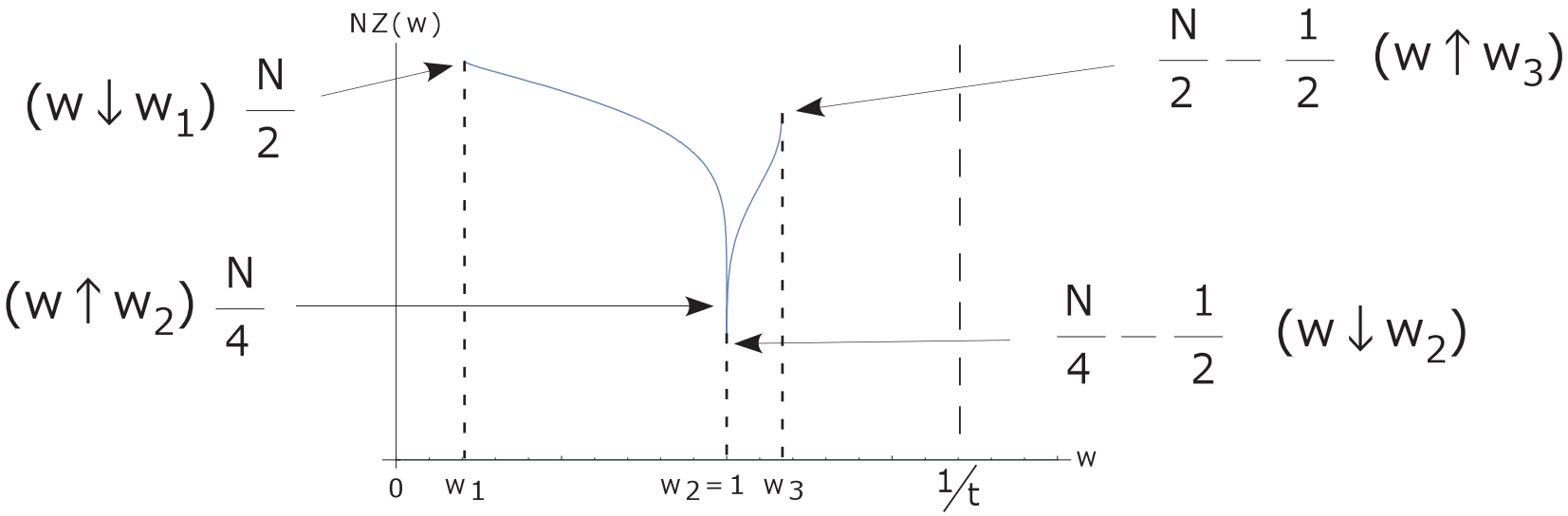


Figure: $\Delta = 1.001$ and $\Delta = \cosh\zeta$

Stable regime



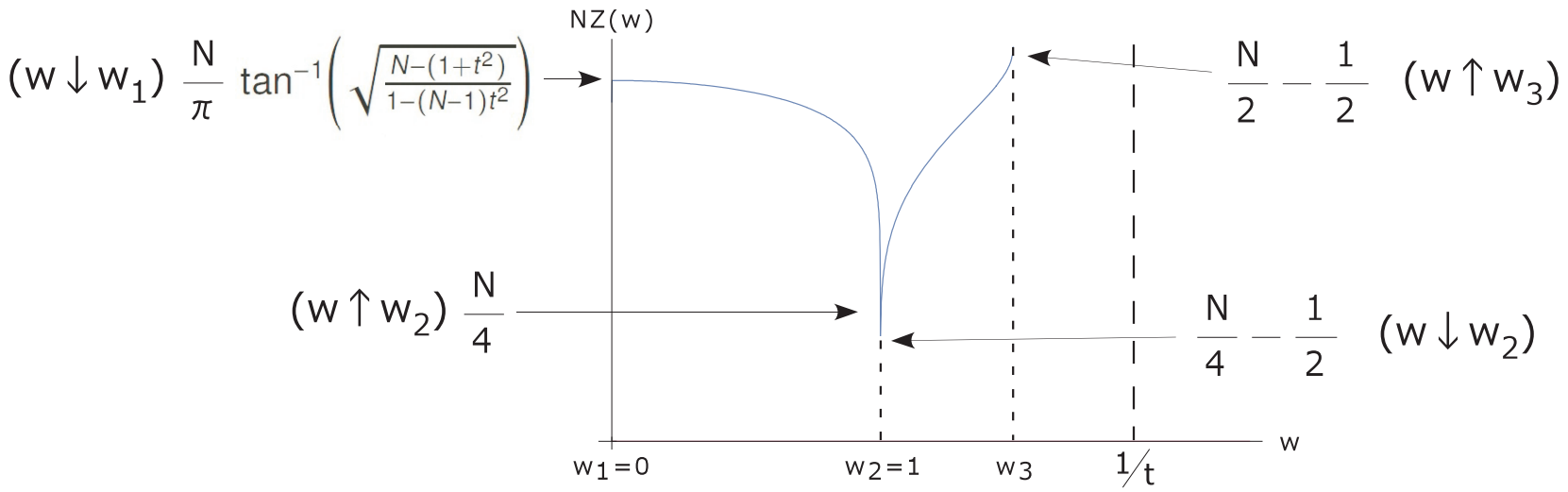
Real part of the Bethe ansatz equation

$$NZ_1(w) = J_1 \tag{25}$$

where $t = \tanh(\zeta/2)$ and $w = \tanh(\zeta/2 + \delta)/\tanh(\zeta/2)$
If $w < 1$, we call the solution of BAE a narrow pair, because δ is negative.
If $w > 1$, we call the solution of BAE a wide pair, because δ is positive.

Given the Bethe quantum number J_1 has the point of intersection with the counting function $Z(w)$, we obtain the solutions of BAE corresponding to the Bethe quantum numbers numerically.

Unstable regime



Real part of the Bethe ansatz equation

$$NZ_1(w) = J_1 \tag{26}$$

where $t = \tanh(\zeta/2)$ and $w = \tanh(\zeta/2 + \delta)/\tanh(\zeta/2)$
If $w < 1$, We call the solution of BAE a narrow pair, because δ is negative.
If $w > 1$, We call the solution of BAE a wide pair, because δ is positive.

Given the Bethe quantum number J_1 has the point of intersection with the counting function $Z(w)$, we obtain the solutions of BAE corresponding to the Bethe quantum numbers numerically.

Bethe quantum numbers

wide pair

$$\frac{N}{4} - \frac{1}{2} < J_1 < \frac{N-1}{2} \quad (\tan x > 0) \qquad -\frac{N+1}{2} < J_1 < -\frac{N}{4} - \frac{1}{2} \quad (\tan x < 0)$$

narrow pair(stable regime)

$$\frac{N}{4} \leq J_1 < \frac{N}{2} \quad (\tan x > 0) \qquad -\frac{N}{2} < J_1 \leq -\frac{N}{4} \quad (\tan x < 0)$$

narrow pair(unstable regime)

$$\frac{N}{4} \leq J_1 < \frac{N}{\pi} \tan^{-1}\left(\sqrt{\frac{N-(1+t^2)}{1-(N-1)t^2}}\right) \quad (\tan x > 0) \qquad -\frac{N}{2} < J_1 \leq -\frac{N}{4} \quad (\tan x < 0)$$

singular solution

$$(J_1, J_2) = (\frac{N}{4} - \frac{1}{2}, \frac{N}{4} + \frac{1}{2}) \text{ and } (-\frac{N}{4} - \frac{1}{2}, -\frac{N}{4} + \frac{1}{2}) \quad \text{for site number } N = 4n \;\; n \in \mathbb{Z}$$

$$(J_1, J_2) = (\frac{N}{4}, \frac{N}{4}) \text{ and } (-\frac{N}{4}, -\frac{N}{4}) \quad \text{for site number } N = 4n+2 \;\; n \in \mathbb{Z}$$

Collapse and extra two string solution

We define $Z_0^{(\zeta,N)}$ as

$$Z_0^{(\zeta,N)} \equiv Z(0) = \frac{1}{\pi}\tan^{-1}\left(\sqrt{\frac{N-(1+t^2)}{1-(N-1)t^2}}\right). \tag{27}$$

The conditions that the collapse of m two-string solutions occurs in the chain of N sites are given by

$$Z_0^{(\zeta,N)} < \frac{N-(1+2m)}{2N} \quad \text{for } m = 1, 2, \cdots . \tag{28}$$

In the limit of sending ζ to 0(the XXX limit)

$$\lim_{\zeta\to 0} Z_0^{(\zeta,N)} = \frac{1}{\pi}\tan^{-1}\left(\sqrt{N-1}\right) < \frac{N-(1+2m)}{2N}. \tag{29}$$

It coincides with that of the XXX chain.[4]

On the other hand, an extra pair of two-strings appears. If the following condition is satisfied, the number of two string-solutions is by two larger than the number due to the string solution. We call it the extra two string solution. The condition that the extra two string solution appear is given by

$$\frac{N-1}{2N} < Z_0^{(\zeta,N)}. \tag{30}$$

[4] T.Deguchi and P. R. Giri (2016)

The following is a summary of the above

Conjecture

If N and ζ satisfy

$$\tanh^2(\zeta/2) < \frac{1-(N-1)\tan^2\Big(\pi(1+2m)/2N\Big)}{(N-1)-\tan^2\Big(\pi(1+2m)/2N\Big)} \tag{31}$$

the collapse of m two-string solutions occurs.

Conjecture

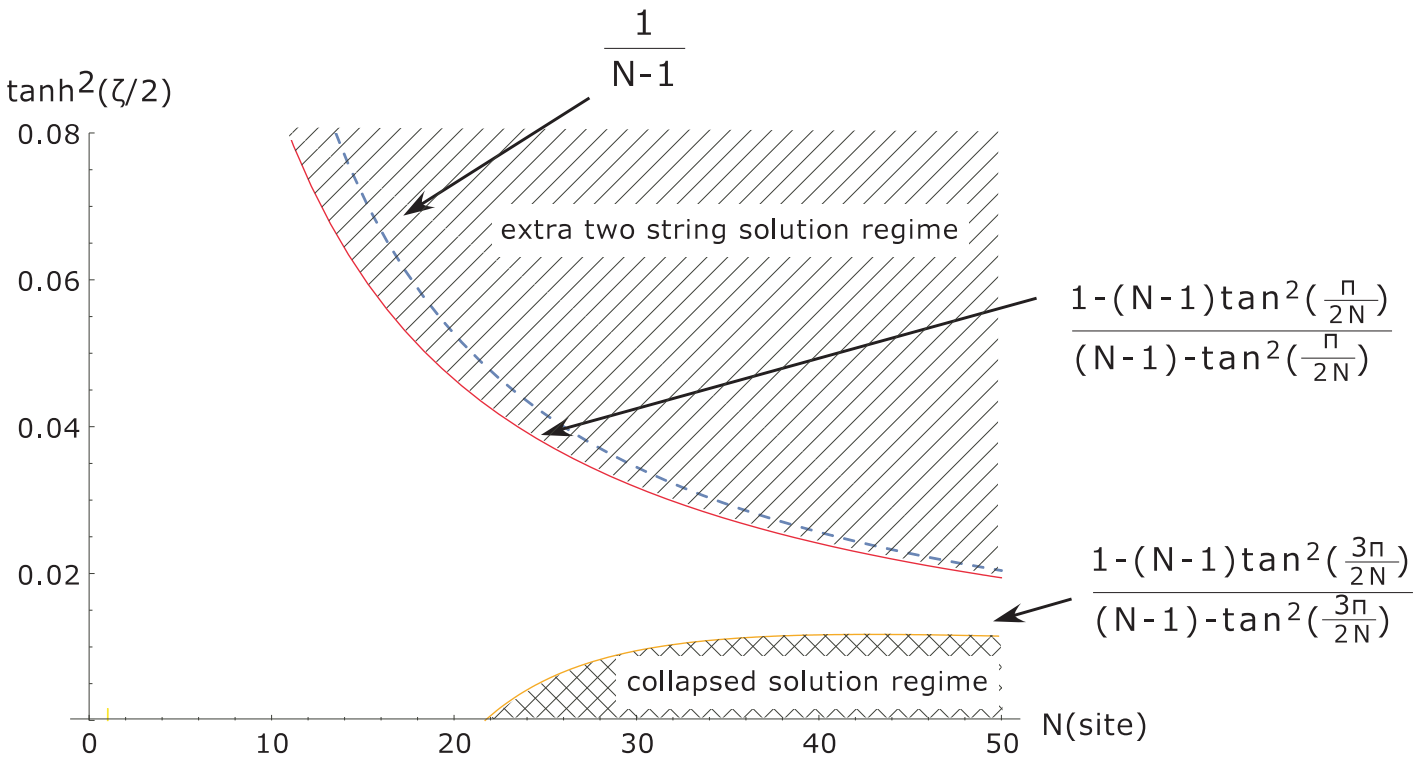
If N and ζ satisfy

$$\tanh^2(\zeta/2) > \frac{1-(N-1)\tan^2\Big(\pi(1+2m)/2N\Big)}{(N-1)-\tan^2\Big(\pi(1+2m)/2N\Big)}, \tag{32}$$

an extra pair of two-string solutions appears.

If we prove the monotonicity of counting function on $[w_1, w_2]$, these conjectures become theorems. We have checked it numerically.

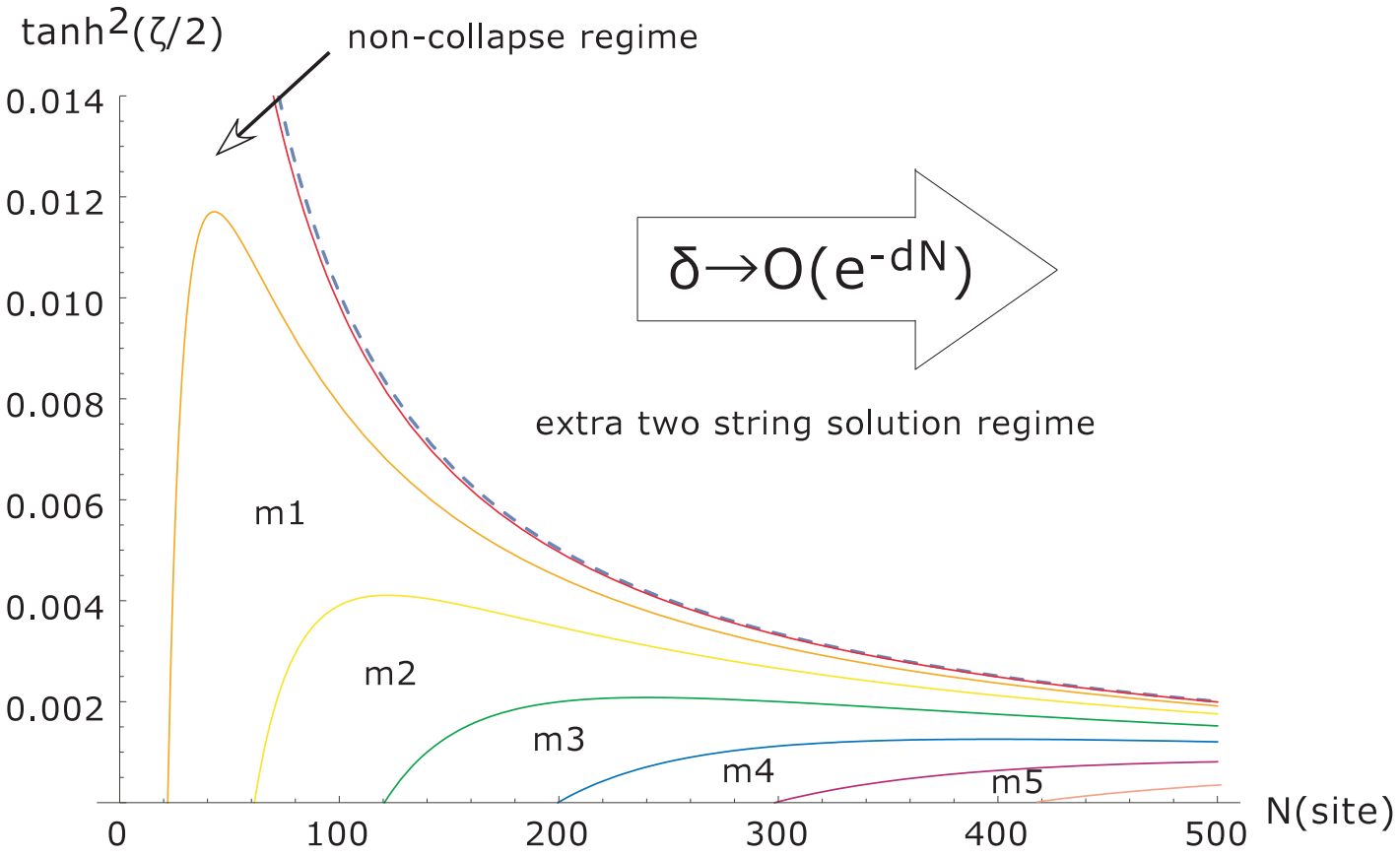
Figure of collapse and extra two-string solutions-part 1



$\Delta = \cosh\zeta$

- $\tanh^2(\zeta/2) \geq \frac{1}{N-1}$:stable regime
- $\tanh^2(\zeta/2) < \frac{1}{N-1}$:unstable regime

Figure of collapse and extra two-string solutions-part 2



$\Delta = \cosh \zeta$

- m1: 1 collapsed regime
- m2: 2 collapsed regime
- mx: x collapsed regime

Numerical values for extra two-string solution

Numerical two-string solutions of the BAE for $N = 12$ in the two down-spin sector, at $\zeta = 0.6$. No.1 and No.6 are extra complex solutions, which are not predicted by the string hypothesis.

No.	J	λ	Energy
1	11/2	1.13537646891480325577+0.16312176718062221300i	−0.42692157141886207577
	11/2	1.13537646891480325577−0.16312176718062221300i	
2	9/2	0.49443316603739513350+0.29840352572689955991i	-0.76659148423211189359
	9/2	0.49443316603739513350−0.29840352572689955991i	
3	7/2	0.14292089534049196825+0.29999999114716871863i	−1.12895830143233665108
	7/2	0.14292089534049196825-0.29999999114716871863i	
4	-7/2	−0.14292089534049196825+0.29999999114716871863i	−1.12895830143233665108
	-7/2	−0.14292089534049196825−0.29999999114716871863i	
5	-9/2	−0.49443316603739513350+0.29840352572689955991i	−0.76659148423211189359
	-9/2	−0.49443316603739513350−0.29840352572689955991i	
6	-11/2	−1.13537646891480325577+0.16312176718062221300i	-0.42692157141886207577
	-11/2	−1.13537646891480325577−0.16312176718062221300i	
7	11/2	0.74045039986314916894+0.31469282216447499146i	−0.54303402832662696979
	9/2	0.74045039986314916894−0.31469282216447499146i	
8	9/2	0.30062425150856577406+0.30002387970572190065i	−0.97443500257666539810
	7/2	0.30062425150856577406−0.30002387970572190065i	
9	-5/2	0.3i	−1.18546521824226770375
	-7/2	−0.3i	
10	−7/2	−0.30062425150856577406+0.30002387970572190065i	−0.97443500257666539810
	-9/2	−0.30062425150856577406−0.30002387970572190065i	
11	-9/2	−0.74045039986314916894+0.31469282216447499146i	−0.54303402832662696979
	-11/2	−0.74045039986314916894−0.31469282216447499146i	

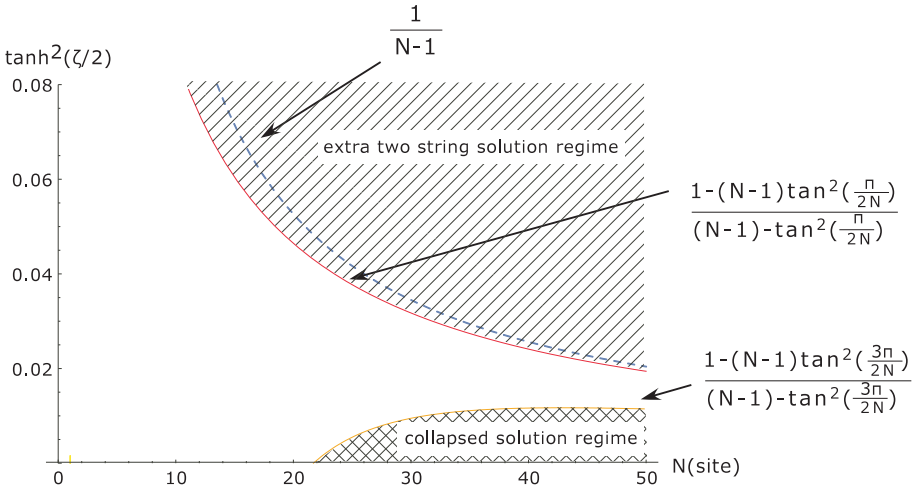
Numerical values for singular solution

Numerical two-string solutions of the BAE for $\underline{N = 12}$ in the two down-spin sector, at $\underline{\zeta = 0.6}$. No.9 is the singular solution. It has another set of quantum number $(5/2, 7/2)$

No.	J	λ	Energy
1	11/2	1.13537646891480325577+0.16312176718062221300i	−0.42692157141886207577
	11/2	1.13537646891480325577−0.16312176718062221300i	
2	9/2	0.49443316603739513350+0.29840352572689955991i	-0.76659148423211189359
	9/2	0.49443316603739513350−0.29840352572689955991i	
3	7/2	0.14292089534049196825+0.29999999114716871863i	−1.12895830143233665108
	7/2	0.14292089534049196825-0.29999999114716871863i	
4	-7/2	−0.14292089534049196825+0.29999999114716871863i	−1.12895830143233665108
	-7/2	−0.14292089534049196825−0.29999999114716871863i	
5	-9/2	−0.49443316603739513350+0.29840352572689955991i	−0.76659148423211189359
	-9/2	−0.49443316603739513350−0.29840352572689955991i	
6	-11/2	−1.13537646891480325577+0.16312176718062221300i	-0.42692157141886207577
	-11/2	−1.13537646891480325577−0.16312176718062221300i	
7	11/2	0.74045039986314916894+0.31469282216447499146i	−0.54303402832662696979
	9/2	0.74045039986314916894−0.31469282216447499146i	
8	9/2	0.30062425150856577406+0.30002387970572190065i	−0.97443500257666539810
	7/2	0.30062425150856577406−0.30002387970572190065i	
9	-5/2	0.3i	−1.18546521824226770375
	-7/2	−0.3i	
10	−7/2	−0.30062425150856577406+0.30002387970572190065i	−0.97443500257666539810
	-9/2	−0.30062425150856577406−0.30002387970572190065i	
11	-9/2	−0.74045039986314916894+0.31469282216447499146i	−0.54303402832662696979
	-11/2	−0.74045039986314916894−0.31469282216447499146i	

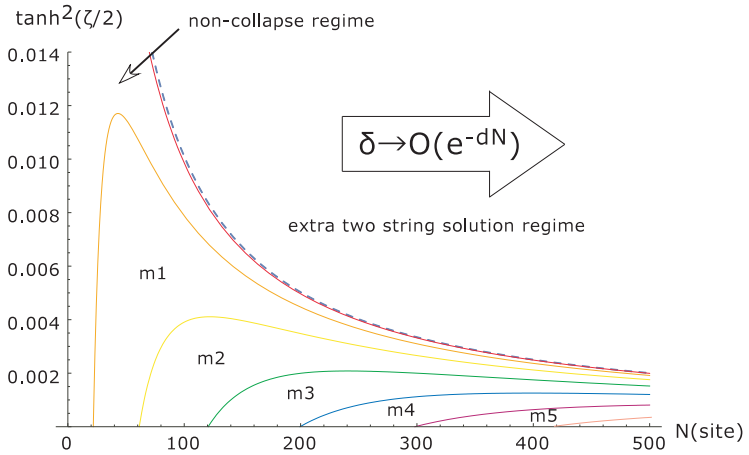
Conclusion

- We have obtained the condition of the collapse. (i.e. Some of two-string solutions predicted by the string-hypothesis become real solutions.)
- We have shown that the extra two string solution appears. (i.e. The number of two-string solutions by two larger than that of the string hypothesis.)
- (We have shown that string deviations are exponentially small with respect to N if N is large.)



$$\frac{1-(N-1)\tan^2(\frac{\pi}{2N})}{(N-1)-\tan^2(\frac{\pi}{2N})}$$

$$\frac{1-(N-1)\tan^2(\frac{3\pi}{2N})}{(N-1)-\tan^2(\frac{3\pi}{2N})}$$



Rigolous proof of complex solutions approaching complete strings exponentially with respect to w

We have obtained next two fact about the complete solutions.

<u>Fact</u>

If N is large enough, w_1 becomes close to 1 exponentially fast with respect to N.

<u>Fact</u>

If N is large enough, w_3 becomes close to 1 exponentially fast with respect to N.

Therefore

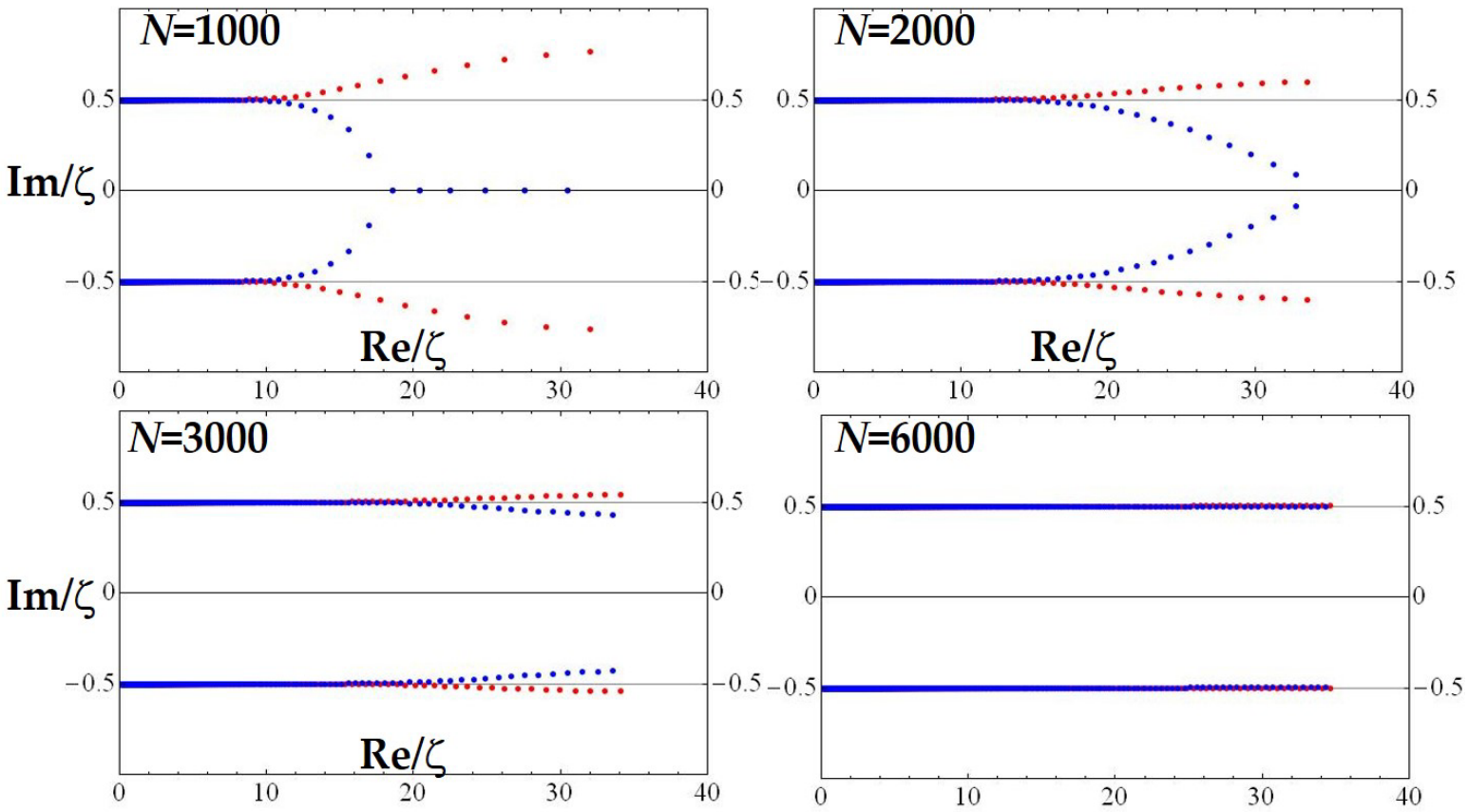
<u>Fact</u>

The string deviations are exponentially small with respect to N if N is large.

$$\lambda 1 = x + \frac{i}{2}\zeta + O(e^{-dN}) \;\; \lambda 2 = x - \frac{i}{2}\zeta + O(e^{-dN}) \qquad (33)$$

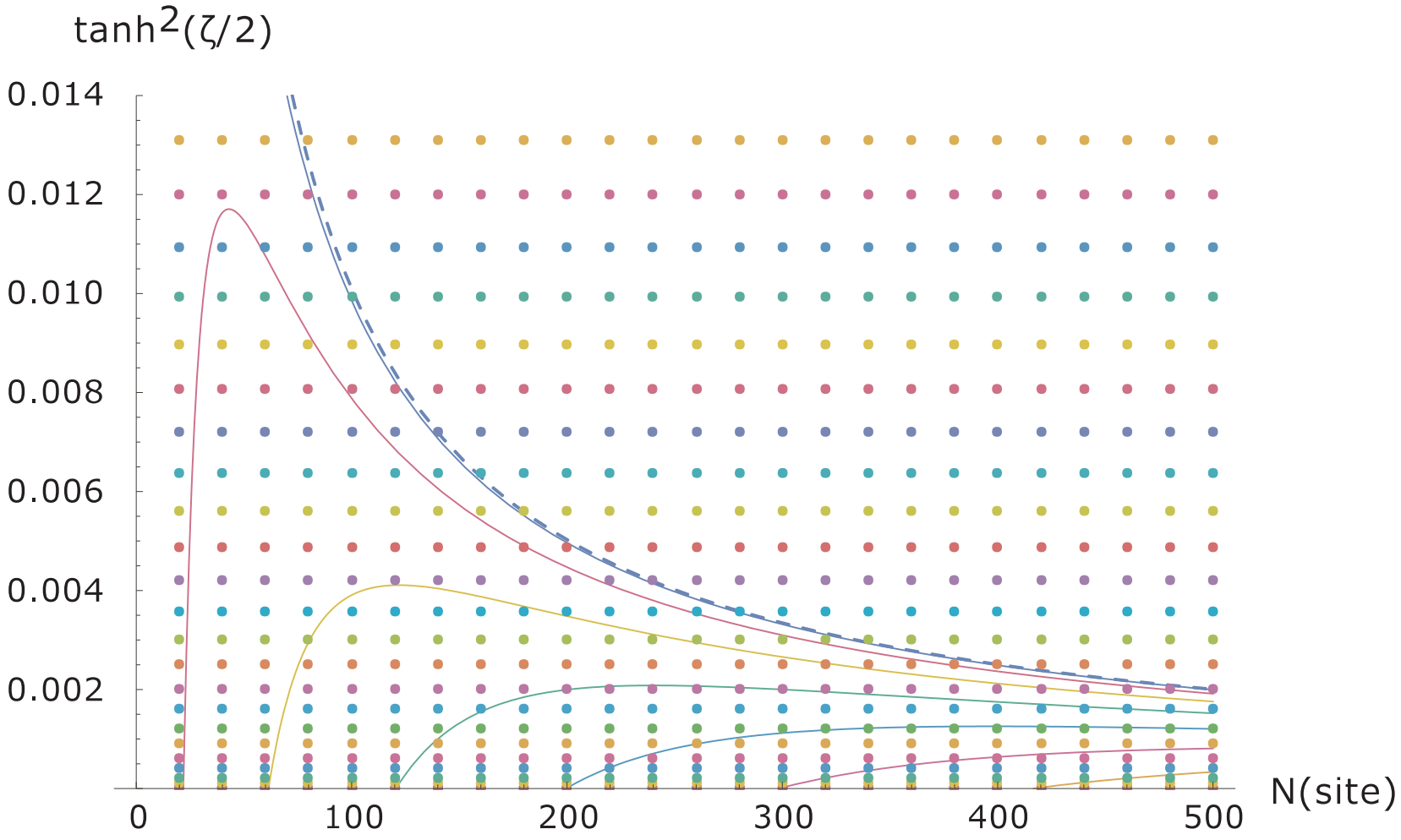
Numerical solutions of two-srings

We check this result with the numerical solutions. Parameters (Δ, N) are $\Delta = 1.001$ and $N = 1000, 2000, 3000, 6000$



When N is large, the string deviation is small.

Check point of monotonicity of counting function



$\Delta = \cosh \zeta$