An algebraic approach to stochastic duality

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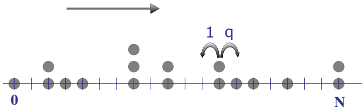
Outline

- Introduction
- Lie algebraic approach to duality theory
- $\mathfrak{su}_q(1,1)$ algebra
- Applications

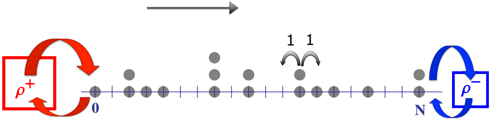
Introduction

Non-equilibrium in 1d: particle transport

- asymmetry

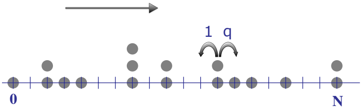


- density reservoirs

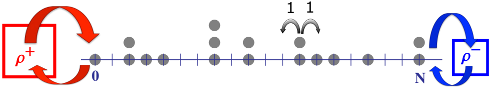


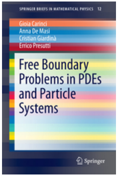
Non-equilibrium in 1d: particle transport

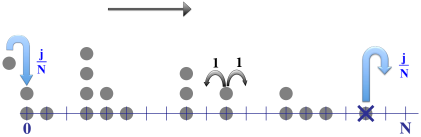
- asymmetry



- density reservoirs

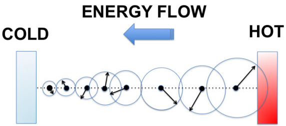






- current reservoirs
  Carinci, De Masi, G., Presutti (2016)

Non-equilibrium in 1d: energy transport



Fourier law $\quad J = \kappa \nabla T$

KMP model (1982)

Energies at every site: $\quad z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N$

$$L^{KMP} f(z) =$$

$$\sum_{i=1}^{N} \int_0^1 dp \left[f(z_1, \ldots, p(z_i + z_{i+1}), (1-p)(z_i + z_{i+1}), \ldots, z_N) - f(z) \right]$$

$\rightarrow$ conductivity $0 < \kappa < \infty$; model solved by duality.

Stochastic Duality

Definition

$(\eta_t)_{t\geq 0}$ Markov process on Ω with generator L,

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual} with generator L_{dual}

ξ_t is dual to η_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

η_t is self-dual if $L_{dual} = L$.

In terms of generators:

$$LD(\cdot, \xi)(\eta) = L_{dual} D(\eta, \cdot)(\xi)$$

Duality for Markov chain

Assume state spaces Ω, Ω_{dual} are countable sets, then the Markov generator L is a matrix $\mathbf{L}(\eta, \eta')$ s.t.

$$\mathbf{L}(\eta, \eta') \geq 0 \quad \text{if } \ \eta \neq \eta', \qquad \qquad \sum_{\eta' \in \Omega} \mathbf{L}(\eta, \eta') = 0$$

$$LD(\cdot, \xi)(\eta) = L_{dual} D(\eta, \cdot)(\xi)$$

amounts to

$$\mathbf{LD} = \mathbf{DL}_{dual}^{T}$$

Indeed

$$\sum_{\eta'} \mathbf{L}(\eta, \eta')\mathbf{D}(\eta', \xi) = \sum_{\xi'} \mathbf{L}_{dual}(\xi, \xi')\mathbf{D}(\eta, \xi')$$

Duality

- A useful tool
  - interacting particle systems [Spitzer, Ligget]
    hydrodynamic limit [Presutti, De Masi]
    KPZ scaling limits [Schütz, Spohn]
    population genetics [Kingman]
    ...
  - the dual process is simpler: “from many to few”.

- Questions
  - how to find a dual process and a duality function?
  - how to construct processes with duality?
    * E.g.: duality for asymmetric partial exclusion process?
    * E.g.: what is the right asymmetric version of KMP?

Lie algebraic approach

to duality theory

Algebraic approach

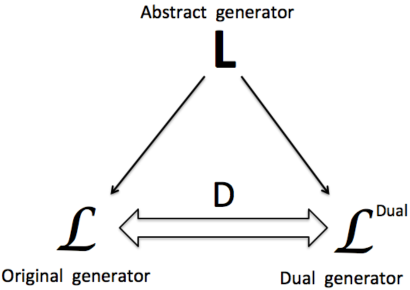
- ⋆ Write the Markov generator in abstract form, i.e. as an element of a universal enveloping algebra of a Lie algebra.

1. Duality is related to a change of representation.
   Duality functions are the intertwiners.

2. Dualities are associated to symmetries.
   Acting with a symmetry on a duality fct. yields another duality fct.

Conversely, the approach can be turned into a constructive method.

1. Change of representation



Example: Wright-Fisher diffusion and Kingman coalescence

Wright-Fisher diffusion $(X(t))_{t\geq 0}$ with state space $[0,1]$

$$L^{WF}f(x) = \frac{1}{2}x(1-x)\frac{\partial^2 f}{\partial x^2}(x)$$

$N(t) =$ number of blocks in the Kingman coalescence at time $t \geq 0$

$$(L^{King}f)(n) = \frac{n(n-1)}{2}(f(n-1) - f(n))$$

Duality Wright-Fisher / Kingman

The process $\{X(t)\}_{t\geq 0}$ with generator L^{WF} and the process $\{N(t)\}_{t\geq 0}$ with generator L^{King} are dual on $D(x,n) = x^n$, i.e.

$$\mathbb{E}_x^{WF}(X(t)^n) = \mathbb{E}_n^{King}(x^{N(t)})$$

Indeed:

$$\begin{aligned} L^{WF} D(\cdot, n)(x) &= \frac{1}{2} x(1-x) \frac{\partial^2}{\partial x^2} x^n \\ &= \frac{n(n-1)}{2}(x^{n-1} - x^n) \\ &= \frac{n(n-1)}{2}(D(x, n-1) - D(x,n)) \\ &= L^{King} D(x, \cdot)(n) \end{aligned}$$

Duality Wright-Fisher / Kingman : algebraic approach

Two representations of the Heisenberg algebra: $[a, a^\dagger] = 1$

$$\begin{cases} a^\dagger = x \\ \\ a = \frac{d}{dx} \end{cases} \qquad\qquad \begin{cases} a^\dagger e^{(n)} = e^{(n+1)} \\ \\ a\, e^{(n)} = n e^{(n-1)} \end{cases}$$

The abstract element $\quad L = \dfrac{1}{2} a^\dagger (1 - a^\dagger)(a)^2$

$L = L^{WF}$ in the first representation

$L^T = L^{King}$ in the second representation

Duality fct. $D(x, n) = x^n$ is the intertwiner:

$$xD(x, n) = D(x, n+1) \qquad\qquad \frac{d}{dx} D(x, n) = nD(x, n-1)$$

2. Symmetries

S: symmetry of the original Markov generator, i.e. $[\mathbf{L}, \mathbf{S}] = 0$

d: duality function between L and L_{dual}

$\longrightarrow$ $\mathbf{D} = \mathbf{Sd}$ is also duality function

Indeed

$$\mathbf{LD} = \mathbf{LSd} = \mathbf{SLd} = \mathbf{SdL}_{dual}^{T} = \mathbf{DL}_{dual}^{T}$$

“Cheap” self-duality

Let μ a reversible measure: $\mu(\eta)\mathbf{L}(\eta,\xi) = \mu(\xi)\mathbf{L}(\xi,\eta)$

A cheap (i.e. diagonal) self-duality is

$$\mathbf{d}(\eta,\xi) = \frac{1}{\mu(\eta)}\delta_{\eta,\xi}$$

Indeed

$$\frac{\mathbf{L}(\eta,\xi)}{\mu(\xi)} = \sum_{\eta'} \mathbf{L}(\eta,\eta')\mathbf{d}(\eta',\xi) = \sum_{\xi'} \mathbf{L}(\xi,\xi')\mathbf{d}(\eta,\xi') = \frac{\mathbf{L}(\xi,\eta)}{\mu(\eta)}$$

“Cheap” duality

Let μ a invariant measure: $\sum_\eta \mu(\eta)\mathbf{L}(\eta,\xi) = 0$

Let the dual process $(\xi_t)_{t\geq 0}$ be the time-reversed process of $(\eta_t)_{t\geq 0}$

$$\mathbf{L}_{dual}(\xi,\xi') = \mu(\xi)^{-1} L(\xi',\xi)\mu(\xi')$$

A cheap (i.e. diagonal) duality is

$$\mathbf{d}(\eta,\xi) = \frac{1}{\mu(\eta)}\delta_{\eta,\xi}$$

Indeed

$$\frac{\mathbf{L}(\eta,\xi)}{\mu(\xi)} = \sum_{\eta'} \mathbf{L}(\eta,\eta')\mathbf{d}(\eta',\xi) = \sum_{\xi'} \mathbf{L}_{dual}(\xi,\xi')\mathbf{d}(\eta,\xi') = \frac{\mathbf{L}(\eta,\xi)}{\mu(\xi)}$$

Construction of Markov generators with algebraic structure

i) (*Lie Algebra*): Start from a Lie algebra $\mathfrak{g}$.

ii) (*Casimir*): Pick an element in the center of $\mathfrak{g}$, e.g. the Casimir C.

iii) (*Co-product*): Consider a co-product $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ making the algebra a bialgebra and conserving the commutation relations.

iv) (*Quantum Hamiltonian*): Compute $H = \Delta(C)$.

v) (*Symmetries*): $S = \Delta(X)$ with $X \in \mathfrak{g}$ is a symmetry of H:

$$[H, S] = [\Delta(C), \Delta(X)] = \Delta([C, X]) = \Delta(0) = 0.$$

vi) (*Markov generator*): Apply a "ground state transformation" to turn H into a Markov generator L.

Quantum $\mathfrak{su}_q(1,1)$ algebra

q-numbers

For $q \in (0, 1)$ and $n \in \mathbb{N}_0$ introduce the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Remark: $\lim_{q \to 1} [n]_q = n$.

The first q-number's are:

$$[0]_q = 0, \qquad [1]_q = 1, \qquad [2]_q = q + q^{-1}, \qquad [3]_q = q^2 + 1 + q^{-2}, \quad \ldots$$

Quantum Lie algebra $\mathfrak{su}_q(1,1)$

For $q \in (0,1)$ consider the algebra with generators K^+, K^-, K^0

$$[K^0, K^\pm] = \pm K^\pm, \qquad [K^+, K^-] = -[2K^0]_q$$

where

$$[2K^0]_q := \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

Irreducible representations are infinite dimensional. E.g., for $n \in \mathbb{N}$

$$\begin{cases} K^+ e^{(n)} &= \sqrt{[n+2k]_q[n+1]_q}\; e^{(n+1)} \\ K^- e^{(n)} &= \sqrt{[n]_q[n+2k-1]_q}\; e^{(n-1)} \\ K^o e^{(n)} &= (n+k)\; e^{(n)} \end{cases}$$

Casimir element

$$C = [K^o]_q[K^o - 1]_q - K^+K^-$$

In this representation

$$C\, e^{(n)} = [k]_q[k-1]_q\, e^{(n)} \qquad\qquad k \in \mathbb{R}_+$$

Co-product

Co-product $\Delta : U_q(\mathfrak{su}(1,1)) \to U_q(\mathfrak{su}(1,1))^{\otimes 2}$

$$\begin{aligned} \Delta(K^{\pm}) &= K^{\pm} \otimes q^{-K^o} + q^{K^o} \otimes K^{\pm} \\ \Delta(K^o) &= K^o \otimes 1 + 1 \otimes K^o \end{aligned}$$

The co-product is an isomorphism s.t.

$$[\Delta(K^o), \Delta(K^{\pm})] = \pm\Delta(K^{\pm}) \qquad [\Delta(K^+), \Delta(K^-)] = -[2\Delta(K^o)]_q$$

From co-associativity $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$

$\Delta^n : U_q(\mathfrak{su}(1,1)) \to U_q(\mathfrak{su}(1,1))^{\otimes(n+1)}$, i.e. for $n \geq 2$

$$\begin{aligned} \Delta^n(K^{\pm}) &= \Delta^{n-1}(K^{\pm}) \otimes q^{-K^o_{n+1}} + q^{\Delta^{n-1}(K^0_i)} \otimes K^{\pm}_{n+1} \\ \Delta^n(K^o) &= \Delta^{n-1}(K^o) \otimes 1 + 1^{\otimes n} \otimes K^0_{n+1} \end{aligned}$$

Quantum Hamiltonian

$$\Delta(C_i) = q^{K_i^0} \left\{ K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1} \right\} q^{-K_{i+1}^0}$$

Quantum Hamiltonian

$$\Delta(C_i) = q^{K_i^0}\left\{K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1}\right\}q^{-K_{i+1}^0}$$

$$\begin{aligned} B_i \otimes B_{i+1} &= \frac{(q^k + q^{-k})(q^{k-1} + q^{-(k-1)})}{2(q-q^{-1})^2}\left(q^{K_i^0} - q^{-K_i^0}\right) \otimes \left(q^{K_{i+1}^0} - q^{-K_{i+1}^0}\right) \\ &+ \frac{(q^k - q^{-k})(q^{k-1} - q^{-(k-1)})}{2(q-q^{-1})^2}\left(q^{K_i^0} + q^{-K_i^0}\right) \otimes \left(q^{K_{i+1}^0} + q^{-K_{i+1}^0}\right) \end{aligned}$$

Quantum Hamiltonian

$$\Delta(C_i) = q^{K_i^0}\left\{K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1}\right\}q^{-K_{i+1}^0}$$

$$H \;:=\; \sum_{i=1}^{L-1}\left(1^{\otimes(i-1)} \otimes \Delta(C_i) \otimes 1^{\otimes(L-i-1)} + c_{q,k}1^{\otimes L}\right)$$

$$c_{q,k} = \frac{(q^{2k} - q^{-2k})(q^{2k-1} - q^{-(2k-1)})}{(q - q^{-1})^2} \qquad s.t. \qquad H \cdot \left(\otimes_{i=1}^{L} e_i^{(0)}\right) = 0$$

Markov processes

with $\mathfrak{su}_q(1,1)$ symmetry

Symmetries of H

Lemma

$$K^{\pm} := \sum_{i=1}^{L} q^{K_1^0} \otimes \cdots \otimes q^{K_{i-1}^0} \otimes K_i^{\pm} \otimes q^{-K_{i+1}^0} \otimes \ldots \otimes q^{-K_L^0}$$

$$K^0 := \sum_{i=1}^{L} \underbrace{1 \otimes \cdots \otimes 1}_{(i-1) \text{ times}} \otimes K_i^0 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i) \text{ times}} .$$

are symmetries of H.

Proof. Let $a \in \{+, -, 0\}$, then $K^a = \Delta^{L-1}(K_1^a)$

For $L = 2$: $[H, K^a] = [\Delta(C_1), \Delta(K_1^a)] = \Delta([C_1, K_1^a]) = \Delta(0) = 0$

For $L > 2$: induction.

Ground state transformation

Lemma

Let H be a matrix with $H(\eta, \eta') \geq 0$ if $\eta \neq \eta'$.
Suppose g is a positive ground state, i.e. $H\,g = 0$ and $g(\eta) > 0$.
Let G be the matrix $G(\eta, \eta') = g(\eta)\delta_{\eta,\eta'}$. Then

$$L = G^{-1} H\, G$$

is a Markov generator.

Indeed

$$L(\eta, \eta') = \frac{H(\eta, \eta')g(\eta')}{g(\eta)}$$

Therefore

$$L(\eta, \eta') \geq 0 \quad \text{if} \quad \eta \neq \eta' \qquad\qquad \sum_{\eta'} L(\eta, \eta') = 0$$

Exponential symmetries

- $g^{(0)} = \otimes_{i=1}^{L} e_i^{(0)}$ is a ground state, i.e. $Hg^{(0)} = 0$.
- For every symmetry $[H, S] = 0$ another ground state is $g = Sg^{(0)}$.
- The exponential symmetry

$$S^+ = \exp_{q^2}(E) = \sum_{n \geq 0} \frac{(E)^n}{[n]_q!} q^{-n(n-1)/2}$$

with

$$E = \Delta^{(L-1)}(q^{K_1^0}) \cdot \Delta^{(L-1)}(K_1^+)$$

gives a positive ground state

$$g = S^+ g^{(0)} = \sum_{\ell_1, \ldots, \ell_L} \otimes_{i=1}^{L} \left(\sqrt{\binom{\ell_i + 2k - 1}{\ell_i}_q} \cdot q^{\ell_i(1-k+2ki)} \right) e^{(\ell_i)}$$

- Remark

$$\lim_{q \to 1} S^+ = e^{\sum_i K_i^+} = \prod_i e^{K_i^+}$$

(1) Asymmetric Inclusion Process: ASIP(q,k)

For $k \in \mathbb{R}_+$ the interacting particle system ASIP(q,k) on $[1,L] \cap \mathbb{Z}$ with state space $\mathbb{N}^L$ is defined by

$$(L^{ASIP(q,k)} f)(\eta) = \sum_{i=1}^{L-1} (L_{i,i+1} f)(\eta)$$

with

$$\begin{aligned}(L_{i,i+1} f)(\eta) &= q^{\eta_i - \eta_{i+1} + (2k-1)} [\eta_i]_q [2k + \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta)) \\ &+ q^{\eta_i - \eta_{i+1} - (2k-1)} [2k + \eta_i]_q [\eta_{i+1}]_q (f(\eta^{i+1,i}) - f(\eta))\end{aligned}$$

- $q \to 1 \Rightarrow SIP(k)$: symmetric inclusion
  jump right at rate $\eta_i(2k + \eta_{i+1})$, jump left at rate $(2k + \eta_i)\eta_{i+1}$

Properties of ASIP(q,k)

- The $ASIP(q,k)$ on $[1,L]\cap\mathbb{Z}$ has a family (labeled by $\alpha>0$) of inhomogeneous reversible product measures with marginals

$$\mathbb{P}_\alpha(\eta_i = x) = \frac{\alpha^x}{Z_{i,\alpha}}\binom{x+2k-1}{x}_q \cdot q^{4kix}$$

- $q\to 1$: the reversible measure is homogeneous and product of Negative Binomials $(2k,\alpha)$

(2) Asymmetric Brownian Energy Process: ABEP(σ, k)

For $\sigma > 0$, let $(\eta^{(\epsilon)}(t))_{t \geq 0}$ be the $ASIP(1 - \epsilon\sigma, k)$ process initialized with ϵ^{-1} particles. The scaling limit (weak asymmetry)

$$z_i(t) := \lim_{\epsilon \to 0} \epsilon \, \eta_i^{(\epsilon)}(t)$$

is the diffusion ABEP(σ, k) with generator $L^{ABEP(\sigma,k)} = \sum_{i=1}^{L-1} \mathcal{L}_{i,i+1}$

$$\mathcal{L}_{i,i+1} =$$

$$- \frac{1}{2\sigma} \left\{ (1 - e^{-2\sigma z_i})(e^{2\sigma z_{i+1}} - 1) + 2k \left(2 - e^{-2\sigma z_i} - e^{2\sigma z_{i+1}}\right) \right\} \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)$$

$$+ \frac{1}{4\sigma^2} (1 - e^{-2\sigma z_i})(e^{2\sigma z_{i+1}} - 1) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2$$

Remark: $\mathcal{L}_{i,i+1}$ conserves $z_i(t) + z_{i+1}(t)$

Properties of ABEP(σ, k)

- $\sigma \neq 0$

  the process is truly asymmetric, i.e. on the 1-d torus it carries a non-zero current.

  on the half-line it has inhomogeneous reversible product measures (labeld by $\gamma > -4\sigma k$) with marginal density

  $$\mu(dz_i) = \frac{1}{Z_{i,\alpha}}(1 - e^{-2\sigma z_i})^{(2k-1)} e^{-(4\sigma k i + \gamma) z_i} dz_i$$

- $\sigma \to 0^+$

  $$\mathcal{L}_{i,i+1} = -2k\,(z_i - z_{i+1})\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right) + z_i z_{i+1}\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}}\right)^2$$

  The reversible measures are given by product of i.i.d. $Gamma(2k;\gamma)$

  $$\mu(dz_i) = \frac{1}{\gamma^{2k}\Gamma(2k)} {z_i}^{(2k-1)} e^{-\gamma z_i} dz_i$$

(3) $KMP(k)$ process

Instantaneous thermalization limit:

$$
\begin{aligned}
L_{i,j}^{KMP(k)} f(z_i, z_j) &:= \lim_{t\to\infty} \left(e^{tL_{i,j}^{BEP(k)}} - 1 \right) f(z_i, z_j) \\
&= \int_0^1 dp\, \nu^{(k)}(p)\, \left[f(p(z_i+z_j), (1-p)(z_i+z_j)) - f(z_i, z_j) \right]
\end{aligned}
$$

$$
Z_i, Z_j \sim \mathrm{Gamma}\,(2k, \theta) \quad \text{i.i.d.} \quad \Longrightarrow \quad P = \frac{Z_i}{Z_i + Z_j} \sim \mathrm{Beta}\,(2k, 2k)
$$

$$
\nu^{(k)}(p) = \frac{p^{2k-1}(1-p)^{2k-1}}{B(2k, 2k)}
$$

For $k = \frac{1}{2}$: uniform redistribution, original KMP

Asymmetric KMP-like processes

- $AKMP(\sigma, k)$

$$L_{i,j}^{AKMP(\sigma,k)} f(z_i, z_j) := \lim_{t\to\infty} \left(e^{tL_{i,j}^{ABEP(\sigma,k)}} - 1 \right) f(z_i, z_j)$$

$$= \int_0^1 dp\, \nu_\sigma^{(k)}(p|z_i + z_j)\, [f(p(z_i + z_j), (1-p)(z_i + z_j)) - f(z_i, z_j)]$$

with

$$\nu_\sigma^{(k)}(p|E) = \frac{1}{\mathcal{N}_{\sigma,k}} e^{2\sigma pE} \left\{ \left(e^{2\sigma pE} - 1 \right) \left(1 - e^{-2\sigma(1-p)E} \right) \right\}^{2k-1}$$

- Th-ASIP(k)

$$(n, m) \to (R_q, n + m - R_q)$$

with R_q a q-deformed Beta-Binomial $(n + m, 2k, 2k)$

Duality relations

Duality between $ABEP(\sigma, k)$ and $SIP(k)$

Theorem [Carinci,G., Redig, Sasamoto (2016)]

- For every σ (including 0^+), the process $\{z(t)\}_{t\geq 0}$ with generator $L^{ABEP(\sigma,k)}$ and the process $\{\eta(t)\}_{t\geq 0}$ with generator $L^{SIP(k)}$ are dual on

$$D(z,\xi) = \prod_{i=1}^{L} \frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} \left(\frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma} \right)^{\xi_i}$$

with

$$E_i(z) = \sum_{l=i}^{L} z_l \qquad E_{L+1}(z) = 0$$

- Same duality holds between $AKMP(\sigma, k)$ and Th-$SIP(k)$

the symmetric case $\sigma = 0^+$

$$L = \sum_{i=1}^{L-1} \Big(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + 2k^2 \Big)$$

Two representations of the $\mathfrak{su}(1,1)$ algebra:

$$\begin{cases} K_i^+ e^{(\eta_i)} = (\eta_i + 2k)\, e^{(\eta_i+1)} \\ K_i^- e^{(\eta_i)} = \eta_i e^{(\eta_i-1)} \\ K_i^o e^{(\eta_i)} = (\eta_i + 4k)\, e^{(\eta_i)} \end{cases} \qquad \begin{cases} \mathcal{K}_i^+ = z_i \\ \mathcal{K}_i^- = z_i\, \partial_{z_i}^2 + 2k\partial_{z_i} \\ \mathcal{K}_i^o = z_i\, \partial_{z_i} + k \end{cases}$$

$L = L^{SIP(k)}$ $\qquad\qquad$ $L = L^{BEP(k)}$

$$\frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} z_i^{\xi_i}$$

Duality fct $\equiv$ intertwiner

the asymmetric case $\sigma \neq 0$

- The $ABEP(\sigma, k)$ can be mapped to $BEP(k)$ via the non-local transformation

$$z \mapsto g(z) \qquad g_i(z) := \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma}$$

Equivalently

$$L^{ABEP(\sigma,k)} = C_g \circ L^{BEP(k)} \circ C_{g^{-1}}$$

with

$$(C_g f)(z) = (f \circ g)(z)$$

- Therefore, despite the asymmetry, the symmetry group of $ABEP(\sigma, k)$ is the same as for $BEP(k)$, namely $\mathfrak{su}(1,1)$. The representation is a non-local conjugation of the differential operator representation.

Self-duality of ASIP(q,k)

Theorem [Carinci,G., Redig, Sasamoto (2016)]

The ASIP(q,k) is self-dual on

$$D(\eta,\xi^{(\ell_1,\dots,\ell_n)}) = \frac{q^{-4k\sum_{m=1}^{n}\ell_m - n^2}}{(q^{2k}-q^{-2k})^n}\cdot\prod_{m=1}^{n}\left(q^{2N_{\ell_m}(\eta)} - q^{2N_{\ell_{m+1}}(\eta)}\right)$$

where $\xi^{(\ell_1,\dots,\ell_n)}$ is the configuration with n particles at sites $\ell_1,\dots,\ell_n$ and

$$N_i(\eta) := \sum_{k=i}^{L}\eta_k$$

- It follows from the explicit knowledge of the reversible measure and from an exponential symmetry.

Applications

1. Bulk driven: current
2. Boundary driven: correlation functions

Example 1: bulk-driven ABEP(σ, k)

Definition

The current $J_i(t)$ during the time interval $[0, t]$ across the bond $(i-1, i)$ is defined as:

$$J_i(t) = E_i(z(t)) - E_i(z(0))$$

where

$$E_i(z) := \sum_{k \geq i} z_k$$

Remark: let $\xi^{(i)}$ be the configuration with 1 dual particle:

$$\xi^{(i)}_m = \begin{cases} 1 & \text{if } m = i \\ 0 & \text{otherwise} \end{cases}$$

then

$$D(z, \xi^{(i)}) = \left(\frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{4k\sigma} \right)$$

Current of bulk-driven ABEP(σ, k)

Using duality between ABEP(σ, k) and SIP(k)

$$\mathbb{E}_z(e^{-2\sigma J_i(t)}) = e^{-4kt} \sum_{n \in \mathbb{Z}} e^{-2\sigma(E_n(z) - E_i(z))} I_{|n-i|}(4kt)$$

$I_n(t)$ modified Bessel function.

The computation requires a single dual SIP particle, which is a simple symmetric random walk X_t jumping at rate $2k$

$$\mathbb{P}_i(X_t = n) = e^{-4kt} I_{|n-i|}(4kt)$$

Example 2: Brownian Momentum Process with reservoirs

Generator

$$L = L_{left} + \sum_{i=1}^{N-1} L_{i,i+1}^{BMP} + L_{right}$$

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$$L^{BMP}_{i,i+1} = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 \qquad \text{Bulk}$$

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$T_L = T_R = T$: equilibrium Gibbs measure $\nu_T = \otimes_{i=1}^{N} \mathcal{N}(0, T)$.
$T_L \neq T_R$: non-equilibrium steady state

SIP with absorbing boundaries

Configurations $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

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$$L^{SIP}_{i,i+1} f(\xi) = \sum_{i=1}^{N-1} \xi_i(\xi_{i+1} + \frac{1}{2})[f(\xi^{i,i+1}) - f(\xi)] \qquad \text{Bulk}$$

$$+ \xi_{i+1}(\xi_i + \frac{1}{2})[f(\xi^{i+1,i}) - f(\xi)]$$

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$$L_{left} f(\xi) = 2\xi_1(f(\xi^{1,0}) - f(\xi)) \qquad \text{Boundary}$$

Duality and correlation functions

$$\mathbb{E}_x[D(x(t),\xi)] = \mathbb{E}^{dual}_\xi[D(x,\xi(t))]$$

with

$$D(x,\xi) = T_L^{\xi_0} \left(\prod_{i=1}^N x_i^{2\xi_i} \frac{\Gamma(\frac{1}{2})}{\Gamma(\xi_i + \frac{1}{2})} \right) T_R^{\xi_{N+1}}$$

As a consequence

$$\int D(x,\xi)\mu(dx) = \sum_{n+m=|\xi|} T_L^a T_R^b \; \mathbb{P}_\xi(n,m)$$

$$\mathbb{P}_\xi(n,m) = \mathbb{P}(n \text{ particles will exit left, } m \text{ exit right} \mid \xi(0) = \xi)$$

μ stationary measure, $|\xi| = \sum_{i=1}^N \xi_i$

Temperature profile 1d linear chain

$$\vec{\xi} = (0,\ldots,0,1,0,\ldots,0) \Rightarrow D(x,\vec{\xi}) = x_i^2$$

site $i \nearrow$ $\Rightarrow$ 1 SIP(1) walker $(X_t)_{t\geq 0}$ with $X_0 = i$

$$\mathbb{E}\left(x_i^2\right) = T_L\, \mathbb{P}_i(X_\infty = 0) + T_R\, \mathbb{P}_i(X_\infty = N+1)$$

$$\mathbb{E}(x_i^2) = T_L + \left(\frac{T_R - T_L}{N+1}\right) i$$ *Linear profile*

$$\mathbb{E}(J) = \mathbb{E}(x_{i+1}^2 - x_i^2) = \frac{T_R - T_L}{N+1}$$ *Fourier's law*

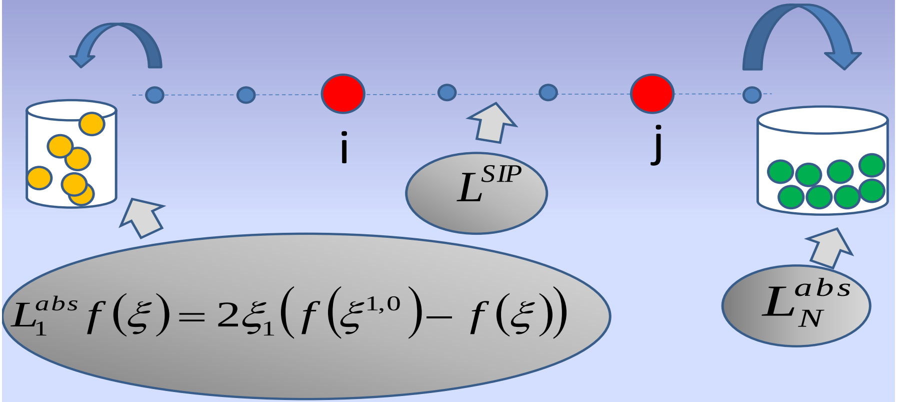
Energy covariance 1d linear chain

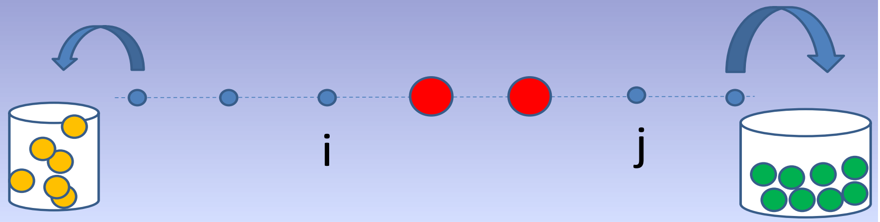
If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \quad \Rightarrow \quad D(x, \vec{\xi}) = x_i^2 x_j^2$

site i $\nearrow$ site j $\nearrow$

In the dual process we initialize two SIP walkers $(X_t, Y_t)_{t \geq 0}$ with $(X_0, Y_0) = (i, j)$

Inclusion Process with absorbing reservoirs

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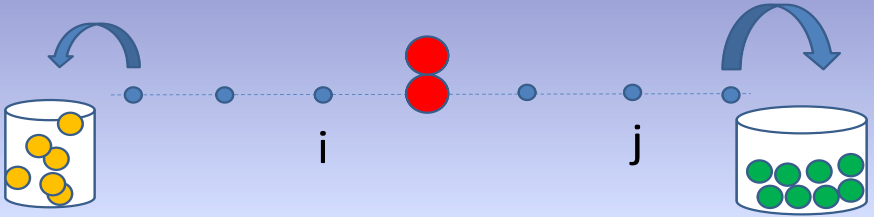
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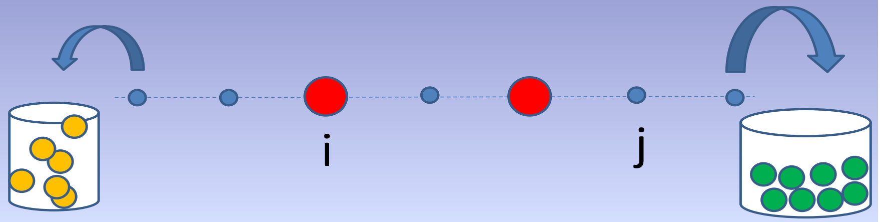
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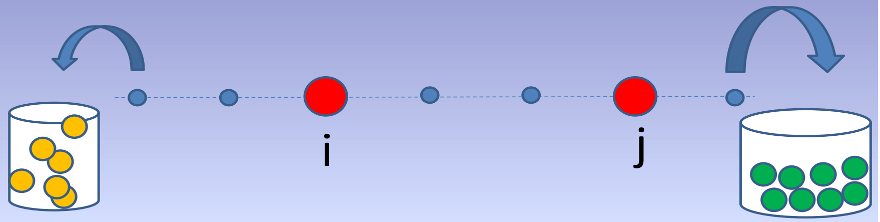
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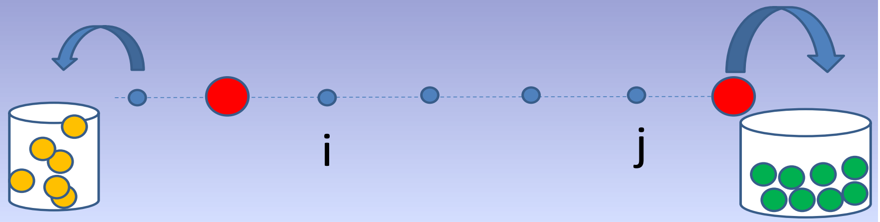
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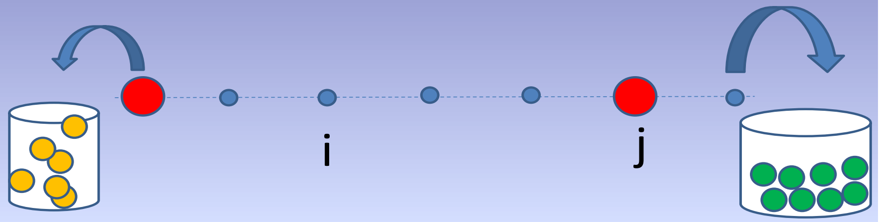
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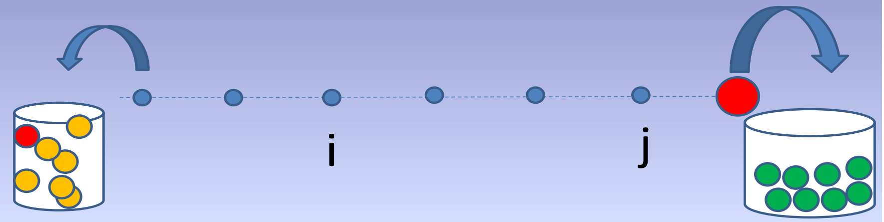
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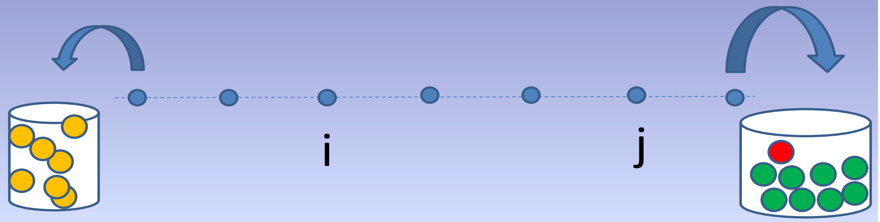
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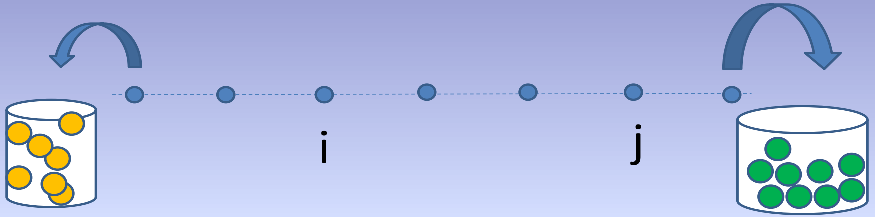

$$\mathbf{E}\left(x_i^2 x_j^2\right) = T_L^2\,\mathrm{P}\left(\begin{smallmatrix}\color{orange}\bullet\\ \color{orange}\bullet\end{smallmatrix}\right) + T_R^2\,\mathrm{P}\left(\begin{smallmatrix}\color{green}\bullet\\ \color{green}\bullet\end{smallmatrix}\right) + T_L\,T_R\left(\mathrm{P}\left(\begin{smallmatrix}\color{orange}\bullet\\ \color{green}\bullet\end{smallmatrix}\right) + \mathrm{P}\left(\begin{smallmatrix}\color{green}\bullet\\ \color{orange}\bullet\end{smallmatrix}\right)\right)$$

Energy covariance

$$\mathbb{E}\left(x_i^2 x_j^2\right) - \mathbb{E}\left(x_i^2\right)\mathbb{E}\left(x_j^2\right) = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2$$

- **Remark 1**: up to a sign, same covariance as in the boundary driven Symmetric Simple Exclusion Process.

- **Remark 2**: Long range correlations

$$\lim_{N\to\infty} N\,Cov(x^2_{\alpha_1 N}, x^2_{\alpha_2 N}) = 2\alpha_1(1-\alpha_2)(T_R - T_L)^2$$

Summary

- Two key ingredients for stochastic duality:
  symmetries and representation theory
- Constructive Lie-algebraic approach to duality theory
- Examples:
  - $\mathfrak{su}_q(1,1)$ algebra
  - $\mathfrak{su}_q(2)$ algebra gives ASEP(q,j) with duality
  - higher rank algebras give multispecies interacting particle systems with duality
- Scaling to KPZ universality class of ABEP and ASIP ?

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Thank you for your attention