An algebraic approach to stochastic duality

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Introduction
Non-equilibrium in 1d: particle transport

- asymmetry

- density reservoirs
Non-equilibrium in 1d: particle transport

- asymmetry

- density reservoirs

- current reservoirs

Carinci, De Masi, G., Presutti (2016)
Non-equilibrium in 1d: energy transport

Fourier law \[ J = \kappa \nabla T \]

KMP model (1982)

Energies at every site: \[ z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N \]

\[ L^{KMP} f(z) = \sum_{i=1}^{N} \int_0^1 dp \left[ f(z_1, \ldots, pz_i + z_{i+1}, (1-p)(z_i + z_{i+1}), \ldots, z_N) - f(z) \right] \]

\[ \rightarrow \text{conductivity } 0 < \kappa < \infty; \text{ model solved by duality.} \]
**Stochastic Duality**

**Definition**

$(\eta_t)_{t \geq 0}$ Markov process on $\Omega$ with generator $L$,

$(\xi_t)_{t \geq 0}$ Markov process on $\Omega_{dual}$ with generator $L_{dual}$

$\xi_t$ is **dual** to $\eta_t$ with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$E_\eta(D(\eta_t, \xi)) = E_\xi(D(\eta, \xi_t)) \quad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

$\eta_t$ is **self-dual** if $L_{dual} = L$.

In terms of generators:

$$LD(\cdot, \xi)(\eta) = L_{dual}D(\eta, \cdot)(\xi)$$
Duality for Markov chain

Assume state spaces $\Omega, \Omega_{dual}$ are countable sets, then the Markov generator $L$ is a matrix $L(\eta, \eta')$ s.t.

$$L(\eta, \eta') \geq 0 \quad \text{if} \quad \eta \neq \eta', \quad \sum_{\eta' \in \Omega} L(\eta, \eta') = 0$$

$$LD(\cdot, \xi)(\eta) = L_{dual}D(\eta, \cdot)(\xi)$$

amounts to

$$LD = DL_{dual}^T$$

Indeed

$$\sum_{\eta'} L(\eta, \eta')D(\eta', \xi) = \sum_{\xi'} L_{dual}(\xi, \xi')D(\eta, \xi')$$
Duality

A useful tool

- interacting particle systems [Spitzer, Ligget]
- hydrodynamic limit [Presutti, De Masi]
- KPZ scaling limits [Schütz, Spohn]
- population genetics [Kingman]
...

- the dual process is simpler: “from many to few”.

Questions

- how to find a dual process and a duality function?
- how to construct processes with duality?
  * E.g.: duality for asymmetric partial exclusion process?
  * E.g.: what is the right asymmetric version of KMP?
Lie algebraic approach to duality theory
Algebraic approach

- Write the Markov generator in **abstract form**, i.e. as an element of a universal enveloping algebra of a Lie algebra.

1. Duality is related to a **change of representation**. Duality functions are the **intertwiners**.

2. Dualities are associated to **symmetries**. Acting with a symmetry on a duality funct. yields another duality funct.

Conversely, the approach can be turned into a constructive method.
1. Change of representation
Example: Wright-Fisher diffusion and Kingman coalescence

Wright-Fisher diffusion \((X(t))_{t \geq 0}\) with state space \([0, 1]\\)

\[
 L^{WF} f(x) = \frac{1}{2} x(1 - x) \frac{\partial^2 f}{\partial x^2}(x)
\]

\(N(t) = \) number of blocks in the Kingman coalescence at time \(t \geq 0\\)

\[
 (L^{King} f)(n) = \frac{n(n - 1)}{2} (f(n - 1) - f(n))
\]
The process \( \{X(t)\}_{t \geq 0} \) with generator \( L^{WF} \) and the process \( \{N(t)\}_{t \geq 0} \) with generator \( L^{King} \) are dual on \( D(x, n) = x^n \), i.e.

\[
E^{WF}_x(X(t)^n) = E^{King}_n(x^{N(t)})
\]

Indeed:

\[
L^{WF} D(\cdot, n)(x) = \frac{1}{2} x(1 - x) \frac{\partial^2}{\partial x^2} x^n
\]

\[
= \frac{n(n - 1)}{2} (x^{n-1} - x^n)
\]

\[
= \frac{n(n - 1)}{2} (D(x, n - 1) - D(x, n))
\]

\[
= L^{King} D(x, \cdot)(n)
\]
Duality Wright-Fisher / Kingman: algebraic approach

Two representations of the Heisenberg algebra: \([a, a^\dagger] = 1\)

\[
\begin{align*}
  a^\dagger &= x \\
  a &= \frac{d}{dx}
\end{align*}
\]

\[
\begin{align*}
  a^\dagger e^{(n)} &= e^{(n+1)} \\
  a e^{(n)} &= ne^{(n-1)}
\end{align*}
\]

The abstract element \(L = \frac{1}{2}a^\dagger(1 - a^\dagger)(a)^2\)

\(L = L^{WF}\) in the first representation

\(L^T = L^{King}\) in the second representation

Duality fct. \(D(x, n) = x^n\) is the intertwiner:

\[
xD(x, n) = D(x, n + 1) \quad \quad \frac{d}{dx}D(x, n) = nD(x, n - 1)
\]
2. Symmetries

\( S \): symmetry of the original Markov generator, i.e. \([L, S] = 0\)

\( d \): duality function between \( L \) and \( L_{\text{dual}} \)

\[ \rightarrow D = Sd \] is also duality function

Indeed

\[ LD = LSd = SLd = SdL_{\text{dual}}^T = DL_{\text{dual}}^T \]
“Cheap” self-duality

Let $\mu$ a reversible measure: $\mu(\eta)L(\eta, \xi) = \mu(\xi)L(\xi, \eta)$

A cheap (i.e. diagonal) self-duality is

$$d(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta, \xi}$$

Indeed

$$\frac{L(\eta, \xi)}{\mu(\xi)} = \sum_{\eta'} L(\eta, \eta')d(\eta', \xi) = \sum_{\xi'} L(\xi, \xi')d(\eta, \xi') = \frac{L(\xi, \eta)}{\mu(\eta)}$$
“Cheap” duality

Let $\mu$ a invariant measure: $\sum_\eta \mu(\eta)L(\eta, \xi) = 0$

Let the dual process $(\xi_t)_{t \geq 0}$ be the time-reversed process of $(\eta_t)_{t \geq 0}$

$L_{dual}(\xi, \xi') = \mu(\xi)^{-1}L(\xi', \xi)\mu(\xi')$

A cheap (i.e. diagonal) duality is

$d(\eta, \xi) = \frac{1}{\mu(\eta)}\delta_{\eta, \xi}$

Indeed

$\frac{L(\eta, \xi)}{\mu(\xi)} = \sum_{\eta'} L(\eta, \eta')d(\eta', \xi) = \sum_{\xi'} L_{dual}(\xi, \xi')d(\eta, \xi') = \frac{L(\eta, \xi)}{\mu(\xi)}$
Construction of Markov generators with algebraic structure

i) (**Lie Algebra**): Start from a Lie algebra \( g \).

ii) (**Casimir**): Pick an element in the center of \( g \), e.g. the Casimir \( C \).

iii) (**Co-product**): Consider a co-product \( \Delta : g \to g \otimes g \) making the algebra a bialgebra and conserving the commutation relations.

iv) (**Quantum Hamiltonian**): Compute \( H = \Delta(C) \).

v) (**Symmetries**): \( S = \Delta(X) \) with \( X \in g \) is a symmetry of \( H \):

\[
[H, S] = [\Delta(C), \Delta(X)] = \Delta([C, X]) = \Delta(0) = 0.
\]

vi) (**Markov generator**): Apply a “ground state transformation” to turn \( H \) into a Markov generator \( L \).
Quantum $\mathfrak{su}_q(1, 1)$ algebra
For $q \in (0, 1)$ and $n \in \mathbb{N}_0$ introduce the $q$-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Remark: $\lim_{q \to 1} [n]_q = n$.

The first $q$-number's are:

$$[0]_q = 0, \quad [1]_q = 1, \quad [2]_q = q + q^{-1}, \quad [3]_q = q^2 + 1 + q^{-2}, \ldots$$
Quantum Lie algebra $\mathfrak{su}_q(1, 1)$

For $q \in (0, 1)$ consider the algebra with generators $K^+, K^-, K^0$

$$[K^0, K^\pm] = \pm K^\pm, \quad [K^+, K^-] = -[2K^0]_q$$

where

$$[2K^0]_q := \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

Irreducible representations are infinite dimensional. E.g., for $n \in \mathbb{N}$

$$\begin{cases} 
K^+ e^{(n)} &= \sqrt{[n + 2k]_q[n + 1]_q} e^{(n+1)} \\
K^- e^{(n)} &= \sqrt{[n]_q[n + 2k - 1]_q} e^{(n-1)} \\
K^0 e^{(n)} &= (n + k) e^{(n)}
\end{cases}$$

Casimir element

$$C = [K^0]_q[K^0 - 1]_q - K^+ K^-$$

In this representation

$$C e^{(n)} = [k]_q[k - 1]_q e^{(n)} \quad k \in \mathbb{R}_+$$
Co-product

Co-product $\Delta : U_q(\mathfrak{su}(1, 1)) \to U_q(\mathfrak{su}(1, 1)) \otimes^2$

$$\Delta(K^\pm) = K^\pm \otimes q^{-K^0} + q^{K^0} \otimes K^\pm$$
$$\Delta(K^o) = K^o \otimes 1 + 1 \otimes K^o$$

The co-product is an isomorphism s.t.

$$[\Delta(K^o), \Delta(K^\pm)] = \pm \Delta(K^\pm) \quad [\Delta(K^+), \Delta(K^-)] = -[2\Delta(K^o)]_q$$

From co-associativity $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$

$\Delta^n : U_q(\mathfrak{su}(1, 1)) \to U_q(\mathfrak{su}(1, 1)) \otimes (n+1)$, i.e. for $n \geq 2$

$$\Delta^n(K^\pm) = \Delta^{n-1}(K^\pm) \otimes q^{-K^0_n} + q^{\Delta^{n-1}(K^0_i)} \otimes K^\pm_{n+1}$$
$$\Delta^n(K^o) = \Delta^{n-1}(K^o) \otimes 1 + 1 \otimes^n \otimes K^0_{n+1}$$
Quantum Hamiltonian

\[ \Delta(C_i) = q^{K_i^0} \left\{ K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1} \right\} q^{-K_{i+1}^0} \]
Quantum Hamiltonian

\[ \Delta(C_i) = q^{K_i^0} \left\{ K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1} \right\} q^{-K_{i+1}^0} \]

\[ B_i \otimes B_{i+1} = \frac{(q^k + q^{-k})(q^{k-1} + q^{-(k-1)})}{2(q - q^{-1})^2} \left( q^{K_i^0} - q^{-K_i^0} \right) \otimes \left( q^{K_{i+1}^0} - q^{-K_{i+1}^0} \right) + \frac{(q^k - q^{-k})(q^{k-1} - q^{-(k-1)})}{2(q - q^{-1})^2} \left( q^{K_i^0} + q^{-K_i^0} \right) \otimes \left( q^{K_{i+1}^0} + q^{-K_{i+1}^0} \right) \]
Quantum Hamiltonian

\[ \Delta(C_i) = q^{K_i^0} \left\{ K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1} \right\} q^{-K_{i+1}^0} \]

\[ H := \sum_{i=1}^{L-1} \left( 1^{\otimes (i-1)} \otimes \Delta(C_i) \otimes 1^{\otimes (L-i-1)} + c_{q,k} 1^{\otimes L} \right) \]

\[ c_{q,k} = \frac{(q^{2k} - q^{-2k})(q^{2k-1} - q^{-(2k-1)})}{(q - q^{-1})^2} \quad \text{s.t.} \quad H \cdot \left( \otimes_{i=1}^{L} e_i^{(0)} \right) = 0 \]
Markov processes

with $\mathfrak{su}_q(1, 1)$ symmetry
Symmetries of $H$

Lemma

$$K^\pm := \sum_{i=1}^{L} q^{K_0^1} \otimes \cdots \otimes q^{K_{i-1}^0} \otimes K_i^\pm \otimes q^{-K_{i+1}^0} \otimes \cdots \otimes q^{-K_L^0}$$

$$K^0 := \sum_{i=1}^{L} 1 \otimes \cdots \otimes 1 \otimes K_i^0 \otimes 1 \otimes \cdots \otimes 1,$$

are symmetries of $H$.

Proof. Let $a \in \{+, -, 0\}$, then $K^a = \Delta^{L-1}(K_1^a)$

For $L = 2$:

$$[H, K^a] = [\Delta(C_1), \Delta(K_1^a)] = \Delta([C_1, K_1^a]) = \Delta(0) = 0$$

For $L > 2$:

induction.
Ground state transformation

Lemma

Let $H$ be a matrix with $H(\eta, \eta') \geq 0$ if $\eta \neq \eta'$.
Suppose $g$ is a positive ground state, i.e. $Hg = 0$ and $g(\eta) > 0$.
Let $G$ be the matrix $G(\eta, \eta') = g(\eta)\delta_{\eta, \eta'}$. Then

$$L = G^{-1}HG$$

is a Markov generator.

Indeed

$$L(\eta, \eta') = \frac{H(\eta, \eta')g(\eta')}{g(\eta)}$$

Therefore

$$L(\eta, \eta') \geq 0 \text{ if } \eta \neq \eta'$$

$$\sum_{\eta'} L(\eta, \eta') = 0$$
Exponential symmetries

- $g^{(0)} = \bigotimes_{i=1}^{L} e^{(0)}_i$ is a ground state, i.e. $Hg^{(0)} = 0$.
- For every symmetry $[H, S] = 0$ another ground state is $g = Sg^{(0)}$.
- The exponential symmetry

$$S^+ = \exp_{q^2}(E) = \sum_{n \geq 0} \frac{(E)^n}{[n]_q q!} q^{-n(n-1)/2}$$

with

$$E = \Delta^{(L-1)}(q^{K_0}) \cdot \Delta^{(L-1)}(K_1^+)$$

gives a positive ground state

$$g = S^+ g^{(0)} = \sum_{\ell_1, \ldots, \ell_L} \bigotimes_{i=1}^{L} \left( \sqrt{\binom{\ell_i + 2k - 1}{\ell_i}} \right)_q q^{\ell_i (1-k+2ki)} e^{(\ell_i)}$$

- Remark

$$\lim_{q \to 1} S^+ = e^{\sum_i K_i^+} = \prod_i e^{K_i^+}$$
(1) Asymmetric Inclusion Process: ASIP\( (q,k) \)

For \( k \in \mathbb{R}_+ \) the interacting particle system ASIP\( (q, k) \) on \([1, L] \cap \mathbb{Z}\) with state space \( \mathbb{N}^L \) is defined by

\[
(L^{\text{ASIP}}(q,k)f)(\eta) = \sum_{i=1}^{L-1} (L_{i,i+1} f)(\eta)
\]

with

\[
(L_{i,i+1} f)(\eta) = q^{n_i-\eta_{i+1}+(2k-1)[\eta_i]} q^{2k+\eta_{i+1}} q \left[ f(\eta^i_{i+1}) - f(\eta) \right] + q^{n_i-\eta_{i+1}-(2k-1)[\eta_i]} q^{2k+\eta_{i+1}} q \left[ f(\eta^i_{i+1}) - f(\eta) \right]
\]

\[\Rightarrow q \to 1 \Rightarrow \text{SIP}(k): \text{symmetric inclusion}\]
jump right at rate \( \eta_i(2k + \eta_{i+1}) \), jump left at rate \( (2k + \eta_i)\eta_{i+1} \)
Properties of ASIP($q,k$)

- The ASIP($q, k$) on $[1, L] \cap \mathbb{Z}$ has a family (labeled by $\alpha > 0$) of inhomogeneous reversible product measures with marginals

\[
P_{\alpha}(\eta_i = x) = \frac{\alpha^x}{Z_{i,\alpha}} \binom{x + 2k - 1}{x} q^{4kix}
\]

- $q \to 1$: the reversible measure is homogeneous and product of Negative Binomials $(2k, \alpha)$
(2) Asymmetric Brownian Energy Process: $ABEP(\sigma, k)$

For $\sigma > 0$, let $(\eta^{(\epsilon)}(t))_{t \geq 0}$ be the $ASIP(1 - \epsilon \sigma, k)$ process initialized with $\epsilon^{-1}$ particles. The scaling limit (weak asymmetry)

$$z_i(t) := \lim_{\epsilon \to 0} \epsilon \eta_i^{(\epsilon)}(t)$$

is the diffusion $ABEP(\sigma, k)$ with generator $L^{ABEP(\sigma,k)} = \sum_{i=1}^{L-1} L_{i,i+1}$

$$L_{i,i+1} = -\frac{1}{2\sigma} \left\{ (1 - e^{-2\sigma z_i})(e^{2\sigma z_{i+1}} - 1) + 2k \left( 2 - e^{-2\sigma z_i} - e^{2\sigma z_{i+1}} \right) \right\} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)$$

$$+ \frac{1}{4\sigma^2} \left( 1 - e^{-2\sigma z_i} \right) (e^{2\sigma z_{i+1}} - 1) \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2$$

Remark: $L_{i,i+1}$ conserves $z_i(t) + z_{i+1}(t)$
Properties of ABEP($\sigma, k$)

- $\sigma \neq 0$
  - the process is truly asymmetric, i.e. on the 1-d torus it carries a non-zero current.
  - on the half-line it has inhomogeneous reversible product measures (labeled by $\gamma > -4\sigma k$) with marginal density
    \[
    \mu(dz_i) = \frac{1}{Z_{i,\alpha}} (1 - e^{-2\sigma z_i})^{(2k-1)} e^{-(4\sigma ki + \gamma)z_i} dz_i
    \]

- $\sigma \to 0^+$
  \[
  \mathcal{L}_{i,i+1} = -2k (z_i - z_{i+1}) \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right) + z_i z_{i+1} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2
  \]

  The reversible measures are given by product of i.i.d. $\text{Gamma}(2k; \gamma)$

  \[
  \mu(dz_i) = \frac{1}{\gamma^{2k} \Gamma(2k)} z_i^{(2k-1)} e^{-\gamma z_i} dz_i
  \]
(3) \textit{KMP}(k) process

Instantaneous \textit{thermalization} limit:

\[
L_{i,j}^{\text{KMP}(k)} f(z_i, z_j) := \lim_{t \to \infty} \left( e^{t L_{i,j}^{\text{BEP}(k)}} - 1 \right) f(z_i, z_j)
\]

\[
= \int_{0}^{1} dp \, \nu^{(k)}(p) \left[ f(p(z_i + z_j), (1 - p)(z_i + z_j)) - f(z_i, z_j) \right]
\]

\[Z_i, Z_j \sim \text{Gamma}(2k, \theta) \quad \text{i.i.d.} \quad \implies \quad P = \frac{Z_i}{Z_i + Z_j} \sim \text{Beta}(2k, 2k)\]

\[
\nu^{(k)}(p) = \frac{p^{2k-1}(1 - p)^{2k-1}}{B(2k, 2k)}
\]

For \( k = \frac{1}{2} \): uniform redistribution, original KMP
Asymmetric KMP-like processes

\[ L_{i,j}^{AKMP(\sigma, k)} f(z_i, z_j) := \lim_{t \to \infty} \left( e^{tL_{i,j}^{ABEP(\sigma, k)}} - 1 \right) f(z_i, z_j) \]

\[ = \int_0^1 dp \ \nu^{(k)}_{\sigma}(p|z_i + z_j) \left[ f(p(z_i + z_j), (1 - p)(z_i + z_j)) - f(z_i, z_j) \right] \]

with

\[ \nu^{(k)}_{\sigma}(p|E) = \frac{1}{\mathcal{N}_{\sigma, k}} e^{2\sigma p E} \left\{ \left( e^{2\sigma p E} - 1 \right) \left( 1 - e^{-2\sigma (1-p) E} \right) \right\}^{2k-1} \]

\[ \text{Th-ASIP}(k) \]

\[ (n, m) \to (R_q, n + m - R_q) \]

with \( R_q \) a q-deformed Beta-Binomial \((n + m, 2k, 2k)\)
Duality relations
Duality between $\text{ABEP}(\sigma, k)$ and $\text{SIP}(k)$

Theorem [Carinci,G., Redig, Sasamoto (2016)]

- For every $\sigma$ (including $0^+$), the process $\{z(t)\}_{t \geq 0}$ with generator $L^{\text{ABEP}(\sigma, k)}$ and the process $\{\eta(t)\}_{t \geq 0}$ with generator $L^{\text{SIP}(k)}$ are dual on

$$D(z, \xi) = \prod_{i=1}^{L} \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} \left( \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma} \right)^{\xi_i}$$

with

$$E_i(z) = \sum_{l=i}^{L} z_l \quad E_{L+1}(z) = 0$$

- Same duality holds between $\text{AKMP}(\sigma, k)$ and Th-$\text{SIP}(k)$
the symmetric case $\sigma = 0^+$

$$L = \sum_{i=1}^{L-1} \left( K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^0 K_{i+1}^0 + 2k^2 \right)$$

Two representations of the $\mathfrak{su}(1,1)$ algebra:

$$\begin{cases} 
K_i^+ e^{(\eta_i)} = (\eta_i + 2k) e^{(\eta_i+1)} \\
K_i^- e^{(\eta_i)} = \eta_i e^{(\eta_i-1)} \\
K_i^0 e^{(\eta_i)} = (\eta_i + 4k) e^{(\eta_i)} 
\end{cases}$$

$$\begin{cases} 
\mathcal{K}_i^+ = z_i \\
\mathcal{K}_i^- = z_i \partial^2_{z_i} + 2k \partial z_i \\
\mathcal{K}_i^0 = z_i \partial z_i + k 
\end{cases}$$

$$L = L^{\text{SIP}(k)}$$

$$L = L^{\text{BEP}(k)}$$

$$\frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} z_i^{\xi_i}$$

Duality fct $\equiv$ intertwiner
the asymmetric case $\sigma \neq 0$

- The $ABEP(\sigma, k)$ can be mapped to $BEP(k)$ via the non-local transformation

$$z \mapsto g(z) \quad g_i(z) := \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma}$$

Equivalently

$$L^{ABEP(\sigma,k)} = C_g \circ L^{BEP(k)} \circ C_{g^{-1}}$$

with

$$(C_g f)(z) = (f \circ g)(z)$$

- Therefore, despite the asymmetry, the symmetry group of $ABEP(\sigma, k)$ is the same as for $BEP(k)$, namely $su(1, 1)$. The representation is a non-local conjugation of the differential operator representation.
Self-duality of ASIP($q, k$)

Theorem [Carinci,G., Redig, Sasamoto (2016)]

The ASIP($q, k$) is self-dual on

$$D(\eta, \xi^{(\ell_1, \ldots, \ell_n)}) = \frac{q^{-4k} \sum_{m=1}^{n} \ell_m - n^2}{(q^{2k} - q^{-2k})^n} \cdot \prod_{m=1}^{n} (q^{2N_{\ell_m}(\eta)} - q^{2N_{\ell_{m+1}}(\eta)})$$

where $\xi^{(\ell_1, \ldots, \ell_n)}$ is the configuration with $n$ particles at sites $\ell_1, \ldots, \ell_n$ and

$$N_i(\eta) := \sum_{k=i}^{L} \eta_k$$

- It follows from the explicit knowledge of the reversible measure and from an exponential symmetry.
Applications

1. Bulk driven: current

2. Boundary driven: correlation functions
Example 1: bulk-driven ABEP($\sigma, k$)

**Definition**
The current $J_i(t)$ during the time interval $[0, t]$ across the bond $(i - 1, i)$ is defined as:

$$J_i(t) = E_i(z(t)) - E_i(z(0))$$

where

$$E_i(z) := \sum_{k \geq i} z_k$$

Remark: let $\xi^{(i)}$ be the configuration with 1 dual particle:

$$\xi^{(i)}_m = \begin{cases} 
1 & \text{if } m = i \\
0 & \text{otherwise}
\end{cases}$$

then

$$D(z, \xi^{(i)}) = \left( \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{4k\sigma} \right)$$
Current of bulk-driven ABEP($\sigma, k$)

Using duality between ABEP($\sigma, k$) and SIP($k$)

\[
\mathbb{E}_z(e^{-2\sigma J_i(t)}) = e^{-4kt} \sum_{n \in \mathbb{Z}} e^{-2\sigma (E_n(z) - E_i(z))} l_{|n-i|}(4kt)
\]

$l_n(t)$ modified Bessel function.

The computation requires a single dual SIP particle, which is a simple symmetric random walk $X_t$ jumping at rate $2k$

\[
\mathbb{P}_i(X_t = n) = e^{-4kt} l_{|n-i|}(4kt)
\]
Example 2: Brownian Momentum Process with reservoirs

Generator

\[ L = L_{left} + \sum_{i=1}^{N-1} L_{i,i+1} + L_{right} \]
Example 2: Brownian Momentum Process with reservoirs

Generator

\[ L = L_{\text{left}} + \sum_{i=1}^{N-1} L_{i,i+1}^{\text{BMP}} + L_{\text{right}} \]

\[ L_{i,i+1}^{\text{BMP}} = \left( x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 \]

Bulk

Bulk

\[ L_{\text{left}} = T L \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \]

Reservoir

\[ T_L = T_R = T_{\nu} = \otimes_{i=1}^{N}(0, T) \]

\[ T_{\text{left}} \neq T_{\text{right}}: \text{non-equilibrium steady state} \]
Example 2: Brownian Momentum Process with reservoirs

Generator

\[
L = L_{\text{left}} + \sum_{i=1}^{N-1} L_{i,i+1}^{\text{BMP}} + L_{\text{right}}
\]

\[
L_{i,i+1}^{\text{BMP}} = \left( x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 \quad \text{Bulk}
\]

\[
L_{\text{left}} = T_L \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \quad \text{Reservoir}
\]
Example 2: Brownian Momentum Process with reservoirs

Generator

\[ L = L_{\text{left}} + \sum_{i=1}^{N-1} L_{i,i+1}^{\text{BMP}} + L_{\text{right}} \]

\[ L_{i,i+1}^{\text{BMP}} = \left( x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1} \frac{\partial}{\partial x_i} \right)^2 \quad \text{Bulk} \]

\[ L_{\text{left}} = T_L \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \quad \text{Reservoir} \]

\[ T_L = T_R = T: \text{ equilibrium Gibbs measure } \nu_T = \otimes_{i=1}^{N} \mathcal{N}(0, T). \]

\[ T_L \neq T_R: \text{ non-equilibrium steady state} \]
SIP with absorbing boundaries

Configurations $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{\text{dual}} = \mathbb{N}^{N+2}$
SIP with absorbing boundaries

Configurations $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Generator

$$L_{dual} = L_{left} + \sum_{i=1}^{N-1} L_{i,i+1}^{SIP} + L_{right}$$
SIP with absorbing boundaries

Configurations $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Generator

$$L_{dual} = L_{left} + \sum_{i=1}^{N-1} L_{i,i+1}^{SIP} + L_{right}$$

$$L_{i,i+1}^{SIP} f(\xi) = \sum_{i=1}^{N-1} \xi_i (\xi_{i+1} + \frac{1}{2}) [f(\xi_{i,i+1}) - f(\xi)]$$

Bulk

$$+ \xi_{i+1} (\xi_{i} + \frac{1}{2}) [f(\xi_{i+1,i}) - f(\xi)]$$
SIP with absorbing boundaries

Configurations $\xi = (\xi_0, \xi_1, \ldots, \xi_N, \xi_{N+1}) \in \Omega_{dual} = \mathbb{N}^{N+2}$

Generator

$$L_{dual} = L_{left} + \sum_{i=1}^{N-1} L_{i,i+1}^{SIP} + L_{right}$$

$$L_{i,i+1}^{SIP} f(\xi) = \sum_{i=1}^{N-1} \xi_i (\xi_{i+1} + \frac{1}{2}) [f(\xi_{i,i+1}) - f(\xi)] + \xi_{i+1} (\xi_i + \frac{1}{2}) [f(\xi_{i+1,i}) - f(\xi)]$$

$$L_{left} f(\xi) = 2\xi_1 (f(\xi_1,0) - f(\xi))$$
Duality and correlation functions

\[ \mathbb{E}_x[D(x(t), \xi)] = \mathbb{E}^{\text{dual}}_\xi[D(x, \xi(t))] \]

with

\[ D(x, \xi) = T^\xi_0 \left( \prod_{i=1}^N x_i^{2\xi_i} \frac{\Gamma(\frac{1}{2})}{\Gamma(\xi_i + \frac{1}{2})} \right) T^\xi_{N+1} \]

As a consequence

\[ \int D(x, \xi) \mu(dx) = \sum_{n+m=|\xi|} T^a_L T^b_R \mathbb{P}_\xi(n, m) \]

\[ \mathbb{P}_\xi(n, m) = \mathbb{P}(n \text{ particles will exit left, } m \text{ exit right } | \xi(0) = \xi) \]

\( \mu \) stationary measure, \( |\xi| = \sum_{i=1}^N \xi_i \)
Temperature profile 1d linear chain

\[ \vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0) \Rightarrow D(x, \vec{\xi}) = x_i^2 \]

\[ \text{site } i \uparrow \Rightarrow 1 \text{ SIP(1) walker } (X_t)_{t \geq 0} \text{ with } X_0 = i \]

\[ \mathbb{E}(x_i^2) = T_L \mathbb{P}_i(X_\infty = 0) + T_R \mathbb{P}_i(X_\infty = N + 1) \]

\[ \mathbb{E}(x_i^2) = T_L + \left( \frac{T_R - T_L}{N + 1} \right) i \quad \text{Linear profile} \]

\[ \mathbb{E}(J) = \mathbb{E}(x_{i+1}^2 - x_i^2) = \frac{T_R - T_L}{N + 1} \quad \text{Fourier's law} \]
Energy covariance 1d linear chain

If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ \implies D(x, \vec{\xi}) = x_i^2 x_j^2$

In the dual process we initialize two SIP walkers $(X_t, Y_t)_{t \geq 0}$ with $(X_0, Y_0) = (i, j)$
Inclusion Process with absorbing reservoirs

\[ L_{i}^{abs} f(\xi) = 2\xi_1 \left( f(\xi^{1,0}) - f(\xi) \right) \]

\[ L_{N}^{abs} \]
\[ E(x_i^2 x_j^2) = T_L^2 P(\bullet) + T_R^2 P(\bullet) + T_L T_R (P(\bullet) + P(\bullet)) \]
Energy covariance

\[ \mathbb{E}(x_i^2 x_j^2) - \mathbb{E}(x_i^2) \mathbb{E}(x_j^2) = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2 \]

▶ **Remark 1**: up to a sign, same covariance as in the boundary driven Symmetric Simple Exclusion Process.

▶ **Remark 2**: Long range correlations

\[ \lim_{N \to \infty} N \text{Cov}(x_{\alpha_1 N}^2, x_{\alpha_2 N}^2) = 2\alpha_1(1 - \alpha_2)(T_R - T_L)^2 \]
Summary

- Two key ingredients for stochastic duality: symmetries and representation theory
- Constructive Lie-algebraic approach to duality theory
- Examples:
  - $\mathfrak{su}_q(1,1)$ algebra
  - $\mathfrak{su}_q(2)$ algebra gives ASEP$(q,j)$ with duality
  - higher rank algebras give multispecies interacting particle systems with duality
- Scaling to KPZ universality class of ABEP and ASIP?
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Thank you for your attention