Nested Bethe ansatz for orthogonal and symplectic open spin chains

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Historical timeline

- $\mathfrak{gl}_2$ closed chain (ABA) - Faddeev–Sklyanin–Takhtadjan'79
- $\mathfrak{gl}_N$ closed chain (NBA) - Kulish–Reshetikhin'81
- $\mathfrak{sp}_{2n}$ closed chain (NBA) - Reshetikhin'85
- $\mathfrak{so}_{2n}$ closed chain (NBA) - de Vega–Karowski'87
- $\mathfrak{gl}_2$ open chain (ABA) - Sklyanin'88
- $\mathfrak{gl}_N$ open chain (NBA) - Martin–Galleas'04; Belliard–Ragoucy'09
- $\mathfrak{osp}_{M|2n}$ open chain (Analytical BA) - Doikou et. al.'03
- $\mathfrak{so}_{2n}$ open chain (NBA) - Gombor–Palla'16

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Notation and definitions

- Throughout, $\pm$ will distinguish the orthogonal and symplectic cases.

$$\mathfrak{g}_{2n} = \begin{cases} \mathfrak{so}_{2n} & \text{with upper sign.} \\ \mathfrak{sp}_{2n} & \text{with lower sign.} \end{cases}$$

- The $\mathfrak{gl}_n$ invariant R-matrix (Yang'68),

$$R(u) := I - \frac{P}{u} \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n).$$

- The $\mathfrak{g}_{2n}$ invariant R-matrix (Zamolodchikov'78),

$$\mathbb{R}(u) := I - \frac{P}{u} - \frac{Q}{\kappa - u} \in \mathrm{End}(\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}),$$

where $Q = P^t$, $Q^2 = 2nQ$ and $\kappa = n \mp 1$.

The orthogonal/symplectic open spin chain

- The state space of the chain is given by,

$$M = L_1(\lambda_1) \otimes \cdots \otimes L_\ell(\lambda_\ell) \otimes M_{\ell+1}(\mu).$$

- Each $L_i(\lambda_i)$ is a highest weight $\mathfrak{g}_{2n}$ module of weight

$$\lambda_i = \begin{cases} (k_i, \underbrace{0, \ldots, 0}_{n-1}) & \text{for } \mathfrak{so}_{2n}, \\ (\underbrace{1, \ldots, 1}_{k_i}, \underbrace{0, \ldots, 0}_{n-k_i}) & \text{for } \mathfrak{sp}_{2n}. \end{cases}$$

- $M_{\ell+1}(\mu)$ is a one-dimensional vector space corresponding to one of two distinct diagonal boundary types

$$K = \begin{cases} \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{2q}, \underbrace{1, \ldots, 1}_{p}) & \mathfrak{g}_{2p} \oplus \mathfrak{g}_{2q} \\ \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{n}) & \mathfrak{gl}_n \end{cases}$$

The monodromy matrix

- The *double-row monodromy matrix* $S(u) \in \mathrm{End}(\mathbb{C}^{2n} \otimes M)$ is

$$S_a(u) \equiv \mathcal{L}_{a1}(u) \cdots \mathcal{L}_{a\ell}(u) K_a(u) \mathcal{L}^t_{a\ell}(\kappa - u) \cdots \mathcal{L}^t_{a1}(\kappa - u)$$

- Lax operators $\mathcal{L}_{ai}(u) \in \mathrm{End}(\mathbb{C}^{2n} \otimes L_i(\lambda_i))$ are constructed via fusion and satisfy

$$\mathbb{R}_{ab}(u-v) \mathcal{L}_{ai}(u) \mathcal{L}_{bi}(v) = \mathcal{L}_{bi}(v) \mathcal{L}_{ai}(u) \mathbb{R}_{ab}(u-v).$$

- Boundary Lax operator $K(u) \in \mathrm{End}(\mathbb{C}^{2n})$ is a diagonal matrix .
- The monodromy matrix $S(u)$ satisfies the *reflection equation*

$$\mathbb{R}_{ab}(u-v) S_a(u) \mathbb{R}_{ab}(u+v) S_b(v) = S_b(v) \mathbb{R}_{ab}(u+v) S_b(u) \mathbb{R}_{ab}(u-v).$$

Problem

Diagonalise $\tau(u) := \mathrm{tr}\, S(u)$ on the spin chain M.

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Diagonalise $\tau(u) := \mathrm{tr}\, S(u)$ on the spin chain M.

The symmetry relation

- The entries of $S(u)$ are not algebraically independent, which is in part summarised by the *symmetry relation* (Guay–Regelskis'16),

$$S^t(u) = \gamma\, S(\kappa - u) \pm \frac{S(u) - S(\kappa - u)}{2u - \kappa} + \frac{\mathrm{tr}(K(u))\, S(u) - \mathrm{tr}(S(u))}{2u - 2\kappa}.$$

  where $\gamma = +1$ or $\gamma = -1$, depending on the boundary type.

- Multiplication by a certain scalar factor $\boldsymbol{S}(u) = g(u)\, S(u)$ leads to a "boundary independent" symmetry relation

$$\boldsymbol{S}^t(u) = -\left(1 \pm \frac{1}{2u - \kappa}\right) \boldsymbol{S}(\kappa - u) \pm \frac{\boldsymbol{S}(u)}{2u - \kappa} - \frac{\mathrm{tr}\, \boldsymbol{S}(u)}{2u - 2\kappa}.$$

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Nesting procedure - $\mathfrak{gl}_n$ open spin chain

- In the $\mathfrak{gl}_n$ case,

$$S^{(\mathfrak{gl}_n)}(u) = \left(\begin{array}{c|c} a(u) & B(u) \\ \hline C(u) & D(u) \end{array}\right).$$

- As an $(n-1)\times(n-1)$ matrix of operators, $D(u)$ satisfies

$$R'_{ab}(u-v)D_a(u)R'_{ab}(u+v)D_b(v) = D_b(v)R'_{ab}(u+v)D_a(u)R'_{ab}(u-v).$$

- Creation operators $B_{a_i}(u_i)$ give rise to the Bethe vector:

$$\Phi(\boldsymbol{u}) = B_{a_1}(u_1)\cdots B_{a_m}(u_m)\cdot\Phi'_{a_1,\ldots,a_m}.$$

where $\boldsymbol{u} = (u_1,\ldots,u_m)$ and $a_1,\ldots,a_m$ label auxiliary spaces, each being a copy of $\mathbb{C}^n$, and $\Phi'_{a_1,\ldots,a_m}$ is a "nested" Bethe vector for the residual $\mathfrak{gl}_{n-1}$ open spin chain (Belliard–Ragoucy'09).

Nesting procedure - $\mathfrak{g}_{2n}$ open spin chain

- For $\mathfrak{g}_{2n}$, we split the matrix $\boldsymbol{S}(u)$ into four $n \times n$ submatrices:

$$\boldsymbol{S}(u) = \left(\begin{array}{c|c} A(u) & B(u) \\ \hline C(u) & D(u) \end{array}\right)$$

- As an $n \times n$ matrix of operators, $A(u)$ satisfies

$$\begin{aligned} R_{ab}(u-v)A_a(u)R_{ab}(u+v)A_b(v) = A_b(v)R_{ab}(u+v)A_a(u)R_{ab}(u-v) \\ +R_{ab}(u-v)\,B_a(u)\,U_{ab}(u+v)\,C_b(v) \\ +B_b(v)\,U_{ab}(u+v)\,C_a(u)\,R_{ab}(u-v), \end{aligned}$$

with the $\mathfrak{gl}_n$-invariant R-matrix $R(u)$, and $U(u) := -P/u - Q/(\kappa - u)$.

Top level of nesting - creation operator

- $B(u)$ matrix contains creation operators for the top-level excitations, that correspond to n^{th} root vectors of $\mathfrak{g}_{2n}$.
- We reinterpret $B(u)$ as a row vector in *two* auxiliary spaces,

$$\beta_{\tilde{a}a}(u) := \sum_{i,j=1}^{n} b_{n-i+1,j}(u) \otimes e_i^* \otimes e_j^* \in B(u) \otimes (\mathbb{C}^n)^* \otimes (\mathbb{C}^n)^*$$

Bethe vector with m top-level excitations,

$$\Psi(\boldsymbol{u}) = \left(\prod_{i=1}^{m} \beta_{\tilde{a}_i a_i}(u_i) \prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_i - u_j) \right) \cdot \Phi_{\tilde{a}_1 a_1, \ldots \tilde{a}_m a_m}.$$

where $\boldsymbol{u} = (u_1, \ldots, u_m)$ and $\Phi_{\tilde{a}_1 a_1, \ldots \tilde{a}_m a_m} \in (\mathbb{C}^n)^{\otimes 2m} \otimes M$.

Symmetry relation in block form

- In block form, the symmetry relation gives a linear relation between the A and D blocks of $\boldsymbol{S}(u)$,

$$D^t(u) = -\left(1 \pm \frac{1}{2u-\kappa}\right)A(\kappa-u) \pm \frac{A(u)}{2u-\kappa} - \left\{\frac{\operatorname{tr} A(u)}{2u-\kappa}\right\}^u,$$

where brace brackets denote symmetrisation

$$\left\{f(u)\right\}^u := f(u) + f(\kappa-u).$$

- In particular, the rescaled transfer matrix may be written in terms of the block A of $\boldsymbol{S}(u)$ only:

$$\tau(u) := \operatorname{tr} \boldsymbol{S}(u) = \frac{2u-2\kappa}{g(u)}\left\{p(u)\operatorname{tr} A(u)\right\}^u.$$

where $p(u) = 1/(2u-\kappa)$.

Exchange relation

Using the symmetry relation, the AB exchange relation may be written

$$\left\{p(v)\,A_a(v)\right\}^v \beta_{\tilde{a}_1 a_1}(u) = \beta_{\tilde{a}_1 a_1}(u)\left\{p(v)\,S'_{a;\tilde{a}_1 a_1}(v;u)\right\}^v + \frac{1}{p(u)}\left\{p(v)\,\frac{\beta_{\tilde{a}_1 a_1}(v)}{u-v}\right\}^v \operatorname*{Res}_{w\to u}\left\{p(w)\,S'_{a;\tilde{a}_1 a_1}(w;u)\right\}^w,$$

where

$$S'_{a;\tilde{a}_1 a_1}(v;u) = R^t_{\tilde{a}_1 a}(u-v)\,R^t_{a_1 a}(\kappa-u-v)\,A_a(v) \times R^t_{a_1 a}(u-v\pm 1)\,R^t_{\tilde{a}_1 a}(\kappa-u-v\pm 1).$$

Exchange relation for multiple excitations

Acting with the transfer matrix on the Bethe vector we find

$$\tau(v)\cdot\Psi(\boldsymbol{u}) = \left(\prod_{i=1}^{m}\beta_{\tilde{a}_i a_i}(u_i)\prod_{j=i-1}^{1}R_{a_j\tilde{a}_i}(-u_i-u_j)\right) \times \left\{p(v)\,\mathrm{tr}\,S'_a(v;\boldsymbol{u})\right\}^{v}\cdot\Phi_{\tilde{a}_1 a_1,\ldots\tilde{a}_m a_m} + UWT,$$

where $S'_a(v;\boldsymbol{u})$ is the *nested monodromy matrix*,

$$S'_a(v;\boldsymbol{u}) := \prod_{i=1}^{m}R^{t}_{\tilde{a}_i a}(u_i-v)\prod_{i=1}^{m}R^{t}_{a_i a}(\kappa-u_i-v) \times A_a(v)\prod_{i=m}^{1}R^{t}_{a_i a}(u_i-v\pm 1)\prod_{i=m}^{1}R^{t}_{\tilde{a}_i a}(\kappa-u_i-v\pm 1)$$

and *UWT* stands for the unwanted terms.

Reduction to $\mathfrak{gl}_n$ open spin chain

The nested spin chain has the state space

$$M' = V_{\tilde{a}_1} \otimes \cdots \otimes V_{a_m} \otimes L'_1(\lambda_1) \otimes \cdots \otimes L'_\ell(\lambda_\ell) \otimes M'_{\ell+1}(\mu).$$

- $V_{\tilde{a}_i}$, V_{a_i} are auxiliary spaces, all being copies of $\mathbb{C}^n$.
- $L'_i(\lambda_i) \subset L_i(\lambda_i)$ are subspaces annihilated by the C block of $\boldsymbol{S}(u)$, and are $\mathfrak{gl}_n$-irreps with the same weight λ_i.
- $M'_{\ell+1}(\mu) = M_{\ell+1}(\mu)$ is the one-dimensional "reduced" boundary space with the same weight μ.

The nested monodromy matrix $S'_a(v;\boldsymbol{u})$ satisfies the reflection equation on the space M',

$$\begin{aligned} R_{ab}(w-v)S'_a(w;\boldsymbol{u})R_{ab}(w+v)S'_b(v;\boldsymbol{u}) \cdot M' = \\ S'_b(v;\boldsymbol{u})R_{ab}(w+v)S'_a(w;\boldsymbol{u})R_{ab}(w-v) \cdot M' \\ + \text{extra terms} \cdot M'. \end{aligned}$$

Solution for the $\mathfrak{gl}_n$ open spin chain

The transfer matrix of the $\mathfrak{gl}_n$ nested system has eigenvalue (Belliard–Ragoucy'09)

$$\Gamma(v;\boldsymbol{u}) = \sum_{k=1}^{n} \frac{2v-n}{2v-k}\mu_k^{\sharp}(v)\tilde{\lambda}_k(v)\tilde{\lambda}'_k(v)$$
$$\times \prod_{j=1}^{m^{(k)}} f^{+}(v-\tfrac{k}{2}, u_j^{(k)}) \prod_{j=1}^{m^{(k-1)}} f^{-}(v-\tfrac{k-1}{2}, u_j^{(k-1)})$$

where

$$\tilde{\lambda}_k(v) = \lambda_k(v) \qquad \tilde{\lambda}_n(v) = \lambda_n(v)\prod_{j=1}^{m} \frac{v-u_i+1}{v-u_i}\,\frac{v-\kappa+u_i+1}{v-\kappa+u_i},$$

and the $u_j^{(k)}$ satisfy Bethe equations given by

$$\operatorname{Res}_{v\to u_j^{(k)}+k/2}\Gamma(v;\boldsymbol{u}) = 0.$$

Results

We have $\tau(v) \cdot \Psi = \Lambda(v)\Psi$ where,

- The eigenvalue of the transfer matrix is given by

$$\Lambda(v) = \left\{p(v)\sigma(v)\Gamma(v)\right\}^{v},$$

- The Bethe vector is given by,

$$\Psi = \left(\prod_{i=1}^{m} \beta_{\tilde{a}_i a_i}(u_i) \prod_{j=i-1}^{1} R_{a_j \tilde{a}_i}(-u_i - u_j) \right) \cdot \Phi_{\tilde{a}_1 a_1, \ldots \tilde{a}_m a_m}.$$

- The u_i satisfy the Bethe equations, obtained by demanding the unwanted terms vanish on the Bethe vector,

$$\operatorname{Res}_{v \to u_i} \Lambda(v) = 0.$$

Conclusions and Outlook

- We presented a nested algebraic Bethe ansatz method that allows us to study the spectral problem of the $\mathfrak{so}_{2n}$ and $\mathfrak{sp}_{2n}$ open spin chains with diagonal boundary conditions at the same.
- We used fusion procedure to construct Lax operators for symmetric $\mathfrak{so}_{2n}$-irreps and skewsymmetric $\mathfrak{sp}_{2n}$-irreps.
- The top-level nesting yields an $\mathfrak{gl}_n$ open spin chain.
- This nesting procedure can not be applied for $\mathfrak{so}_{2n+1}$ open spin chains.
- Work in progress: a trace formula for the Bethe vector.

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Thank you !

$K(u)$ matrices

- $\mathfrak{g}_{2n} \to \mathfrak{gl}_n$ case

$$\mu_1(u) = \ldots = \mu_n(u) = 1 + \frac{a}{u},$$

- $\mathfrak{g}_{2n} \to \mathfrak{g}_{2p} \oplus \mathfrak{g}_{2q}$ cases

$$\mu_i(u) = \frac{p - q - 2u\,K_{ii}}{p - q - 2u} \quad \text{for} \quad 1 \leq i \leq n,$$

- $\mathfrak{so}_4 \to \mathfrak{so}_2 \oplus \mathfrak{so}_2$ case

$$\mu_1(u) = \left(1 + \frac{a}{u}\right)\left(1 + \frac{b}{u}\right), \quad \mu_2(u) = \left(1 + \frac{a}{u}\right)\left(-1 + \frac{b}{u}\right),$$

- $\mathfrak{so}_{2n} \to \mathfrak{so}_{2n-2} \oplus \mathfrak{so}_2$ case

$$\mu_1(u) = \ldots = \mu_{n-1}(u) = \frac{(u - a)(u + a - (n-2))}{(u - \frac{n-2}{2})^2},$$

$$\mu_n(u) = -\frac{(u + a)(u + a - (n-2))}{(u - \frac{n-2}{2})^2}.$$

Bethe equations

$$\frac{\mu^\sharp(u_j)\lambda_n(u_j)\lambda'_n(u_j)}{\mu^\sharp(\kappa-u_j)\lambda_n(\kappa-u_j)\lambda'_n(\kappa-u_j)} =$$

$$-\frac{\sigma(\kappa-u_j)}{\sigma(u_j)}\frac{2u_j-\kappa-1}{2u_j-\kappa+1}\prod_{i\neq j}\frac{f^-(u_j,u_i)}{f^+(u_j,u_i)}\frac{f^+(\kappa-u_j,u_i)}{f^-(\kappa-u_j,u_i)}$$

$$\times\prod_{i=1}^{m}\frac{f^-(\kappa-u_j-(n-1),u_i)}{f^+(\kappa-u_j+(n-1),u_i)}\cdot\frac{f^+(u_j+n-1,u_i)}{f^-(u_j-(n-1),u_i)}$$

$$\times\prod_{i=1}^{m^{(n-1)}}\frac{f^-(\kappa-u_j-\frac{n-1}{2},u_i^{(n-1)})}{f^-(u_j-\frac{n-1}{2},u_i^{(n-1)})}$$