Stochastic vertex models and generalized Macdonald polynomials

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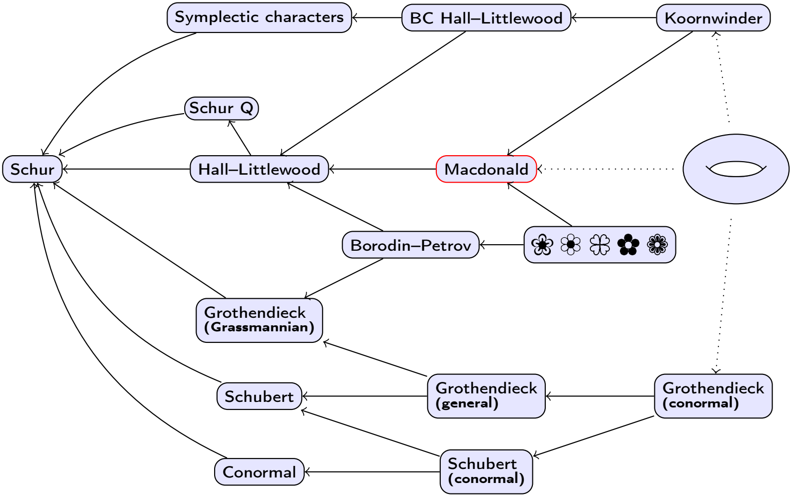
26 August, 2016

Outline

1. Macdonald polynomials
2. Borodin–Petrov polynomials
3. Unification

Charting the territory



Why are Macdonald polynomials interesting?

- Macdonald polynomials can be considered as grandparents of a host of symmetric functions (monomial, Schur, Hall–Littlewood, Jack, zonal). They are two parameter (q,t) generalizations of Schur polynomials.
- They have led to fascinating problems in pure mathematics, such as the constant term conjecture (proved in full generality by Cherednik) and the positivity conjecture (proved by Haiman).
- They have appeared in the physics literature, in connection with the 5D AGT correspondence.
- They were also the inspiration for the Macdonald process of Borodin and Corwin, which generalizes the Schur process introduced by Okounkov and Reshetikhin.
- In the algebraic Bethe Ansatz for quantum integrable models, symmetric functions arise very naturally:

$$\Psi_{i_1,\ldots,i_n}(x_1,\ldots,x_n) = \langle i_1,\ldots,i_n|B(x_1)\ldots B(x_n)|0\rangle \qquad \text{is symmetric in } x_1,\ldots,x_n.$$

A key point of this work is to realize Macdonald polynomials via such a fomula.

Macdonald polynomials: first definition

- Let λ be a partition, $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. The monomial symmetric functions are given by

$$m_\lambda(x_1,\ldots,x_n) = \sum_{\sigma\in S_\lambda} \prod_{i=1}^{n} x_{\sigma(i)}^{\lambda_i}, \qquad S_\lambda = S_n / S_n^\lambda.$$

- For example:

$$m_{2,2,1,1}(x_1,x_2,x_3,x_4) = \\ x_1^2x_2^2x_3x_4 + x_1^2x_2x_3^2x_4 + x_1^2x_2x_3x_4^2 + x_1x_2^2x_3^2x_4 + x_1x_2^2x_3x_4^2 + x_1x_2x_3^2x_4^2$$

- The Macdonald polynomial $P_\lambda(x;q,t)$ is the unique homogeneous, symmetric function which satisfies

$$P_\lambda(x_1,\ldots,x_n;q,t) = m_\lambda(x_1,\ldots,x_n) + \sum_{\mu<\lambda} c_{\lambda,\mu}(q,t) m_\mu(x_1,\ldots,x_n),$$

$$\langle P_\lambda, P_\mu\rangle = 0,\ \lambda \neq \mu,$$

with respect to a certain bilinear form defined on power sums:

$$\langle p_\lambda, p_\mu\rangle = \delta_{\lambda,\mu} \times (\text{some rational function in } q,t).$$

Macdonald polynomials: second definition

- Introduce a commuting family of difference operators [Macdonald 87], and their generating series:

$$D_n^r = t^{r(r-1)/2} \sum_{\substack{S \subseteq [1,\dots,n] \\ |S|=r}} \prod_{i \in S} \prod_{j \notin S} \left(\frac{t x_i - x_j}{x_i - x_j} \right) \prod_{k \in S} \mathcal{T}_{q,x_k}, \qquad \mathbb{D}_n(z) = \sum_{r=0}^{n} D_n^r z^r.$$

- The Macdonald polynomials are the unique eigenfunctions of $\mathbb{D}_n(z)$:

$$\mathbb{D}_n(z) P_\lambda(x_1,\dots,x_n;q,t) = \prod_{i=1}^{n} (1 + z q^{\lambda_i} t^{n-i}) P_\lambda(x_1,\dots,x_n;q,t).$$

- This definition is more appealing for a mathematical physicist, but it gives no clear insight into the structure of $P_\lambda(x_1,\dots,x_n;q,t)$.

Macdonald polynomials in integrable probability

- Macdonald processes [Borodin, Corwin 11] are a very general class of stochastic processes which include many others as special cases: Schur processes, totally asymmetric simple exclusion processes, last passage percolation, random directed polymers.
- They are based on the Macdonald measure on partitions

$$\mathcal{M}_\lambda(\rho_1;\rho_2) := \frac{P_\lambda(\rho_1)Q_\lambda(\rho_2)}{\Pi(\rho_1;\rho_2)}, \quad \Pi(\rho_1;\rho_2) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\frac{1-t^n}{1-q^n} p_n(\rho_1)p_n(\rho_2)\right),$$

  where ρ_1 and ρ_2 are two specializations of the ring of symmetric functions.
- The action of difference operators on Macdonald polynomials then leads to a natural class of observables for study:

$$\mathbb{E}\left[e_m(q^{\lambda_1}t^{n-1},\dots,q^{\lambda_n}t^0)\right] = \left[\frac{\sum_\lambda \prod_{i=1}^n (1+zq^{\lambda_i}t^{n-i}) P_\lambda(x)Q_\lambda(y)}{\Pi(x;y)}\right]_{z^m}$$
$$= \left[\frac{\mathbb{D}_n(z)\Pi(x;y)}{\Pi(x;y)}\right]_{z^m}$$

  where we have taken $\rho_1 = (x_1,\dots,x_n)$ and $\rho_2 = (y_1,\dots,y_n)$ for simplicity.

Non-symmetric Macdonald polynomials

- The Hecke algebra of type A_{n-1} is generated by $T_1, \dots, T_{n-1}$ modulo the relations

$$(T_i - t)(T_i + 1) = 0, \qquad T_i T_{i\pm1} T_i = T_{i\pm1} T_i T_{i\pm1}, \qquad T_i T_j = T_j T_i,\ |i-j| > 1.$$

- We consider a polynomial representation of the algebra, given by

$$T_i = t - \frac{t x_i - x_{i+1}}{x_i - x_{i+1}} (1 - \sigma_i), \qquad 1 \leqslant i \leqslant n-1.$$

- Consider the following (commuting) elements of the Hecke algebra:

$$Y_i = T_i \cdots T_{n-1} \omega T_1^{-1} \cdots T_{i-1}^{-1}, \qquad \omega g(x_1, \dots, x_n) = g(q x_n, x_1, \dots, x_{n-1}).$$

- Non-symmetric Macdonald polynomials E_μ [Cherednik 95], [Opdam 95], [Macdonald 95] are defined as the unique eigenfunctions of these operators:

$$Y_i E_\mu = y_i(\mu) E_\mu, \qquad y_i(\mu) = t^{\rho_i(\mu)} q^{\mu_i}.$$

Theorem

The Macdonald polynomial $P_\lambda(x; q, t)$ is given by

$$P_\lambda(x_1, \dots, x_n; q, t) = \sum_{\sigma \in S_\lambda} \kappa_{\sigma(\lambda)} E_{\sigma(\lambda)}(x_1, \dots, x_n; q, t),$$

where the sum is over all distinct permutations of λ.

Another non-symmetric basis

- In this work we are interested in another basis, $f_\mu(x_1,\dots,x_n;q,t)$, defined by

$$f_{\delta_1,\dots,\delta_n} := E_{\delta_1,\dots,\delta_n} \qquad \text{when} \quad \delta_1 \leqslant \cdots \leqslant \delta_n$$
$$f_{\mu_1,\dots,\mu_i,\mu_{i+1},\dots,\mu_n} := T_i^{-1} f_{\mu_1,\dots,\mu_{i+1},\mu_i,\dots,\mu_n} \qquad \text{when} \quad \mu_i > \mu_{i+1}.$$

- The transition matrix between f_μ and E_μ cannot easily be written down, but it is a triangular change of basis.

Theorem

The Macdonald polynomial $P_\lambda(x;q,t)$ is given by

$$P_\lambda(x_1,\dots,x_n;q,t) = \sum_{\sigma\in S_\lambda} f_{\sigma(\lambda)}(x_1,\dots,x_n;q,t),$$

where the sum is over all distinct permutations of λ.

- For reasons that will become clear, we call this the ASEP basis, after the asymmetric simple exclusion process.

Matrix product solution of Knizhnik–Zamolodchikov equations

- These polynomials satisfy the Knizhnik–Zamolodchikov equations [Kasatani, Takeyama 07]:

$$T_i f_{\mu_1,\dots,\mu_i,\mu_{i+1},\dots,\mu_n}(x_1,\dots,x_n) = \begin{cases} t f_{\mu_1,\dots,\mu_{i+1},\mu_i,\dots,\mu_n}(x_1,\dots,x_n), & \mu_i = \mu_{i+1} \\ f_{\mu_1,\dots,\mu_{i+1},\mu_i,\dots,\mu_n}(x_1,\dots,x_n), & \mu_i > \mu_{i+1} \end{cases}$$

$$f_{\mu_n,\mu_1,\dots,\mu_{n-1}}(qx_n,x_1,\dots,x_{n-1}) = q^{\mu_n} f_{\mu_1,\dots,\mu_n}(x_1,\dots,x_n).$$

- We seek a matrix product solution of the above equations:

$$\Omega_{\mu^+}(q,t) f_\mu(x_1,\dots,x_n) = \mathrm{Tr}\Big[A_{\mu_1}(x_1)\dots A_{\mu_n}(x_n)\mathbb{S}\Big]$$

- This Ansatz works provided that

$$A_i(x)A_i(y) = A_i(y)A_i(x)$$

$$tA_j(x)A_i(y) - \frac{tx-y}{x-y}\Big(A_j(x)A_i(y) - A_j(y)A_i(x)\Big) = A_i(x)A_j(y)$$

$$\mathbb{S}A_i(qx) = q^i A_i(x)\mathbb{S}.$$

Zamolodchikov–Faddeev algebra

- The previous relations can be written more succinctly as

$$R_{ab}(x/y)\mathbb{A}_a(x)\mathbb{A}_b(y) = \mathbb{A}_b(y)\mathbb{A}_a(x), \qquad \mathbb{A}_a(x) = \begin{pmatrix} A_0(x) \\ A_1(x) \\ \vdots \\ A_r(x) \end{pmatrix}_a$$

- The R matrix is (a stochastic) higher rank version of the six-vertex model:

$$R_{ab}(x/y) = (1 - tx/y)\sum_{i=0}^{r} E_a^{(ii)} E_b^{(ii)} + (1 - x/y) \sum_{0 \leqslant i < j \leqslant r} \left(E_a^{(ii)} E_b^{(jj)} + t\ E_a^{(jj)} E_b^{(ii)} \right)$$
$$+ (1 - t) \sum_{0 \leqslant i < j \leqslant r} \left(E_a^{(ij)} E_b^{(ji)} + x/y\ E_a^{(ji)} E_b^{(ij)} \right).$$

- The operator $\mathbb{S}$ satisfies

$$\mathbb{S}\mathbb{A}(qx) = q^{\sum_{i=0}^{r} iE^{(ii)}} \mathbb{A}(x)\mathbb{S}.$$

Matrix product formula

Theorem (Cantini–de Gier–Wheeler 2015)

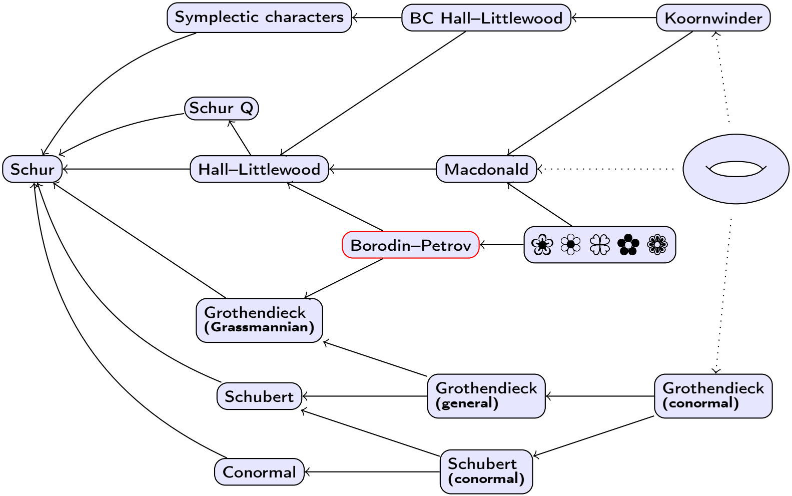
Let $A_i(x)$ be the i^{th} component of $\mathbb{A}(x)$, and $\mathbb{S}$ be as above. Then

$$f_\mu(x_1,\ldots,x_n;q,t) = \prod_{1\leqslant i<j\leqslant r} \left(1 - q^{j-i} t^{(\mu^+)'_i - (\mu^+)'_j}\right) \mathrm{Tr}\left[A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)\mathbb{S}\right]$$

$$P_\lambda(x_1,\ldots,x_n;q,t) = \prod_{1\leqslant i<j\leqslant r} \left(1 - q^{j-i} t^{\lambda'_i - \lambda'_j}\right) \sum_{\mu\in S_\lambda\cdot\lambda} \mathrm{Tr}\left[A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)\mathbb{S}\right]$$

- The non-symmetric polynomials f_μ can be viewed as generalized ASEP configuration probabilities.
- Symmetric Macdonald polynomials P_λ are the normalizations of these probabilities.

A reminder of the landscape



Borodin–Petrov polynomials: sum formula

- Recently, Borodin and Petrov introduced a family of inhomogeneous symmetric functions which generalize Hall–Littlewood polynomials by the inclusion of a new parameter s:

$$F_\lambda(x_1,\dots,x_n;t;s) = \frac{1}{v_\lambda(t)} \sum_{\sigma\in S_n} \sigma\left[\prod_{1\leqslant i<j\leqslant n}\left(\frac{x_i - tx_j}{x_i - x_j}\right)\prod_{i=1}^{n}\left(\frac{x_i - s}{1 - x_i s}\right)^{\lambda_i}\right].$$

- At the special values $s = t^{-m/2}$, $m \in \mathbb{N}$, these functions reduce to wavefunctions of a spin-$m/2$ semi-infinite XXZ Heisenberg chain.

- At $s = 0$ (the limit of infinite spin), one recovers the Hall–Littlewood polynomial $P_\lambda(x_1,\dots,x_n;t)$.

- They satisfy all the nice properties that one could hope for: branching formulae, Pieri rules and Cauchy identities [Borodin 14].

Borodin–Petrov polynomials: as an expectation value

- The Borodin–Petrov polynomials can be constructed in the framework of the algebraic Bethe Ansatz. The necessary ingredients are the L and T matrices:

$$L(x) = \frac{1}{1-xs} \begin{pmatrix} 1-xsk & (1-sk)\phi \\ x\phi^{\dagger} & x-sk \end{pmatrix}, \quad T(x) = L^{(1)}(x) \cdots L^{(r)}(x),$$

  an integrable deformation of the t-boson model of [Bogoliubov, Izergin, Kitanine 97].

- The entries of the L matrix are elements of the t-boson algebra $\mathfrak{B}$. It is generated by $\{\phi, \phi^{\dagger}, k\}$, modulo the relations

$$\phi\phi^{\dagger} - t\phi^{\dagger}\phi = 1-t, \qquad \phi k = tk\phi, \qquad t\phi^{\dagger}k = k\phi^{\dagger}.$$

- A well known representation of this algebra is the Fock representation:

$$\phi^{\dagger}|m\rangle = (1-t^{m+1})|m+1\rangle, \qquad \phi|m\rangle = |m-1\rangle, \qquad k|m\rangle = t^m|m\rangle \quad \forall m \geqslant 0.$$

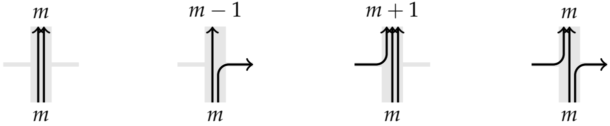
- We then have

$$F_{\lambda}(x_1,\ldots,x_n;t;s) = \langle\lambda|\, T_{10}(x_1)\ldots T_{10}(x_n)\,|0\rangle\,.$$

  A similar result was already known for Hall–Littlewood polynomials [Tsilevich 06], [Korff 13].

Stochastic vertex models in integrable probability

- The main motivation for introducing these polynomials is their interpretation as a probability distribution on lattice path configurations in a stochastic vertex model.
- The vertex model is recovered by calculating all possible local expectation values $\langle n| L_{ij}(x) |m\rangle$:



$$\frac{1-xst^m}{1-xs} \qquad \frac{1-s^2t^{m-1}}{1-xs} \qquad \frac{x(1-t^{m+1})}{1-xs} \qquad \frac{x-st^m}{1-xs}$$

All other vertices have zero weight.

- It can be made stochastic by a conjugation of the vertices:

$$w_x(i,m|j,n) = \frac{(s^2;t)_n}{(t;t)_n} \langle n| (-s)^j x^{j-i} L_{ij}(x) |m\rangle \frac{(t;t)_m}{(s^2;t)_m}.$$

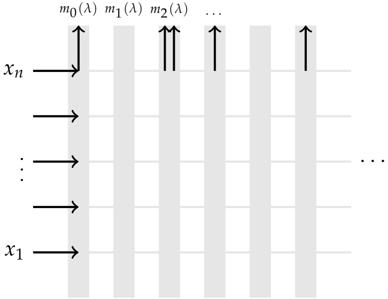
- Taking the two inputs of such a vertex (left and bottom) to be fixed, one has

$$\sum_{0\leqslant j\leqslant 1} \sum_{n\geqslant 0} w_x(i,m|j,n) = 1.$$

Stochastic vertex models in integrable probability

- The Borodin–Petrov polynomials can be expressed as a partition function in this model:

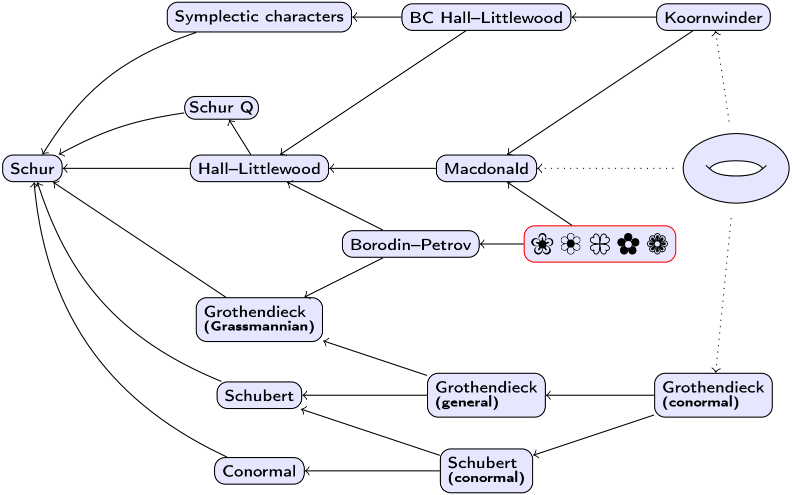
$$F_\lambda(x_1,\dots,x_n;t;s) = \sum_{\substack{\text{configs}\\ \mapsto \lambda}}$$



- The height function $\mathfrak{h}(i,j)$ is defined as the number of lattice paths which pass through or to the right of the vertex (i,j).
- With respect to the measure we are using, one then seeks to calculate the expectation of z-moments of $\mathfrak{h}$ [Borodin, Petrov 16]:

$$\mathbb{E}\left[\prod_{a=1}^{m} z^{\mathfrak{h}(i_a,n)}\right] \propto \oint_{w_1}\cdots\oint_{w_m} \prod_{1\leqslant a<b\leqslant m} \frac{w_a-w_b}{w_a-tw_b} \prod_{a=1}^{m}\left[\frac{1}{w_a}\left(\frac{1-sw_a}{1-w_a/s}\right)^{i_a-1}\prod_{b=1}^{n}\left(\frac{1-tw_a x_b}{1-w_a x_b}\right)\right]$$

Final destination



Higher-rank bosonic L-matrices

- The key idea is to obtain a higher-rank version of the L matrix used in the Borodin–Petrov construction:

$$R_{ab}(x/y)L_a(x)L_b(y) = L_b(y)L_a(x)R_{ab}(x/y),$$

  where $L_a(x)$ is an $(r+1)\times(r+1)$ operator-valued matrix.
- The desired L matrix can be obtained via an algebra homomorphism of a universal R matrix in [Jimbo 86]. Its entries are given by

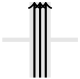
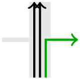
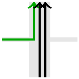
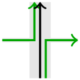
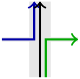
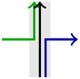
$$L_{00}(x) = 1 - xs\prod_{l=1}^{r} k_l,$$

$$L_{0j}(x) = \left(1 - s^2\prod_{l=1}^{r} k_l\right)\phi_j, \qquad L_{i0}(x) = x\left(\prod_{l=i+1}^{r} k_l\right)\phi_i^{\dagger},$$

$$L_{ij}(x) = \begin{cases} (x - sk_i)\prod_{l=i+1}^{r} k_l, & i = j \\ x\left(\prod_{l=i+1}^{r} k_l\right)\phi_i^{\dagger}\phi_j, & i > j \\ s\left(\prod_{l=i+1}^{r} k_l\right)\phi_i^{\dagger}\phi_j, & i < j \end{cases} \quad \text{for } 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant r.$$

Higher-rank stochastic vertex model

- The resulting vertex model is given by

$\{m_1,\dots,m_r\}$  $\{m_1,\dots,m_r\}$	$\{\dots,m_i-1,\dots\}$  $\{\dots,m_i,\dots\}$	$\{\dots,m_i+1,\dots\}$  $\{\dots,m_i,\dots\}$
$1-xst^{\lvert m\rvert}$	$1-s^2t^{\lvert m\rvert-1}$	$x(1-t^{m_i+1})t^{\lvert m\rvert_i}$
$\{m_1,\dots,m_r\}$  $\{m_1,\dots,m_r\}$	$\{\dots,m_i-1,\dots,m_j+1,\dots\}$  $\{\dots,m_i,\dots,m_j,\dots\}$	$\{\dots,m_i+1,\dots,m_j-1,\dots\}$  $\{\dots,m_i,\dots,m_j,\dots\}$
$(x-st^{m_i})t^{\lvert m\rvert_i}$	$x(1-t^{m_j+1})t^{\lvert m\rvert_j}$	$s(1-t^{m_i+1})t^{\lvert m\rvert_i-1}$

where $|m| := \sum_{\ell=1}^{r} m_\ell$ and $|m|_i := \sum_{\ell=i+1}^{r} m_\ell$.

A colour-invariance theorem

Theorem

Let $|m_1,\ldots,m_r\rangle$ *denote a generic bosonic state in* $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_r$ *and define*

$$\|M\rangle\!\rangle = \sum_{\substack{\{m_1,\ldots,m_r\} \\ |m|=M}} |m_1,\ldots,m_r\rangle, \qquad \text{where } |m| = \sum_{i=1}^{r} m_i.$$

In particular, $\|0\rangle\!\rangle = |0,\ldots,0\rangle$. *The following four relations are valid for any* $M \geqslant 0$*:*

$$\begin{aligned} L_{00}(x)\,\|M\rangle\!\rangle &= (1 - xst^{M})\,\|M\rangle\!\rangle\,, \\ L_{0j}(x)\,\|M\rangle\!\rangle &= (1 - s^2 t^{M-1})\,\|M-1\rangle\!\rangle\,, \quad \forall\, j \in \{1,\ldots,r\}, \\ \sum_{i=1}^{r} L_{i0}(x)\,\|M\rangle\!\rangle &= x(1 - t^{M+1})\,\|M+1\rangle\!\rangle\,, \\ \sum_{i=1}^{r} L_{ij}(x)\,\|M\rangle\!\rangle &= (x - st^{M})\,\|M\rangle\!\rangle\,, \quad \forall\, j \in \{1,\ldots,r\}. \end{aligned}$$

- By virtue of this theorem, our vertex model can be made stochastic (again) by a simple conjugation.

Assembling the ingredients

- Construct a monodromy matrix in the rank r model, of length r, then extract its first column:

$$T(x) = L^{(1)}(x) \dots L^{(r)}(x),$$

$$\mathbb{A}(x) = (A_0(x), A_1(x), \dots, A_r(x))^{\mathrm{T}} = (T_{00}(x), T_{10}(x), \dots, T_{r0}(x))^{\mathrm{T}}, \quad \mathcal{A}(x) := \sum_{i=0}^{r} A_i(x).$$

  Note that $T(x)$ depends on r^2 copies $\mathfrak{B}_i^{(j)}$ of the t-boson algebra!
- Similarly define a factorized twist operator as follows:

$$\mathbb{S} = S^{(1)} \cdots S^{(r)}, \qquad S^{(i)} = \left(\prod_{j=i+1}^{r} k_j^{(j-i)\alpha} \right)^{(i)} \quad \text{where} \quad t^{\alpha} := q.$$

Definition

Let $\langle \cdot \rangle_\lambda : \otimes_{i,j=1}^{r} \mathfrak{B}_i^{(j)} \to \mathbb{C}$ be a linear form constructed as follows:

1. *Trace over the Fock representation of all algebras $\mathfrak{B}_i^{(j)}$ such that $i > j$.*
2. *Sandwich between vacuum states $\langle 0|_i^{(j)}$ and $|0\rangle_i^{(j)}$ for all algebras $\mathfrak{B}_i^{(j)}$ such that $i < j$.*
3. *Sandwich between the states $\langle m_i(\lambda)|_i^{(i)}$ and $|0\rangle_i^{(i)}$ for all algebras $\mathfrak{B}_i^{(i)}$.*

The new family of polynomials

Definition

Construct a family of symmetric polynomials as follows:

$$P_\lambda(x_1,\dots,x_n;q,t;s) := \langle \mathcal{A}(x_1)\dots\mathcal{A}(x_n)\mathbb{S}\rangle_\lambda .$$

Theorem (Garbali–de Gier–Wheeler 2016)

Up to harmless differences in conventions, one has

$$P_\lambda(x_1,\dots,x_n;q,t;s)\Big|_{s=0} = P_\lambda(x_1,\dots,x_n;q,t) \quad (\text{Macdonald})$$

$$P_\lambda(x_1,\dots,x_n;q,t;s)\Big|_{q=0} = F_\lambda(x_1,\dots,x_n;t;s) \quad (\text{Borodin}-\text{Petrov})$$

Proof.

The $s=0$ case reduces manifestly to the matrix product formula of [Cantini, de Gier, Wheeler 15] for Macdonald polynomials.

The $q=0$ case is more subtle. At $q=0$ all traces become trivial, and we recover a (flat) partition function in the higher-rank vertex model. Using the colour-invariance theorem, one can show that this partition function is essentially "colour-independent" and recovers exactly that of [Borodin 14] [Borodin, Petrov 16]. □

Concluding remarks

- Both Macdonald processes and integrable stochastic vertex models can be used to define and calculate the expectations of important observables, in a form amenable to asymptotic analysis.
- In the scaling limit, the height functions obtained are solutions of the KPZ equation. Both types of processes produce observables in the KPZ universality class, without being directly related, so it is natural to search for a structure which unites them.
- The polynomials in [Garbali, de Gier, Wheeler 16] may constitute a first step along this road.
- Although we have an explicit formula for $P_\lambda(x_1, \ldots, x_n; q, t; s)$, at this stage it remains an open problem to obtain branching formulae, Pieri rules and Cauchy identities, or to study their behaviour under suitable difference operators.