

Painlevé monodromy varieties: geometry and quantisation

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Based on Chekhov-Mazzocco-R. [arXiv:1511.03851](#)

Talk at the
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Plan:

- Painlevé equations, Confluence, Isomonodromy and Affine cubics;
- Decorated character varieties;
- Quantisation;

Painlevé list

$$\frac{d^2 w}{dz^2} = 6w^2 + z$$

$$\frac{d^2 w}{dz^2} = 2w^3 + zw + \alpha$$

$$\frac{d^2 w}{dz^2} = \frac{1}{w} \left(\frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{d^2 w}{dz^2} = \frac{1}{2w} \left(\frac{dw}{dz} \right)^2 + \frac{3}{2} w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\frac{d^2 w}{dz^2} = \frac{3w-1}{2w(w-1)} w_z^2 - \frac{1}{z} \frac{dw}{dz} + \frac{\gamma w}{z} + \frac{(w-1)^2}{z^2} \frac{\alpha w^2 + \beta}{w} + \frac{\delta w(w+1)}{w-1}$$

$$\begin{aligned} \frac{d^2 w}{dz^2} = & \frac{1}{2} \left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z} \right) w_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z} \right) w_z + \\ & + \frac{w(w-1)(w-z)}{z^2(z-1)^2} \left[\alpha + \beta \frac{z}{w^2} + \gamma \frac{z-1}{(w-1)^2} + \delta \frac{z(z-1)}{(w-z)^2} \right] \end{aligned}$$

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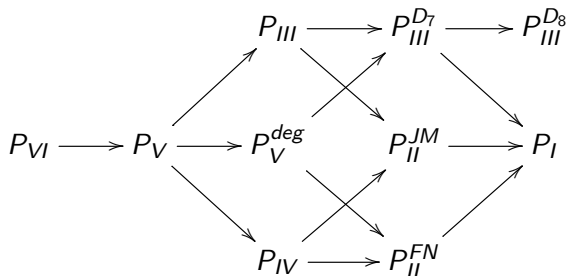
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- More recently: P_{IV} - has a (non genuine) fully NC analogue (M. Cafasso-M. Iglesias)

Confluences of the Painlevé equations-1



Confluences of the Painlevé equations-2

Example

Take $w(z) = \epsilon \tilde{w}(\tilde{z}) + \frac{1}{\epsilon^5}$, $z = \epsilon^2 \tilde{z} - \frac{6}{\epsilon^{10}}$, $\alpha = \frac{4}{\epsilon^{15}}$ then *PII*

$$\frac{d^2 w}{dz^2} = 2w^3 + zw + \alpha$$

becomes

$$\frac{d^2 \tilde{w}}{d\tilde{z}^2} = 6\tilde{w}^2 + \tilde{z} + \epsilon^2(2\tilde{w}^3 + \tilde{z}\tilde{w}),$$

that for $\epsilon \rightarrow 0$ is *PI*.

All Painlevé equations are **isomonodromic deformation equations** (Jimbo-Miwa1980)

$$\frac{dB}{d\lambda} - \frac{dA}{dz} = [A, B]$$

$$A = A(\lambda; z, w, w_z), B = B(\lambda; z, w, w_z) \in \mathfrak{sl}_2.$$

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The monodromy data are encoded in an **affine cubic surfaces** called *monodromy varieties*.

Painlevé monodromy manifolds

Saito and van der Put

$$M_\varphi := \text{Spec}(\mathbb{C}[x_1, x_2, x_3] / \langle \varphi = 0 \rangle)$$

$$PVI \quad x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

$$PV \quad x_1 x_2 x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$$

$$PIV \quad x_1 x_2 x_3 + x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_2 x_3 + 1 = \omega_4$$

$$PIII \quad x_1 x_2 x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 = \omega_1 - 1$$

$$PII \quad x_1 x_2 x_3 + x_1 + x_2 + x_3 = \omega_4$$

$$PI \quad x_1 x_2 x_3 + x_1 + x_2 + 1 = 0$$

PVI as isomonodromic deformation-I

Painlevé sixth equation

- The Painlevé VI equation describes the isomonodromic deformations of the rank 2 meromorphic connections on \mathbb{P}^1 with simple poles.

$$\frac{dY}{d\lambda} = \left(\frac{A_1(z)}{\lambda} + \frac{A_2(z)}{\lambda - t} + \frac{A_3(z)}{\lambda - 1}, \right) Y, \lambda \in \mathbb{C} \setminus \{0, t, 1\} \quad (1)$$

where $A_1, A_2, A_3 \in \mathfrak{sl}_2(\mathbb{C})$, $A_1 + A_2 + A_3 = -A_\infty$, diagonal.

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- Fundamental matrix: $Y_\infty(\lambda) = (1 + O(\frac{1}{\lambda}))\lambda^{A_\infty}$.
- Monodromy matrices** $\gamma_j(Y_\infty) = Y_\infty M_j$
- Describes by generators of the fundamental group under the anti-isomorphism

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}, \lambda_1) \rightarrow \mathrm{SL}_2(\mathbb{C}).$$

PVI as isomonodromic deformation-II

- $\text{eigen}(M_j) = \text{eigen}(\exp(2\pi i A_j))$
- We fix the base point λ_1 at infinity and the generators of the fundamental group to be $\gamma_1, \gamma_2, \gamma_3$ such that γ_j encircles only the pole i once and are oriented in such a way that

$$M_1 M_2 M_3 M_\infty = \mathbb{I}, \quad M_\infty = \exp(2\pi i A_\infty). \quad (2)$$

- Eigenvalues of A_j are $(\theta_j, -\theta_j)$, $j = 0, t, 1, \infty$.

-

$$\alpha := (\theta_\infty - 1/2)^2; \quad \beta := -\theta_0^2;$$

$$\gamma := \theta_1^2; \quad \delta := (1/4 - \theta_t)^2.$$

PVI as isomonodromic deformation-III

Let:

$$G_j := \text{Tr}(M_j) = 2 \cos(\pi \theta_j), \quad j = 0, t, 1, \infty,$$

The [Riemann-Hilbert correspondence](#)

$$\mathcal{F}(\theta_0, \theta_t, \theta_1, \theta_\infty)/\mathcal{G} \rightarrow \mathcal{M}(G_1, G_2, G_3, G_\infty)/\text{SL}_2(\mathbb{C}),$$

where \mathcal{G} is the gauge group, is defined by associating to each Fuchsian system its monodromy representation class. The representation space $\mathcal{M}(G_1, G_2, G_3, G_\infty)$ is realised as an affine cubic surface (Jimbo)

$$x_1 x_2 x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0, \quad (3)$$

where:

PVI as isomonodromic deformation-IV

$$x_1 = \operatorname{Tr}(M_2 M_3), \quad x_2 = \operatorname{Tr}(M_1 M_3), \quad x_3 = \operatorname{Tr}(M_1 M_2).$$

and

$$\begin{aligned} -\omega_i &:= G_k G_j + G_i G_\infty, i \neq k, j, \\ \omega_\infty &= G_1^2 + G_2^2 + G_3^2 + G_\infty^2 + G_1 G_2 G_3 G_\infty - 4. \end{aligned}$$

Iwasaki proved that the triple (x_1, x_2, x_3) satisfying the cubic relation (3) provides a set of coordinates on a large open subset

$$S \subset \mathcal{M}(G_1, G_2, G_3, G_\infty).$$

In what follows, we restrict to such open set.

General Affine Cubic

The main object studied in this talk is the affine irreducible cubic surface $M_\varphi := \text{Spec}(\mathbb{C}[x_1, x_2, x_3]/\langle \varphi=0 \rangle)$ where

$$\varphi = x_1 x_2 x_3 + \epsilon_1^{(d)} x_1^2 + \epsilon_2^{(d)} x_2^2 + \epsilon_3^{(d)} x_3^2 + \omega_1^{(d)} x_1 + \omega_2^{(d)} x_2 + \omega_3^{(d)} x_3 + \omega_4^{(d)} = 0, \quad (4)$$

According to Saito and Van der Put, all monodromy manifolds $\mathcal{M}^{(d)}$ have the form of M_φ for φ from the list above.

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- In algebraic geometry - projective completion:

$$\begin{aligned} \overline{M}_\varphi := \{ (u, v, w, t) \in \mathbb{P}^3 \mid & x_1^2 t + x_2^2 t + x_3^2 t - x_1 x_2 x_3 + \\ & + \omega_3 x_1 t^2 + \omega_2 x_2 t^2 + \omega_3 x_3 t^2 + \omega_4 t^3 = 0 \} \end{aligned}$$

is a del Pezzo surface of degree three and differs from it by three smooth lines at infinity forming a triangle [Oblomkov]
 $t = 0, \quad x_1 x_2 x_3 = 0.$

This family of cubics is a variety

$M_\varphi = \{(\bar{x}, \bar{\omega}) \in \mathbb{C}^3 \times \Omega) : \varphi(\bar{x}, \bar{\omega}) = 0\}$ where

$\bar{x} = (x_1, x_2, x_3)$, $\bar{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and the " \bar{x} -forgetful"

projection $\pi : M_\varphi \rightarrow \Omega : \pi(\bar{x}, \bar{\omega}) = \bar{\omega}$. This projection defines a family of affine cubics with generically non-singular fibres $\pi^{-1}(\bar{\omega})$

The cubic surface M_φ has a volume form $\vartheta_{\bar{\omega}}$ given by the Poincaré residue formulae:

$$\vartheta_{\bar{\omega}} = \frac{dx_1 \wedge dx_2}{(\partial\varphi_{\bar{\omega}})/(\partial x_3)} = \frac{dx_2 \wedge dx_3}{(\partial\varphi_{\bar{\omega}})/(\partial x_1)} = \frac{dx_3 \wedge dx_1}{(\partial\varphi_{\bar{\omega}})/(\partial x_2)}. \quad (5)$$

The volume form is a holomorphic 2-form on the non-singular part of M_φ and it has singularities along the singular locus. This form defines the Poisson brackets on the surface in the usual way as

$$\{x_1, x_2\}_{\bar{\omega}} = \frac{\partial \varphi_{\bar{\omega}}}{\partial x_3} \quad (6)$$

The other brackets are defined by circular transposition of x_1, x_2, x_3 . For $(i, j, k) = (1, 2, 3)$:

$$\{x_i, x_j\}_{\bar{\omega}} = \frac{\partial \varphi_{\bar{\omega}}}{\partial x_k} = x_i x_j + 2\epsilon_i^d x_k + \omega_i^d \quad (7)$$

and the volume form (5) reads as

$$\vartheta_{\bar{\omega}} = \frac{dx_i \wedge dx_j}{(\partial \varphi_{\bar{\omega}} / \partial x_k)} = \frac{dx_i \wedge dx_j}{(x_i x_j + 2\epsilon_i^d x_k + \omega_i^d)}. \quad (8)$$

Observe that for any $\varphi \in \mathbb{C}[x_1, x_2, x_3]$ the following formulae define a Poisson bracket on $\mathbb{C}[x_1, x_2, x_3]$:

$$\{x_i, x_{i+1}\} = \frac{\partial \varphi}{\partial x_{i+2}}, \quad x_{i+3} = x_i, \quad (9)$$

and φ itself is a central element for this bracket, so that the quotient space

$$M_\varphi := \text{Spec}(\mathbb{C}[x_1, x_2, x_3] / \langle \varphi=0 \rangle)$$

inherits the Poisson algebra structure [Nambu \sim 70].

Today we are going to re-parametrize and to quantize it.

Affine Cubic as it is -2:

- In the Painlevé context (the dynamic on the family of PVI monodromy surfaces) this cubic was considered by S. Cantat et F. Loray and by M. Inaba, K. Iwasaki and M.Saito.

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- PVI (\tilde{D}_4) cubic with only $\omega_4 \neq 0$ admits the **log-canonical** symplectic structure $\bar{\vartheta} = \frac{du \wedge dv}{uv}$ under isomorphism $\mathbb{C}^* \times \mathbb{C}^* / \iota \rightarrow M_\varphi$ by

$$(u, v) \rightarrow (x_1 = -(u + \frac{1}{u}), x_2 = -(v + \frac{1}{v}), x_3 = -(uv + \frac{1}{uv}))$$

and $\iota : \mathbb{C}^* \rightarrow \mathbb{C}^*$ is the involution $\iota(u) = \frac{1}{u}$, $\iota(v) = \frac{1}{v}$.

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- The family (3) can be "uniformize" by some analogues of theta-functions related to toric mirror data on log-Calabi-Yau surfaces (M. Gross, P. Hacking and S.Keel (see Example 5.12 of "Mirror symmetry for log-Calabi-Yau varieties I, arXiv:1106.4977)).

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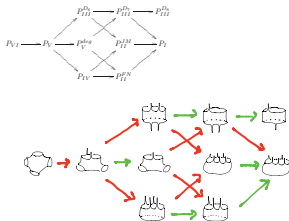
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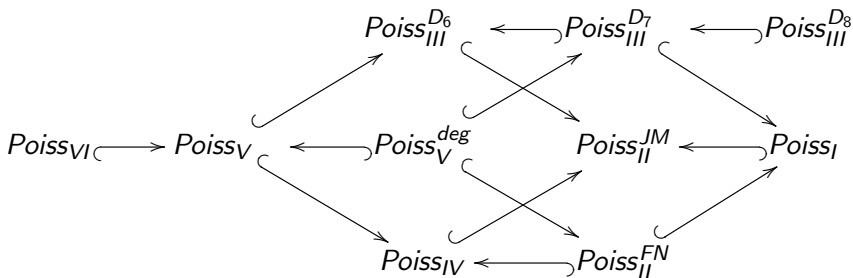
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Use the confluence scheme of the Painlevé equations.



"Confluent" Poisson algebras



Poincaré-Katz invariants and Stokes rays

Painlevé eqs	no. of cusps	Katz invariants	no. Stokes rays	pole-orders for φ
PVI	$(0, 0, 0, 0)$	$(0, 0, 0, 0)$	$(0, 0, 0, 0)$	$(2, 2, 2, 2)$
PV	$(0, 0, 2)$	$(0, 0, 1)$	$(0, 0, 2)$	$(2, 2, 4)$
PV_{deg}	$(0, 0, 1)$	$(0, 0, 1/2)$	$(0, 0, 1)$	$(2, 2, 3)$
PIV	$(0, 4)$	$(0, 2)$	$(0, 4)$	$(2, 6)$
$PIII^{D_6}$	$(0, 2, 2)$	$(0, 1, 1)$	$(0, 2, 2)$	$(2, 4, 4)$
$PIII^{D_7}$	$(0, 1, 2)$	$(0, 1/2, 1)$	$(0, 1, 2)$	$(2, 3, 4)$
$PIII^{D_8}$	$(0, 1, 1)$	$(0, 1/2, 1/2)$	$(0, 1, 1)$	$(2, 3, 3)$
PII^{FN}	$(0, 3)$	$(0, 3/2)$	$(0, 3)$	$(2, 5)$
PII^{MJ}	6	3	6	8
PI	5	$5/2$	5	7

Table: For each Painlevé isomonodromic problem, this table displays the number of cusps on each hole for the corresponding Riemann surface, the Katz invariants associated to the corresponding flat connection, the number of Stokes rays in the linear system defined by the flat connection and the number of poles of the quadratic differential φ defined by the linear system.

Diagonalizable (non-ramified) case

Notation: the fundamental matrix at an irregular singular point λ_k has the form

$$Y_k = G_k(\lambda) \lambda^{\Lambda_k} \begin{pmatrix} e^{Q_k(\lambda)} & 0 \\ 0 & e^{-Q_k(\lambda)} \end{pmatrix}$$

	$A(\lambda)$	Casimirs	extended exponents	$\dim(\mathcal{C})$
PV	$\frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} + A_\infty$	$\text{eigen}(A_0), \text{eigen}(A_1), \Lambda_\infty$	$Q_\infty = \frac{t}{2}\lambda$	7
PIV	$\frac{A_0}{\lambda} + A_1 + A_\infty \lambda$	$\text{eigen}(A_0), \Lambda_\infty$	$Q_\infty = \lambda^2 + \frac{t}{2}\lambda$	8
$PIII^{D_6}$	$\frac{A_0}{\lambda^2} + \frac{A_1}{\lambda} + A_\infty$	$\Lambda_0, \Lambda_\infty$	$Q_\infty = \frac{t}{2}\lambda, Q_0 = \frac{t}{2}\frac{1}{\lambda}$	8
PII^{MJ}	$A_0 + A_1 \lambda + A_\infty \lambda^2$	Λ_∞	$Q_\infty = \lambda^3 + \frac{t}{2}\lambda$	9

Table: Here $Q_k(\lambda)$ is polynomial in $(\lambda - \lambda_k)$ of order $n - 1$ with n being the order of λ_k and Λ_k is the formal monodromy (diagonal). Expand $A(\lambda)$ near λ_k to calculate $Q_k(\lambda)$ and Λ_k , then diagonalize it using the gauge transformation $G_k(\lambda)$.

Non-diagonalizable (ramified) case

The same notations.

	$A(\lambda)$	Casimirs	extended exponents	$\dim(\mathcal{C})$
PV_{deg}	$\frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} + A_\infty$	$\text{eigen}(A_0), \text{eigen}(A_1), \Lambda_\infty$	$Q_\infty = \frac{t}{2}\sqrt{\lambda}$	5
$PIII^{D_7}$	$\frac{A_0}{\lambda^2} + \frac{A_1}{\lambda} + A_\infty$	$\Lambda_0, \Lambda_\infty$	$Q_0 = \frac{1}{\sqrt{\lambda}}, Q_\infty = \frac{t}{2}\lambda$	6
$PIII^{D_8}$	$\frac{A_0}{\lambda^2} + \frac{A_1}{\lambda} + A_\infty$	$\Lambda_0, \Lambda_\infty$	$Q_\infty = \sqrt{\lambda}, Q_0 = \frac{1}{\sqrt{\lambda}}$	4
$P I I^{FN}$	$\frac{A_0}{\lambda} + A_1 + A_\infty \lambda$	$\text{eigen}(A_0), \Lambda_\infty$	$Q_\infty = \lambda^{3/2} + \frac{t}{2}\sqrt{\lambda}$	6
PI	$A_0 + A_1 \lambda + A_\infty \lambda^2$	Λ_∞	$Q_\infty = \lambda^{5/2} + \frac{t}{2}\sqrt{\lambda}$	7

Table: Here

$$\dim PV_{deg} = \dim PV - 2, \quad \dim PIII^{D_7} = \dim PIII - 2, \quad \dim PIII^{D_8} = \dim PIII^{D_7} - 2, \quad \dim PII^{FN} = \dim PIV - 2, \quad \dim PI = \dim PII^{JM} - 2.$$

*Geometric confluence scheme of the Painlevé equations:
application.* (P. Gavrylenko, O. Lisovyy, arXiv:1608.00958)

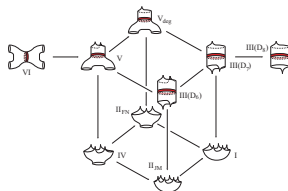


Figure 3: CMR confluence diagram for Painlevé equations.



Figure 4: Some solvable RHPs in rank $N = 2$: Gauss hypergeometric (3 regular punctures), Whittaker (1 regular + 1 of Poincaré rank 1) and Bessel (1 regular + 1 of rank $\frac{1}{2}$).

Basic ideas:history

- The character variety of a Riemann sphere with 4 holes $\text{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}); \text{SL}_2(\mathbb{C}))/\text{SL}_2(\mathbb{C})$ is the monodromy cubic of the Painlevé VI (Goldman-Toledo).

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- The "wild" character varieties, fissions and meromorphic connections (Ph. Boalch 2011-2015).

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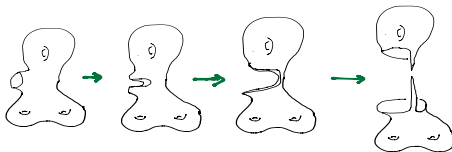
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- The real slice of the $SL_2(\mathbb{C})$ character variety is the Teichmüller space.
- The shear coordinates on the Teichmüller space can be complexified \Rightarrow coordinate description for the character variety.
- To visualize the confluence and the "decoration" we shall introduce **two moves** correspond to certain asymptotics in the (complexified) shear coordinates.

Two moves

- **Hooking holes:**



- **Pinching two sides of the same hole:**



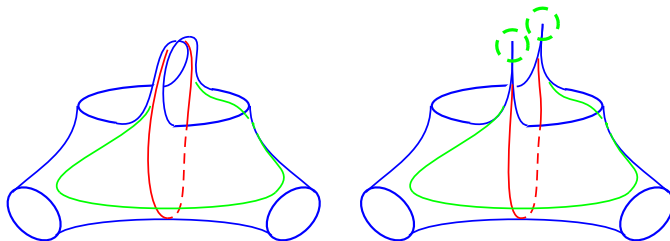


Figure: The process of confluence of two holes on the Riemann sphere with four holes. **Chewing-gum move:** hook two holes together and stretch to infinity by keeping the area between them finite (see Fig.). As a result we obtain a Riemann sphere with one less hole, but with two new cusps on the boundary of this hole. The red geodesic line which was initially closed becomes infinite, therefore two horocycles (the green dashed circles) must be introduced in order to measure its length.

Teichmüller space

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$$\operatorname{Hom}(\pi_1(\Sigma), \mathbb{P}SL_2(\mathbb{R})) / GL_2(\mathbb{R}).$$

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This will provide a **cluster algebra of geometric type**

Coordinates: geodesic lengths

Theorem

The geodesic length functions ($G_\gamma := \text{Tr} \gamma = 2 \cosh(l_\gamma)$) form an algebra with multiplication:

$$G_\gamma G_{\tilde{\gamma}} = G_{\gamma\tilde{\gamma}} + G_{\gamma\tilde{\gamma}^{-1}}.$$

$$(\text{Tr}(AB) + \text{Tr}(AB^{-1}) = \text{Tr}(A)\text{Tr}(B) \quad \forall A, B \in \text{SL}_2(\mathbb{C}))$$

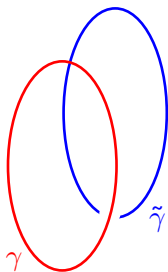
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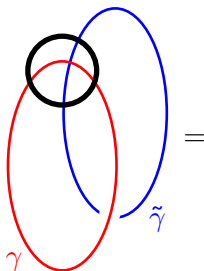
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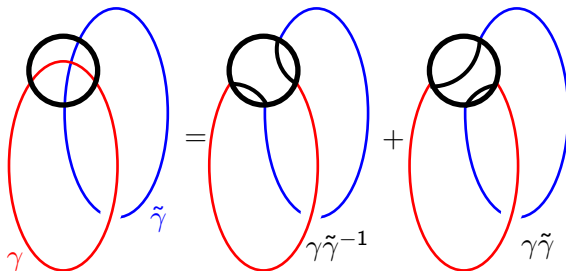
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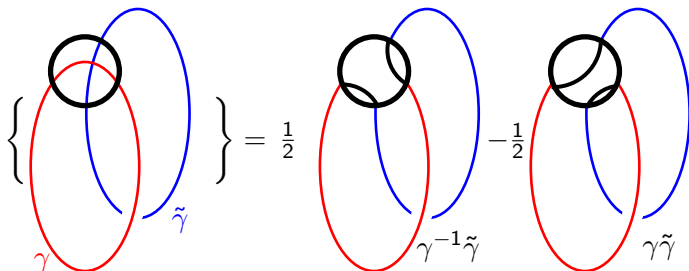
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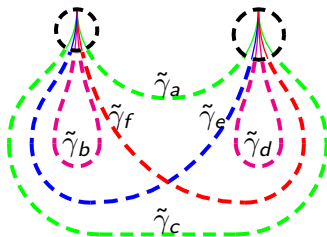


Poisson structure

$$\{G_\gamma, G_{\tilde{\gamma}}\} = \frac{1}{2}G_{\gamma\tilde{\gamma}} - \frac{1}{2}G_{\gamma\tilde{\gamma}^{-1}}.$$



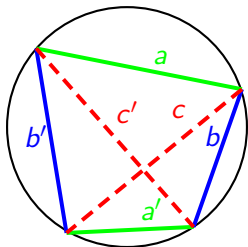
Ptolemy Relation-1



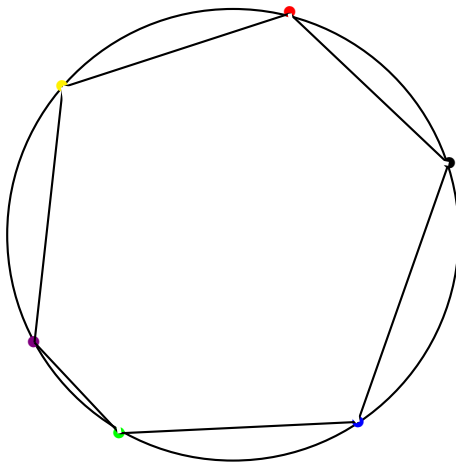
$$G_{\tilde{\gamma}_e} G_{\tilde{\gamma}_f} = G_{\tilde{\gamma}_a} G_{\tilde{\gamma}_c} + G_{\tilde{\gamma}_b} G_{\tilde{\gamma}_d}$$

Ptolemy Relation-2

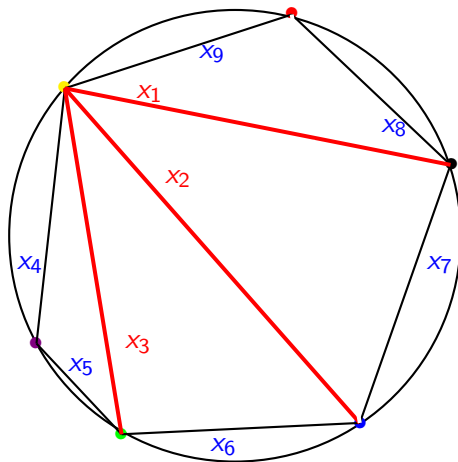
$$aa' + bb' = cc'$$



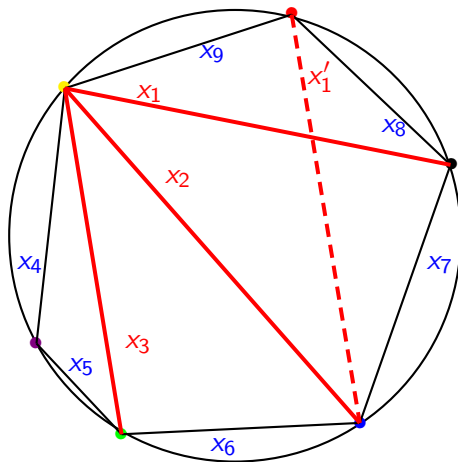
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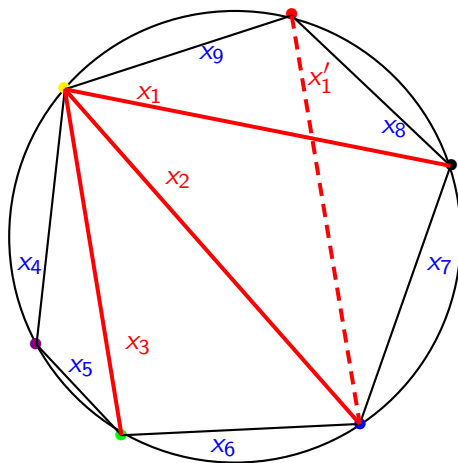


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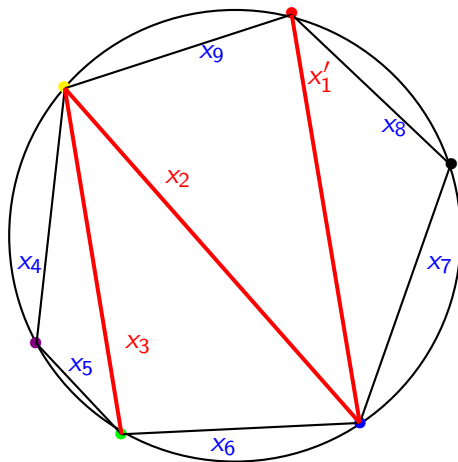
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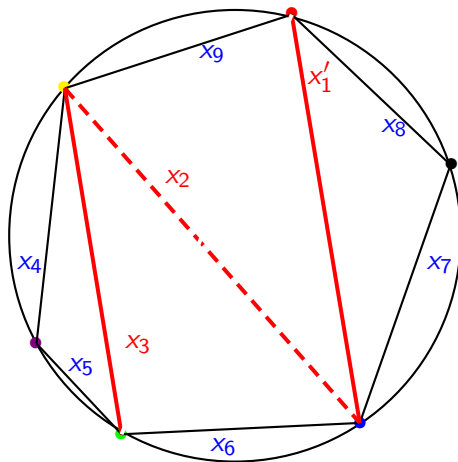
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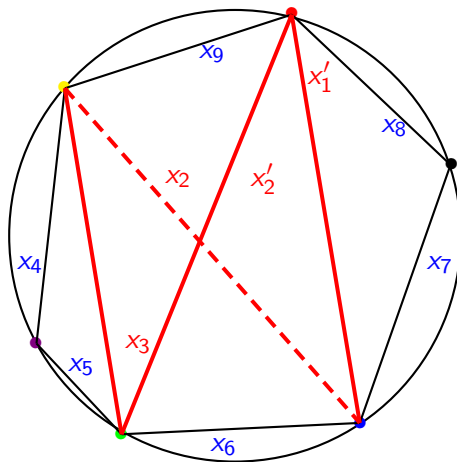
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$\mu_i : (x_1, x_2, \dots, x_n) \rightarrow (x'_1, x'_2, \dots, x'_n)$, $\mu_i : B \rightarrow B'$ where

$$x_i x'_i = \prod_{j: b_{ij} > 0} x_j^{b_{ij}} + \prod_{j: b_{ij} < 0} x_j^{-b_{ij}}, \quad x'_j = x_j \quad \forall j \neq i.$$

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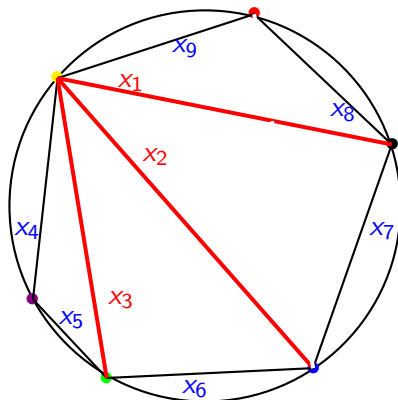
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Definition

A cluster algebra of rank n is a set of all seeds (x_1, \dots, x_n, B) related to one another by sequences of mutations μ_1, \dots, μ_k . The cluster variables x_1, \dots, x_k are called **exchangeable**, while x_{k+1}, \dots, x_n are called **frozen**. [Fomin-Zelevinsky 2002].

Example

Cluster algebra of rank 9 with 3 exchangeable variables x_1, x_2, x_3 and 6 frozen ones x_4, \dots, x_9 .



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All Berenstein-Zelevinsky cluster algebras are geometric

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Poisson structure

Theorem

The Poisson algebra of the λ -lengths of a complete cusped lamination is a Poisson cluster algebra [Chekhov-Mazzocco. ArXiv:1509.07044].

$$\{g_{s_i, t_j}, g_{p_r, q_l}\} = g_{s_i, t_j} g_{p_r, q_l} \mathcal{I}_{s_i, t_j, p_r, q_l}$$

$$\mathcal{I}_{s_i, t_j, p_r, q_l} = \frac{\epsilon_{i-r} \delta_{s,p} + \epsilon_{j-r} \delta_{t,p} + \epsilon_{i-l} \delta_{s,q} + \epsilon_{j-l} \delta_{t,q}}{4}$$

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- Replace tr by two characters: tr and tr_K .

Cusped fat graphs

Cusped fat graph (a graph with the prescribed cyclic ordering of edges entering each vertex) $\mathcal{G}_{g,s,n}$ a *spine of the Riemann surface* $\Sigma_{g,s,n}$ with g handles, s holes and $n > 0$ decorated bordered cusps if

- (a) this graph can be embedded without self-intersections in $\Sigma_{g,s,n}$;
- (b) all vertices of $\mathcal{G}_{g,s,n}$ are three-valent except exactly n one-valent vertices (endpoints of the open edges), which are placed at the corresponding bordered cusps;
- (c) upon cutting along all **non-open** edges of $\mathcal{G}_{g,s,n}$ the Riemann surface $\Sigma_{g,s,n}$ splits into s polygons each containing exactly one hole and being simply connected upon contracting this hole.

Geometric laminations

We call geometric *cusped geodesic lamination* (CGL) on a bordered cusped Riemann surface a set of nondirected curves up to a homotopy equivalence such that

- (a) these curves are either closed curves (γ) or *arcs* (α) that start and terminate at bordered cusps (which can be the same cusp);
- (b) these curves have no (self)intersections inside the Riemann surface (but can be incident to the same bordered cusp);
- (c) these curves are not empty loops or empty loops starting and terminating at the same cusp.

Decorated character variety-2

We introduce *the fundamental groupoid of arcs* \mathcal{G} as the set of all directed paths $\gamma_{ij} : [0, 1] \rightarrow \tilde{\Sigma}_{g,s,n}$ such that $\gamma_{ij}(0) = m_i$ and $\gamma_{ij}(1) = m_j$ modulo homotopy. The groupoid structure is dictated by the usual path-composition rules.

For each m_j , $j = 1, \dots, n$, the isotopy group

$$\Pi_j = \{\gamma_{jj} | \gamma_{jj} : [0, 1] \rightarrow \tilde{\Sigma}_{g,s,n}, \gamma_{jj}(0) = m_j, \gamma_{jj}(1) = m_j\} / \{\text{homotopy}\}$$

is isomorphic to the usual fundamental group and $\Pi_j = \gamma_{ij}^{-1} \Pi_i \gamma_{ij}$ for any arc $\gamma_{ij} \in \mathcal{G}$.

The decoration assigns to each arc γ_{ij} a matrix $M_{ij} \in SL_2(\mathbb{R})$, for example $M_{ij} = X(k_j) L X(z_n) R \cdots L X(z_1) R X(k_i)$.

Decorated character variety-3

To associate a matrix in $SL_2(\mathbb{C})$ - complexify the coordinates. The decorated character variety is:

$$\mathrm{Hom}(\mathcal{G}, SL_2(\mathbb{C})) / \prod_{j=1}^n B_j,$$

where B_j is the (unipotent) Borel subgroup in $SL_2(\mathbb{C})$ (one Borel subgroup for each cusp) with the characters:

$$\mathrm{Tr}_K : SL_2(\mathbb{C}) \rightarrow \mathbb{C}$$

$$M \mapsto \mathrm{Tr}(MK), \quad \text{where} \quad K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

PVI

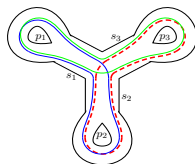


FIGURE 4. The fat graph of the 4-holed Riemann sphere. The dashed geodesic is x_1 ; the solid geodesics are x_2 and x_3 .

Shear coordinates in the Teichmüller space

Fatgraph:



FIGURE 4. The fat graph of the 3-bulb Brieskorn sphere. The dashed gradient is γ_a , the solid gradients are γ_b and γ_c .

Decompose each hyperbolic element in Right,

Left and Edge matrices Fock, Thurston

$$R := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

$$X_y := \begin{pmatrix} 0 & -\exp\left(\frac{y}{2}\right) \\ \exp\left(-\frac{y}{2}\right) & 0 \end{pmatrix}.$$



FIGURE 5. The fat graph of the 3-sheet Riemann sphere. The dashed graph is as the solid graph is on \mathbb{C}^* and \mathbb{C} .

The three geodesic lengths: $x_i = \text{Tr}(\gamma_{jk})$

$$x_1 = e^{s_2+s_3} + e^{-s_2-s_3} + e^{-s_2+s_3} + \left(e^{\frac{p_2}{2}} + e^{-\frac{p_2}{2}}\right)e^{s_3} + \left(e^{\frac{p_3}{2}} + e^{-\frac{p_3}{2}}\right)e^{-s_2}$$

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$$\{x_1, x_2\} = x_1x_2 + 2x_3 + \omega_3, \quad \{x_2, x_3\} =$$

$$x_2x_3 + 2x_1 + \omega_1, \quad \{x_3, x_1\} = x_3x_1 + 2x_2 + \omega_2.$$

The confluence from the cubic associated to PVI to the one associated to PV is realized by

$$p_3 \rightarrow p_3 - 2 \log[\epsilon],$$

in the limit $\epsilon \rightarrow 0$. We obtain the following shear coordinate description for the *PV* cubic:

$$\begin{aligned} x_1 &= -e^{s_2+s_3+\frac{p_2}{2}+\frac{p_3}{2}} - G_3 e^{s_2+\frac{p_2}{2}}, \\ x_2 &= -e^{s_3+s_1+\frac{p_3}{2}+\frac{p_1}{2}} - e^{s_3-s_1+\frac{p_3}{2}-\frac{p_1}{2}} - G_3 e^{-s_1-\frac{p_1}{2}} - G_1 e^{s_3+\frac{p_3}{2}}, \\ x_3 &= -e^{s_1+s_2+\frac{p_1}{2}+\frac{p_2}{2}} - e^{-s_1-s_2-\frac{p_1}{2}-\frac{p_2}{2}} - e^{s_1-s_2+\frac{p_1}{2}-\frac{p_2}{2}} - G_1 e^{-s_2-\frac{p_2}{2}} - G_\infty \end{aligned}$$

where

$$G_i = e^{\frac{p_i}{2}} + e^{-\frac{p_i}{2}}, \quad i = 1, 2, \quad G_3 = e^{\frac{p_3}{2}}, \quad G_\infty = e^{s_1+s_2+s_3+\frac{p_1}{2}+\frac{p_2}{2}+\frac{p_3}{2}}.$$

These coordinates satisfy the following cubic relation:

$$x_1 x_2 x_3 + x_1^2 + x_2^2 - (G_1 G_\infty + G_2 G_3) x_1 - (G_2 G_\infty + G_1 G_3) x_2 - G_3 G_\infty x_3 + G_\infty^2 + G_3^2 + G_1 G_2 G_3 G_\infty = 0. \quad (11)$$

Note that the parameter p_3 is now redundant, we can eliminate it by rescaling. To obtain the correct PV- cubic, we need to pick $p_3 = -p_1 - p_2 - 2s_1 - 2s_2 - 2s_3$ so that $G_\infty = 1$.

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$$\{x_1, x_2\} = x_1 x_2 - G_3 G_\infty, \quad \{x_2, x_3\} = x_2 x_3 + 2x_1 - (G_1 G_\infty + G_2 G_3), \quad \{x_3, x_1\} = x_3 x_1 + 2x_2 - (G_2 G_\infty + G_1 G_3).$$

Geometrically speaking, sending the perimeter p_3 to infinity means that we are performing a **chewing-gum move**:

two holes, one of perimeter p_3 and the other of perimeter $s_1 + s_2 + s_3 + \frac{p_1}{2} + \frac{p_2}{2} + \frac{p_3}{2}$, become infinite, but the area between them remains finite.

This is leading to a Riemann sphere with three holes and two cusps on one of them. In terms of the fat-graph, this is represented by Figure 2.

The geodesic x_3 corresponds to the closed loop obtained going around p_1 and p_2 (green and red loops), while x_1 and x_2 are "asymptotic geodesics" starting at one cusp, going around p_1 and p_2 respectively, and coming back to the other cusp.

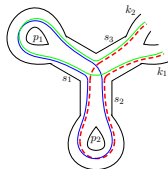


FIGURE 5. The fat graph corresponding to PV. The geodesic x_3 remains closed, while x_1 (the dashed line) and x_2 become arcs.

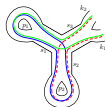


FIGURE 5. The fat graph corresponding to PV. The geodesic s_3 remains closed, while s_1 (the dashed line) and s_2 become arcs.

$$\gamma_b = X(k_1)RX(s_3)RX(s_2)RX(p_2)RX(s_2)LX(s_3)LX(k_1) - \text{BUT its length}$$

$$\text{is } b = \text{tr}_K(\gamma_b) = \text{tr}(bK), \quad K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$



FIGURE 5. The fat graph corresponding to PV. The geodesic x_2 remains closed, while x_1 (the dashed line) and x_3 become arcs.

$$\begin{aligned}
 \{g_{s_i, t_j}, g_{p_r, q_l}\} &= g_{s_i, t_j} g_{p_r, q_l} \frac{\epsilon_{i-r} \delta_{s,p} + \epsilon_{j-r} \delta_{t,p} + \epsilon_{i-l} \delta_{s,q} + \epsilon_{j-l} \delta_{t,q}}{4} \\
 \{b, d\} &= \{g_{13,14}, g_{21,18}\} \\
 &= g_{13,14} g_{21,18} \frac{\epsilon_{3-1} \delta_{1,2} + \epsilon_{4-1} \delta_{1,2} + \epsilon_{3-8} \delta_{1,1} + \epsilon_{4-8} \delta_{1,1}}{4} \\
 &= -bd \frac{1}{2}
 \end{aligned}$$

The character variety of a Riemann sphere with three holes and two cusps on one boundary is 7-dimensional (rather than 2-dimensional like in PVI case). The fat-graph admits a complete cusped lamination as displayed in the figure below. A full set of coordinates on the character variety is given by the five elements in the lamination and the two parameters G_1 and G_2 which determine the perimeter of the two non-cusped holes.

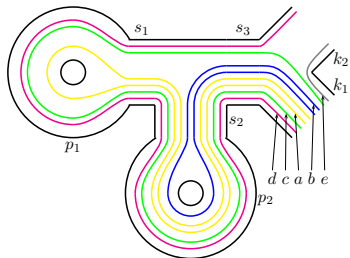


FIGURE 6. The system of arcs for PV.

Notice that there are two shear coordinates associated to the two cusps, they are denoted by k_1 and k_2 , their sum corresponds to what we call p_3 above.

These shear coordinates do not commute with the other ones, they satisfy the following relations:

$$\{s_3, k_1\} = \{k_1, k_2\} = \{k_2, s_3\} = 1.$$

As a consequence in the character variety, the elements G_3 and G_∞ are not Casimirs.

In terms of shear coordinates, the elements in the lamination are expressed as follows:

$$\begin{aligned} a &= e^{k_1 + s_1 + 2s_2 + s_3 + \frac{p_1}{2} + p_2}, & b &= e^{k_1 + s_2 + s_3 + \frac{p_2}{2}}, & e &= e^{\frac{k_1}{2} + \frac{k_2}{2}}, \\ c &= e^{k_1 + s_1 + s_2 + s_3 + \frac{p_1}{2} + \frac{p_2}{2}}, & d &= e^{\frac{k_1}{2} + \frac{k_2}{2} + s_1 + s_2 + s_3 + \frac{p_1}{2} + \frac{p_2}{2}}. \end{aligned} \quad (12)$$

They satisfy the following Poisson relations:

$$a \quad \{a, b\} = ab, \quad \{a, c\} = 0, \quad \{a, d\} = -\frac{1}{2}ad, \quad \{a, e\} = \frac{1}{2}ad, \quad (13)$$

$$\{b, c\} = 0, \quad \{b, d\} = -\frac{1}{2}bd, \quad \{b, e\} = \frac{1}{2}be, \quad (14)$$

$$\{c, d\} = -\frac{1}{2}cd, \quad \{c, e\} = \frac{1}{2}ce, \quad \{d, e\} = 0, \quad (15)$$

so that the element $G_3 G_\infty = de$ is a Casimir.

The symplectic leaves are determined by the values of the three Casimirs G_1, G_2 and $G_3 G_\infty$.

On each symplectic leaf, the PV monodromy manifold (11) is the subspace defined by those functions of a, b, c (and of the Casimir values $G_1, G_2, G_3 G_\infty$) which commute with $G_3 = e$. To see this, we can use relations (12) to determine the exponentiated shear coordinates in terms of a, b, c, d, e and then deduce the expressions of x_1, x_2, x_3 in terms of the lamination. We obtain the following expressions:

$$x_1 = -e \frac{a}{c} - d \frac{b}{c}, \quad x_2 = -e \frac{b}{c} - G_1 d \frac{b}{a} - d \frac{b^2}{ac} - d \frac{c}{a} \quad (16)$$

$$x_3 = -G_2 \frac{c}{b} - G_1 \frac{c}{a} - \frac{b}{a} - \frac{c^2}{ab} - \frac{a}{b}, \quad (17)$$

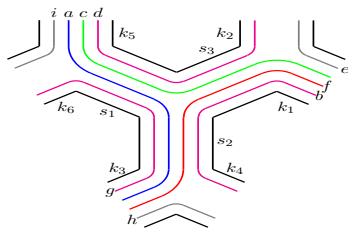
which automatically satisfy (11).

Due to the Poisson relations (13) the functions that commute with e are exactly the functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$. Such functions may depend on the Casimir values G_1, G_2 and $G_3 G_\infty$ and e itself, so that $d = G_\infty$ becomes a parameter now. With this in mind, it is easy to prove that x_1, x_2, x_3 are algebraically independent functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ so that x_1, x_2, x_3 form a basis in the space of functions which commute with e .

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$$\begin{aligned} \pi : \text{Spec}(\mathbb{C}[G_1, G_2, G_3, G_3^{-1}, x_1, x_2, x_3] / x_1 x_2 x_3 + x_1^2 + x_2^2 - \\ -(G_1 + G_2 G_3)x_1 - (G_2 + G_1 G_3)x_2 - G_3 x_3 + 1 + G_3^2 + G_1 G_2 G_3) - \\ \mapsto \text{Spec}(\mathbb{C}[G_1, G_2, G_3, G_3^{-1}]). \end{aligned} \quad (18)$$

The decorated character variety associated with PII^{JM} has 6 cusps on the boundary is 9-dimensional. The fat-graph admits a complete cusped lamination as displayed in the figure below.



Quantisation

For standard geodesic lengths $G_\gamma \rightarrow G_\gamma^{\hbar}$ [Chekhov-Fock '99]:

$$[G_\gamma^{\hbar}, G_{\tilde{\gamma}}^{\hbar}] = q^{-\frac{1}{2}} G_{\gamma^{-1}\tilde{\gamma}}^{\hbar} + q^{\frac{1}{2}} G_{\gamma\tilde{\gamma}}^{\hbar}$$

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For arcs $g_{s_i, t_j} \rightarrow g_{s_i, t_j}^\hbar$:

$$q^{\mathcal{I}_{s_i, t_j, p_r, q_l}} g_{s_i, t_j}^\hbar g_{p_r, q_l}^\hbar = g_{p_r, q_l}^\hbar g_{s_i, t_j}^\hbar q^{\mathcal{I}_{p_r, q_l, s_i, t_j}}$$

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$$q^{\mathcal{I}_{s_i, t_j, p_r, q_l}} g_{s_i, t_j}^\hbar g_{p_r, q_l}^\hbar = g_{p_r, q_l}^\hbar g_{s_i, t_j}^\hbar q^{\mathcal{I}_{p_r, q_l, s_i, t_j}}$$

This identifies the geometric basis of the quantum cluster algebras introduced by Berenstein - Zelevinsky.

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Many thanks for your attention!!!