CORRELATION FUNCTIONS OF INTEGRABLE MODELS, COMBINATORICS, AND RANDOM WALKS.

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ABSTRACT

Certain quantum integrable models solvable by the Quantum Inverse Scattering Method demonstrate a relationship with combinatorics, partition theory, and random walks. Special thermal correlation functions of the models in question are related with the generating functions for plane partitions and self-avoiding lattice paths. The correlation functions of the integrable models admit interpretation in terms of nests of the lattice paths made by random walkers.

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THE TALK IS BASED ON THE PAPERS

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[6] N. Bogoliubov, C. Malyshev, *Integrable models and combinatorics*, Russian Math. Surveys **70** (2015), 789.

[5] N. M. Bogoliubov, C. L. Malyshev, *Correlation functions of $XX0$ Heisenberg chain, q-binomial determinants, and random walks*, Nucl. Phys. B, **879** (2014), 268-291.

[4] N. Bogoliubov, *Calculation of correlation functions in totally asymmetric exactly solvable models on a ring*, Theoretical and Mathematical Physics **175** (2013), 755.

[3] N. M. Bogoliubov, C. Malyshev, *The correlation functions of the XXZ Heisenberg chain in the case of zero or infinite anisotropy, and random walks of vicious walkers*, St.-Petersburg Math. J. **22** (2011), 359-377.

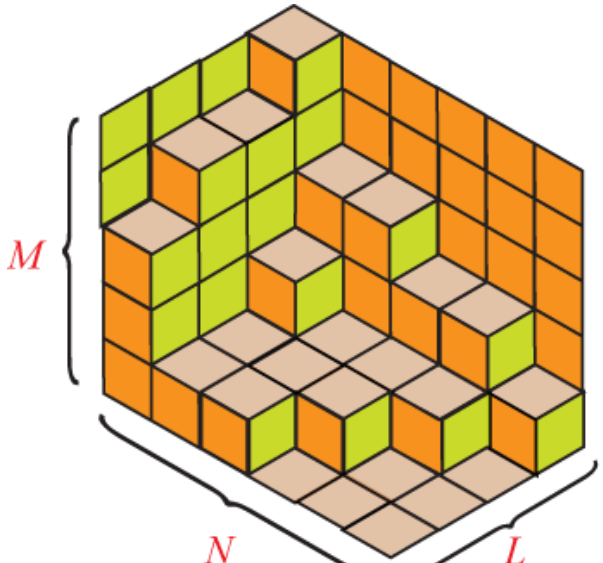
[2] N. M. Bogoliubov, C. Malyshev, *The correlation functions of the XX Heisenberg magnet and random walks of vicious walkers*, Theor. Math. Phys., **159** (2009), No. 2, 179-192.

[1] N. M. Bogoliubov, T. Nassar, *On the spectrum of the non-Hermitian phase-difference model*, Phys. Lett. A, **234** (1997), 345-350.

I. COMBINATORIAL PRELIMINARIES: q-BINOMIAL DETERMINANTS AND GENERATING FUNCTIONS OF PLANE PARTITIONS

• • • Let us begin with the definitions of the generating functions of *boxed plane partitions* and *self-avoiding lattice walks*.

An array $(\pi_{i,j})_{i,j\geq 1}$ of non-negative integers that are non-increasing as functions of both i and j $(i,j=1,2,\dots)$ is called boxed plane partition π. Plane partition is represented by cubes arranged as stacks with coordinates (i,j) and with height equal to $\pi_{i,j}$. The plane partition is contained inside a box $\mathcal{B}(L,N,M)$ provided that $i\leq L$, $j\leq N$ and $\pi_{i,j}\leq M$ for all cubes of the plane partition.



The generating function of plane partitions inside $\mathcal{B}(L,N,P)$ is defined as formal series $Z_q(L,N,P) \equiv \sum_{\{\boldsymbol{\pi}\}} q^{|\boldsymbol{\pi}|}$ (summation over all partitions inside the box), and it takes the form:

$$Z_q(L,N,P) = \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \prod_{j=1}^{L}\prod_{k=1}^{N}\frac{1-q^{P+j+k-1}}{1-q^{j+k-1}}\,.$$

The limit $q \to 1$ leads to the MacMahon formula:

$$A(L,N,P) = \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P}\frac{i+j+k-1}{i+j+k-2} = \prod_{j=1}^{L}\prod_{k=1}^{N}\frac{P+j+k-1}{j+k-1}\,.$$

• • • To calculate the correlation functions, we need the determinant of a block-matrix $(\bar{\mathsf{T}})_{1\le j,k\le N}$ given by entries:

$$\bar{\mathsf{T}}_{kj} = \frac{1-q^{(P+1)(j+k-1)}}{1-q^{j+k-1}}\,, \quad 1\le k\le L, \qquad 1\le j\le N\,,$$

$$\bar{\mathsf{T}}_{kj} = q^{j(N-k)}\,, \qquad L+1\le k\le N, \quad 1\le j\le N\,,$$

where P and $L\le N$ are arbitrary. The matrix $(\bar{\mathsf{T}})_{1\le j,k\le N}$ consists of two blocks of the sizes $L\times N$ and $(N-L)\times N$.

Several definitions are in order.

The q-*binomial determinant* $\binom{\mathbf{a}}{\mathbf{b}}_q$ is defined by

$$\binom{\mathbf{a}}{\mathbf{b}}_q \equiv \begin{pmatrix} a_1, & a_2, & \dots & a_S \\ b_1, & b_2, & \dots & b_S \end{pmatrix}_q \equiv \det\left(\begin{bmatrix} a_j \\ b_i \end{bmatrix}\right)_{1\le i,j\le S},$$

where $\mathbf{a}$ and $\mathbf{b}$ are ordered tuples: $0 \le a_1 < a_2 < \dots < a_S$ and $0 \le b_1 < b_2 < \dots < b_S$. The entries $\begin{bmatrix} a_j \\ b_i \end{bmatrix}$ are the q-*binomial coefficients*:

$$\begin{bmatrix} N \\ r \end{bmatrix} \equiv \frac{(1-q^N)(1-q^{N-1})\dots(1-q^{N-r+1})}{(1-q)(1-q^2)\dots(1-q^r)}, \qquad q \in \mathbb{R}.$$

The q-binomial coefficients are replaced at $q \to 1$ by the binomial coefficients $\binom{a_j}{b_i}$. The q-binomial determinant is transformed to the *binomial determinant*:

$$\binom{\mathbf{a}}{\mathbf{b}} \equiv \begin{pmatrix} a_1, & a_2, & \dots & a_S \\ b_1, & b_2, & \dots & b_S \end{pmatrix} = \det\left(\binom{a_j}{b_i}\right)_{1\le i,j\le S}.$$

The binomial determinant is positive at $b_i \le a_i$, $\forall i$.

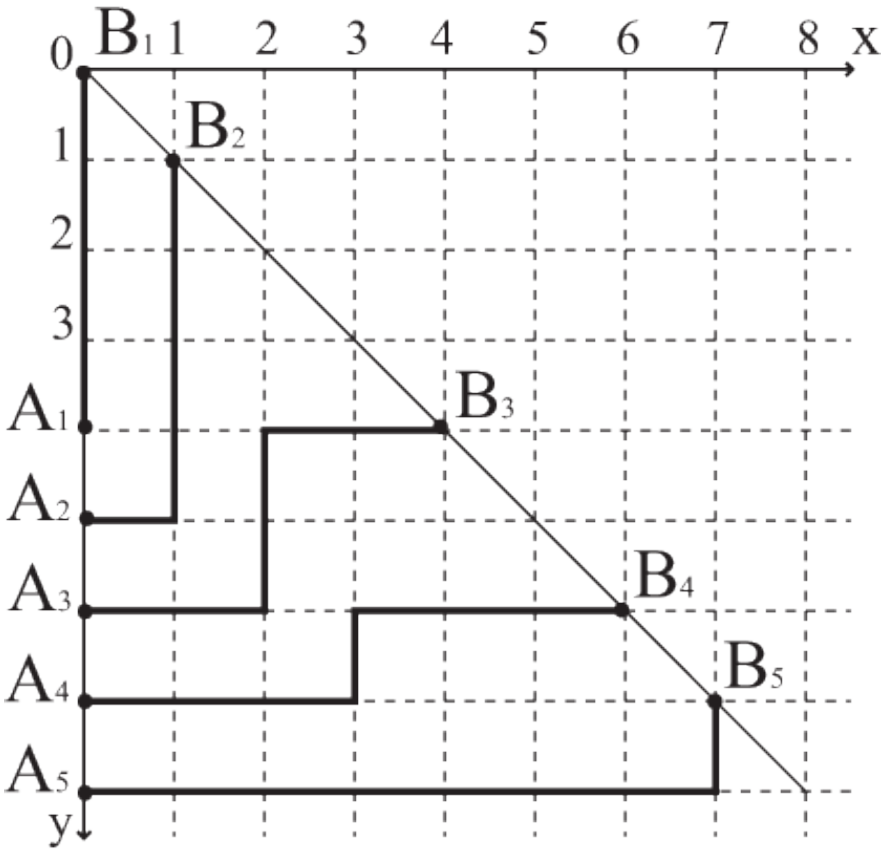


Рис.: S-Tuple $(w_1, w_2, \ldots, w_S)$ of self-avoiding walks for $S = 5$.

Binomial determinant gives the number of self-avoiding walks across two-dimensional lattice. Each path w_i from a tuple $(w_1, w_2, \ldots, w_S)$ goes from $A_i = (0, a_i)$ to $B_i = (b_i, b_i)$, $1 \leq i \leq S$. Only steps to north and to east are allowed.

So, let us consider $(\bar{\mathsf{T}})_{1\le j,k\le N}$ given by the entries:

$$\begin{aligned}
\bar{\mathsf{T}}_{kj} &= \frac{1-q^{(P+1)(j+k-1)}}{1-q^{j+k-1}}\,, && 1\le k\le L, && 1\le j\le N\,,\\
\bar{\mathsf{T}}_{kj} &= q^{j(N-k)}, && L+1\le k\le N, && 1\le j\le N\,,
\end{aligned}$$

where P and $L\le N$ are arbitrary. Now we formulate the following

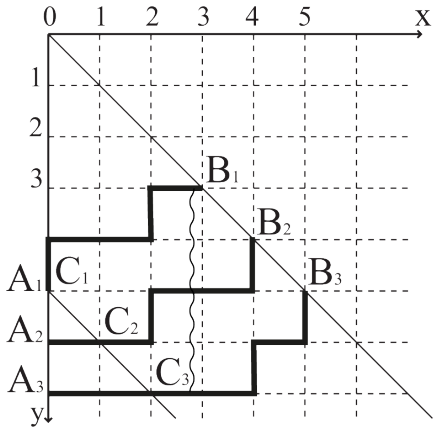
PROPOSITION 1 *Let the matrix $(\bar{\mathsf{T}})_{1\le j,k\le N}$, be defined by the entries* (32) *with $\frac{P}{2}<N<P$. The determinant of $(\bar{\mathsf{T}})_{1\le j,k\le N}$ is given as:*

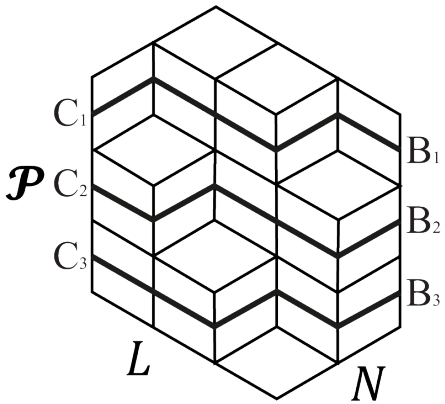
$$\begin{aligned}
& q^{-\frac{L}{2}(L-1)(N-L)}\,\frac{\det(\bar{\mathsf{T}})_{1\le j,k\le N}}{\mathcal{V}(\mathbf{q}_N)\mathcal{V}(\mathbf{q}_L/q)}\\
&= q^{-\frac{N}{2}(\mathcal{P}-1)\mathcal{P}}\begin{pmatrix} L+N, & L+N+1, & \dots & L+N+\mathcal{P}-1\\ L, & L+1, & \dots & L+\mathcal{P}-1\end{pmatrix}_q\\
&= \prod_{k=1}^{\mathcal{P}}\prod_{j=1}^{L}\frac{1-q^{j+k+N-1}}{1-q^{j+k-1}} = Z_q(L,N,\mathcal{P})\,,
\end{aligned}$$

where $\mathcal{P}\equiv P-N+1$, and $Z_q(L,N,\mathcal{P})$ is the generating function of plane partitions.

The *PROPOSITION 1* relates $\det \bar{\mathsf{T}}$ to the q-binomial determinant, which is transformed at $q \to 1$ to the binomial determinant equal, in turn, to the number of $\mathcal{P}$-tuples of lattice self-avoiding paths between the *end points* $A_l = (0, N + L + l - 1)$ and $B_l = (L + l - 1, L + l - 1)$, $1 \leq l \leq \mathcal{P}$.

The Figure gives appropriate picture with *end points* A_l and B_l at $\mathcal{P} = L = 3$ and $N = 2$.





The generating function $Z_q(L,N,\mathcal{P})$ gives at $q \to 1$ the number of plane partitions $A(L,N,\mathcal{P})$ inside $\mathcal{B}(L,N,\mathcal{P})$:

$$Z_q(L,N,\mathcal{P}) = \prod_{j=1}^{L}\prod_{k=1}^{N} \frac{1-q^{\mathcal{P}+j+k-1}}{1-q^{j+k-1}} \underset{q\to 1}{\longrightarrow}$$

$$\underset{q\to 1}{\longrightarrow} A(L,N,\mathcal{P}) = \det\left(\left(\begin{array}{c} N+L+i-1 \\ L+j-1 \end{array}\right)\right)_{1\le i,j\le \mathcal{P}}.$$

RHS expresses the fact that number of plane partitions $A(L,N,\mathcal{P})$ is equal to the number of self-avoiding lattice paths. Just the paths constituting a $\mathcal{P}$-tuple are in bi-jection with, so-called, gradient lines corresponding to a plane partition inside $\mathcal{B}(L,N,\mathcal{P})$.

I.1 FREE FERMION LIMIT OF XXZ HEISENBERG MODEL

• • • Consider $XX0$ model describing free-fermion limit of XXZ spin-$\frac{1}{2}$ Heisenberg chain. The $XX0$ Hamiltonian in absence of magnetic field takes the form:

$$\hat{H}_{XX} \equiv -\frac{1}{2}\sum_{k=0}^{M}(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-}) = -\frac{1}{2}\sum_{n,m=0}^{M}\Delta_{nm}\sigma_{n}^{-}\sigma_{m}^{+}\,,$$

where Δ_{nm} is the hopping matrix with the entries

$$\Delta_{nm} = \delta_{|n-m|,1} + \delta_{|n-m|,M}\,,$$

or, less formally,

$$\Delta = \begin{pmatrix} 0 & 1 & & & & 1 \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & & \dots & & \\ & & & 1 & 0 & 1 \\ 1 & & & & 1 & 0 \end{pmatrix}.$$

The local spin operators $\sigma_k^{\pm} = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$, $k \in \{0, 1, \ldots, M\}$ are defined on sites of periodic chain of "length" $M+1$, and they are the tensor products:

$$\sigma_k^{\#} = \sigma^0 \otimes \cdots \otimes \sigma^0 \otimes \underbrace{\sigma^{\#}}_{k} \otimes \sigma^0 \otimes \cdots \otimes \sigma^0 \,,$$

where σ^0 is 2×2 unit matrix, and $\sigma^{\#}$ at kth site is a Pauli matrix, $\sigma^{\#} \in \mathfrak{su}(2)$ ($\#$ is either x, y, z or $\pm$). The commutation rules are:

$$[\,\sigma_k^+, \sigma_l^-\,] \,=\, \delta_{k,l}\,\sigma_l^z \,, \quad [\,\sigma_k^z, \sigma_l^{\pm}\,] \,=\, \pm 2\,\delta_{k,l}\,\sigma_l^{\pm} \,.$$

We introduce spin "up" and spin "down" states, $|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The Pauli operators act on $|\uparrow\rangle$ and $|\downarrow\rangle$ as follows:

$$\sigma^- \, |\uparrow\rangle = |\downarrow\rangle, \qquad \sigma^- \, |\downarrow\rangle = 0\,, \qquad \sigma^- = \begin{pmatrix} 00 \\ 10 \end{pmatrix}\,,$$

$$\sigma^+ \, |\uparrow\rangle = 0, \qquad \sigma^+ \, |\downarrow\rangle = |\uparrow\rangle\,, \qquad \sigma^+ = \begin{pmatrix} 01 \\ 00 \end{pmatrix}\,.$$

The lattice spin operators defined above act over the the state-vectors $\bigotimes_{k=0}^{M} |s\rangle_k$, where $s = \uparrow, \downarrow$. The state-vectors are elements of the state-space $\mathfrak{H}_{M+1} = \bigotimes\limits_{k=0}^{M} \left(\mathbb{C}^2\right)_k$.

The periodic boundary conditions $\sigma^{\#}_{k+(M+1)} = \sigma^{\#}_k$ are imposed.

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Let the sites with spin "down" states are labeled by decreasing coordinates $M \geq \mu_1 > \mu_2 > \ldots > \mu_N \geq 0$, which constitute strict partition $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$.
In the free-fermion limit, we define N-excitation state-vectors $|\Psi_N(\mathbf{u})\rangle$:

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \left(\prod_{k=1}^{N} \sigma^-_{\mu_k} \right) |\Uparrow\rangle \,, \quad |\Uparrow\rangle \equiv \bigotimes_{n=0}^{M} |\uparrow\rangle_n \,,$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is $\boldsymbol{\lambda} = \boldsymbol{\mu} - \boldsymbol{\delta}_N$, where $\boldsymbol{\delta}_N = (N-1, N-2, \ldots, 1, 0)$. Besides, $\mathcal{M} \equiv M + 1 - N \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$.

The parameters $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ and $\mathbf{u}^2 \equiv (u_1^2, u_2^2, \ldots, u_N^2)$ correspond to arbitrary complex numbers.

The coefficients of the state-vector $|\Psi_N(\mathbf{u})\rangle$ are given by the *Schur functions* defined by the Jacobi-Trudi relation:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}_N) \equiv S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_N) \equiv \frac{\det(x_j^{\lambda_k+N-k})_{1\leq j,k\leq N}}{\mathcal{V}(\mathbf{x}_N)}\,,$$

in which $\mathcal{V}(\mathbf{x}_N)$ is the Vandermonde determinant

$$\mathcal{V}(\mathbf{x}_N) \equiv \det(x_j^{N-k})_{1\leq j,k\leq N} = \prod_{1\leq m<l\leq N}(x_l - x_m)\,.$$

There is a natural correspondence between the coordinates of the spin "down" states $\boldsymbol{\mu}$ and the partition $\boldsymbol{\lambda}$ expressed by the Young diagram:

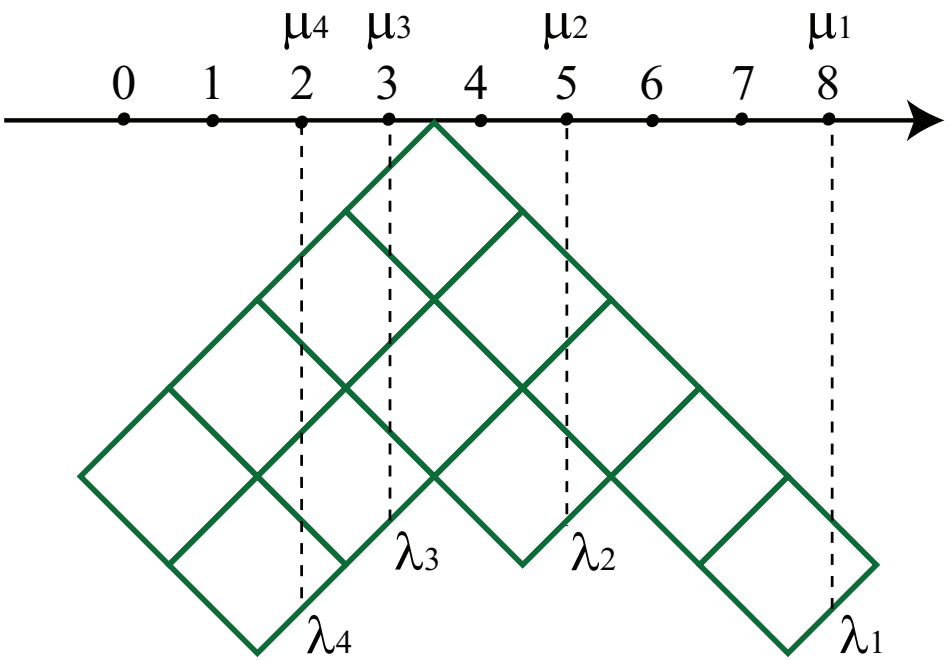

Рис.: Relation of the spin "down" coordinates $\boldsymbol{\mu} = (8, 5, 3, 2)$ and partition $\boldsymbol{\lambda} = (5, 3, 2, 2)$ for $M = 8$, $N = 4$.

The states $|\Psi_N(\mathbf{u})\rangle$ are the eigen-states,

$$\hat{H}_{XX}\,|\Psi_N(\mathbf{u}_N)\rangle = E_N\,|\Psi_N(\mathbf{u}_N)\rangle\,,$$

with eigen-values $E_N \equiv E_N^{XX}(I_1, I_2, \ldots, I_N) = -\sum_{l=1}^{N}\cos\left(\frac{2\pi I_l}{M+1}\right)$, if and only if u_l $(1 \leq l \leq N)$ satisfy the *Bethe equations*:

$$u_j^{2(M+1)} = (-1)^{N-1}\,, \qquad u_j^2 = e^{i\frac{2\pi}{M+1}I_j}\,, \qquad 1 \leq j \leq N\,.$$

where I_j are integers or half-integers: $M \geq I_1 > I_2 > \cdots > I_N \geq 0$.

The scalar products of the state-vectors are calculated by means of the Binet–Cauchy formula:

$$\mathcal{P}_{L/n}(\mathbf{y},\mathbf{x}) \equiv \sum_{\boldsymbol{\lambda}\subseteq\{(L/n)^N\}} S_{\boldsymbol{\lambda}}(x_1^2,\ldots,x_N^2)S_{\boldsymbol{\lambda}}(y_1^2,\ldots,y_N^2)$$

$$= \frac{\left(\prod_{l=1}^{N} y_l^n x_l^n\right)\det(T_{jk})_{1\le j,k\le N}}{\prod_{1\le k<j\le N}\left(y_j^2-y_k^2\right)\prod_{1\le m<l\le N}\left(x_l^2-x_m^2\right)},$$

where summation $\sum_{\boldsymbol{\lambda}\subseteq\{(L/n)^N\}}$ is over non-strict partitions $\boldsymbol{\lambda}$: $L \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N \ge n$, $n \ge 0$. The entries T_{jk} take the form:

$$T_{kj} = \frac{1-(x_k y_j)^{N+L-n}}{1-x_k y_j}.$$

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PROPOSITION 2 *The following sums of products of the Schur functions take place*:

$$\sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^{N-n}\}} S_{\hat{\boldsymbol{\lambda}}}(\mathbf{v}_N^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}_{N-n}^2) = \left(\prod_{l=1}^{N-n} u_l^{-2n}\right) \frac{\det(\bar{T}_{kj})_{1\le k,j\le N}}{\mathcal{V}(\mathbf{u}_{N-n}^2)\mathcal{V}(\mathbf{v}_N^{-2})},$$

$$\sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^{N-n}\}} S_{\boldsymbol{\lambda}}(\mathbf{v}_{N-n}^{-2}) S_{\hat{\boldsymbol{\lambda}}}(\mathbf{u}_N^2) = \left(\prod_{l=1}^{N-n} v_l^{2n}\right) \frac{\det(\tilde{T}_{kj})_{1\le k,j\le N}}{\mathcal{V}(\mathbf{v}_{N-n}^{-2})\mathcal{V}(\mathbf{u}_N^2)},$$

where the entries of the matrices $(\bar{T}_{kj})_{1\le k,j\le N}$ *and* $(\tilde{T}_{kj})_{1\le k,j\le N}$ *are*:

$$\begin{aligned}
\bar{T}_{kj} &= T^{\mathrm{o}}_{kj}\,, && 1\le k\le N-n, && 1\le j\le N\,,\\
\bar{T}_{kj} &= v_j^{-2(N-k)}\,, && N-n+1\le k\le N, && 1\le j\le N\,,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{T}_{kj} &= T^{\mathrm{o}}_{kj}\,, && 1\le k\le N, && 1\le j\le N-n\,,\\
\tilde{T}_{kj} &= u_j^{2(N-k)}\,, && 1\le k\le N, && N-n+1\le j\le N\,.
\end{aligned}$$

I.2 CORRELATION FUNCTIONS AND VICIOUS WALKERS

• • • Consider the simplest one-particle correlation function

$$G\left(j,l|t\right) \equiv \langle \Uparrow | \, \sigma_j^+ e^{-t\mathcal{H}} \sigma_l^- \, | \Uparrow \rangle \, ,$$

where $\mathcal{H}$ is the Hamiltonian. Differentiating $G\left(j,l|t\right)$ over t and using the commutators obtain:

$$\frac{d}{dt} G\left(j,l|t\right) = \sum_n \Delta_{nl} \langle \Uparrow | \, \sigma_j^+ e^{-t\mathcal{H}} \sigma_n^- \, | \Uparrow \rangle = \sum_n \Delta_{nl} G\left(j,n|t\right) \, .$$

The correlation function satisfies:

$$\frac{d}{dt} G\left(j,l|t\right) = G\left(j,l-1|t\right) + G\left(j,l+1|t\right) \, ,$$

for fixed j (and analogous eqn at fixed l). The "initial condition" is: $G\left(j,l|0\right) = \delta_{jl}$.

The correlator $G\left(j,l|t\right)$ may be considered as the generating function of random walks. Expanding in powers of t one has:

$$G\left(j,l|t\right)=\sum_{K=0}^{\infty}\frac{t^K}{K!}\langle\Uparrow|\,\sigma_j^+(-\mathcal{H})^K\sigma_l^-\,|\Uparrow\rangle\,.$$

This equality may be interpreted as follows. Position of the walker on lattice is labelled by the spin "down" state, while the spin "up" states correspond to empty sites. The relation enables to enumerate all admissible trajectories of the walker starting from l^{th} site. The state $\langle\Uparrow|\,\sigma_j^+$ acting on from the left allows to fix the ending point of the trajectories because of orthogonality of the spin states. Hence,

$$\langle\Uparrow|\,\sigma_j^+(-\mathcal{H})^K\sigma_l^-\,|\Uparrow\rangle=\left(\Delta^K\right)_{jl}\,,$$

i.e., we obtain the entry of Kth power of the hopping matrix $\{\Delta_{nl}\}_{0\leq n,l\leq M}$.

Consider an infinite chain ($M \to \infty$). In this case the solution respecting $G(j,\, l|0) = \delta_{j,\,l}$ is the modified Bessel function $I_{j-l}(t)$:

$$G(j,\, l|t) \;=\; I_{j-l}(t) \;=\; \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} e^{t\cos\theta} e^{i(j-l)\theta} d\theta\,.$$

Let K be such that $K \geq |l-j|$ and $K + |j-l| = 0 (\mathrm{mod}\, 2)$. Then differentiation gives the binomial formula $|P_K(l \to j)| = C_K^L$ for the number of all lattice paths of "length" K between two sites on an infinite axis:

$$|P_K(l \to j)| \;\equiv \left(\frac{d}{dt}\right)^K \left[G(j,\, l|t)\right]\Big|_{t=0} = \frac{(m+2L)!}{L!\,(m+L)!}\,,$$

where $L \equiv (K-m)/2$ is half the number of turns.

• • • Consider now the multi-particle correlation function

$$G(\mathbf{j};\mathbf{l}|t) = \langle\Uparrow| \left(\prod_{n=1}^{N} \sigma_{j_n}^{+}\right) e^{-t\mathcal{H}} \left(\prod_{k=1}^{N} \sigma_{l_k}^{-}\right) |\Uparrow\rangle \,,$$

which is parametrized by multi-indices $\mathbf{j} \equiv (j_1, j_2, \ldots, j_N)$ and $\mathbf{l} \equiv (l_1, l_2, \ldots, l_N)$. This correlator is the generation function of N random turns vicious walkers. The average $\langle\Uparrow| \left(\prod_{n=1}^{N} \sigma_{j_n}^{+}\right) (-\mathcal{H})^K \left(\prod_{k=1}^{N} \sigma_{l_k}^{-}\right) |\Uparrow\rangle$ is equal to the number of nests of trajectories of N random turns walkers being initially located on the sites $l_1 > l_2 > \cdots > l_N$ and arrived at the positions at $j_1 > j_2 > \cdots > j_N$ after K steps. The condition that vicious walkers do not touch each other up to N steps is guaranteed by the Pauli matrices $(\sigma_k^{\pm})^2 = 0$.

Differentiating by t and using commutators one obtains:

$$\frac{d}{dt}\, G(\mathbf{j};\mathbf{l}|t) = \sum_{k=1}^{N} \big(G(\mathbf{j}; l_1, l_2, \ldots, l_k + 1, \ldots, l_N|t) + G(\mathbf{j}; l_1, l_2, \ldots, l_k - 1, \ldots, l_N|t) \big)$$

($\mathbf{j}$ is fixed). The non-intersection condition means that $G(\mathbf{j};\mathbf{l}|t) = 0$ if $l_k = l_p$ (or $j_k = j_p$) for any $1 \le k, p \le N$. The solution is represented in the determinantal form:

$$G(\mathbf{j};\mathbf{l}|t) \;=\; \det\big(G(j_r, l_s|t)\big)_{1\le r,s\le N}\,,$$

where $G(j, l|t)$ is the one-particle correlator. The "initial condition" is: $G(\mathbf{j};\mathbf{l}|0) = \prod_{m=1}^{N} \delta_{j_m, l_m}$.

Let $|P_K(\mathbf{l} \to \mathbf{j})|$ is the number of K-step trajectories traced by N vicious walkers in the random turns model. A typical configuration of paths for three walkers is shown in figure:

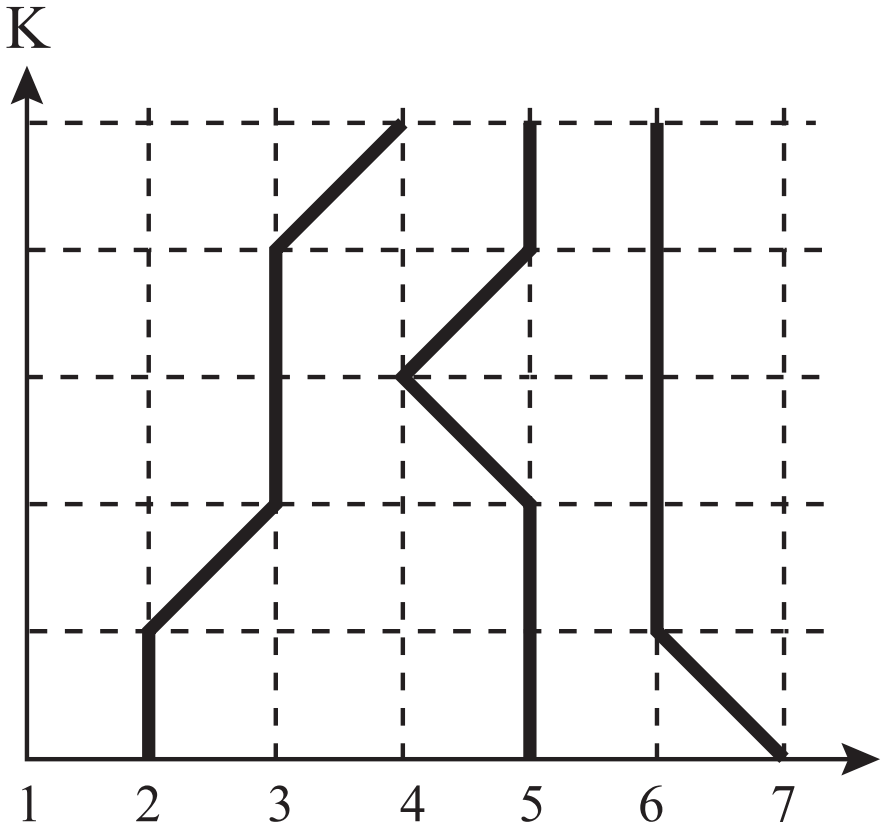


Рис.: A configuration of paths of three vicious walkers ($\mathbf{l} = (2, 5, 7)$, $\mathbf{j} = (4, 5, 6)$) in the random turns model.

The generating function $G(\mathbf{j};\mathbf{l}|t)$ may be expressed in the form:

$$G(\mathbf{j};\mathbf{l}|t) = \frac{1}{(M+1)^N} \sum_{\{\boldsymbol{\phi}_N\}} e^{-\beta E_N(\boldsymbol{\phi}_N)} |\mathcal{V}_N(e^{i\boldsymbol{\phi}_N})|^2 \, S_{\boldsymbol{\lambda}^L}(e^{i\boldsymbol{\phi}_N}) S_{\boldsymbol{\lambda}^R}(e^{-i\boldsymbol{\phi}_N}) \,,$$

where the parametrization $\phi_{k_m} = \frac{2\pi}{M+1}\left(k_m - \frac{M}{2}\right)$, $1 \leq m \leq N$, is used. Summation is over N-tuples $\boldsymbol{\phi}_N \equiv (\phi_{k_1}, \phi_{k_2}, \ldots, \phi_{k_N})$, parametrized by integers k_i, $1 \leq i \leq N$, respecting $M \geq k_1 > k_2 \cdots > k_N \geq 0$. Here $E_N(\boldsymbol{\phi}_N)$ is the energy.

I.3 RANDOM WALKS AS THE CORRELATION FUNCTIONS

The thermal correlation function can be considered which is related to the projection operator $\bar{\Pi}_n$ that forbids spin "down" states on the first n sites of the chain and we shall call it the *persistence of ferromagnetic string*:

$$\mathcal{T}(\boldsymbol{\theta}_N^{\mathrm{g}}, n, t) \equiv \frac{\langle \Psi(\boldsymbol{\theta}_N^{\mathrm{g}}) \mid \bar{\Pi}_n \, e^{-tH_{\mathrm{XX}}} \, \bar{\Pi}_n \mid \Psi(\boldsymbol{\theta}_N^{\mathrm{g}}) \rangle}{\langle \Psi(\boldsymbol{\theta}_N^{\mathrm{g}}) \mid e^{-tH_{\mathrm{XX}}} \mid \Psi(\boldsymbol{\theta}_N^{\mathrm{g}}) \rangle}, \qquad \bar{\Pi}_n \equiv \prod_{j=0}^{n-1} \frac{\mathbb{I} + \sigma_j^z}{2},$$

where $t \in \mathbb{C}$. The correlator is calculated on the ground state solution of the Bethe equation.
We shall consider the nominator separately under arbitrary parametrization:

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\Pi}_n \, e^{-\beta \hat{H}_{XX}} \, \bar{\Pi}_n \mid \Psi_N(\mathbf{u}) \rangle$$

$$= \sum_{\boldsymbol{\lambda}^L, \boldsymbol{\lambda}^R \subseteq \{(M-N-n)^N\}} S_{\boldsymbol{\lambda}^L}(\mathbf{v}^{-2}) S_{\boldsymbol{\lambda}^R}(\mathbf{u}^2) G_{\boldsymbol{\mu}^L; \boldsymbol{\mu}^R}(\beta) \, .$$

To give the combinatorial interpretation, we need a relationship between Schur functions and lattice paths.
A combinatorial description of the Schur functions may be given in terms of *semistandard Young tableaux*. A filling of the cells of the Young diagram of $\boldsymbol{\lambda}$ with positive integers $n \in \mathbb{N}^+$ is called a *semistandard tableau of shape* $\boldsymbol{\lambda}$ provided it is weakly increasing along rows and strictly increasing along columns. A definition of the Schur function is given:

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_m) = \sum_T \mathbf{x}^T \,, \quad \mathbf{x}^T \equiv \prod_{i,j} x_{T_{ij}} \,,$$

where $m \geq N$, the product is over all entries T_{ij} of the tableau T, and the sum is over all tableaux T of shape $\boldsymbol{\lambda}$ with the entries being numbers $\{1, 2, \ldots, m\}$.
Each semistandard tableau of shape $\boldsymbol{\lambda}$ with entries not exceeding N is a nest of self-avoiding lattice paths with fixed start and end points. Let T_{ij} be an entry in i^{th} row and j^{th} column of the semistandard tableau T. The ith lattice path of the nest C (counted from the top of the nest) encodes the ith row of the tableau ($i = 1, \ldots, N$). It goes from $C_i = (N - i + 1, N - i)$ to $(1, \mu_i = \lambda_i + N - i)$. It makes λ_i steps to the north so that

the step along the line x_j corresponds to the occurrences of the letter $N-j+1$ in the ith row of T. The power l_j of x_j in the weight of any particular nest of paths is the number of steps to north taken along the vertical line x_j. Thus, the Schur function takes the form:

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_N) = \sum_{C} \prod_{j=1}^{N} x_j^{l_j},$$

where summation is over all admissible nests C.

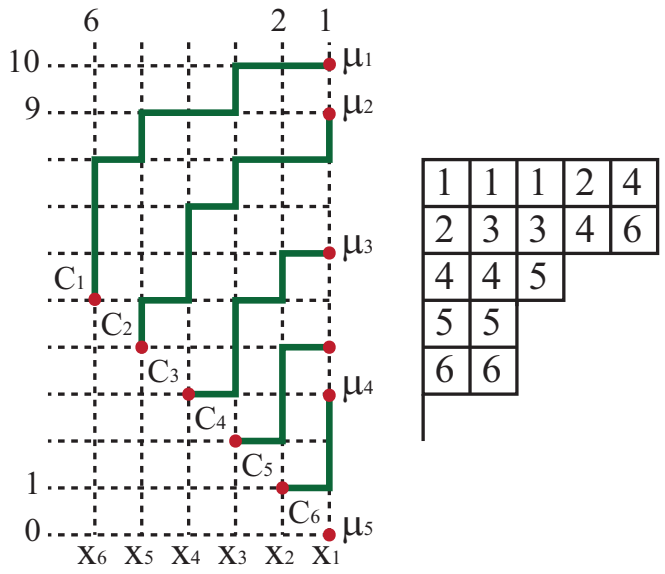
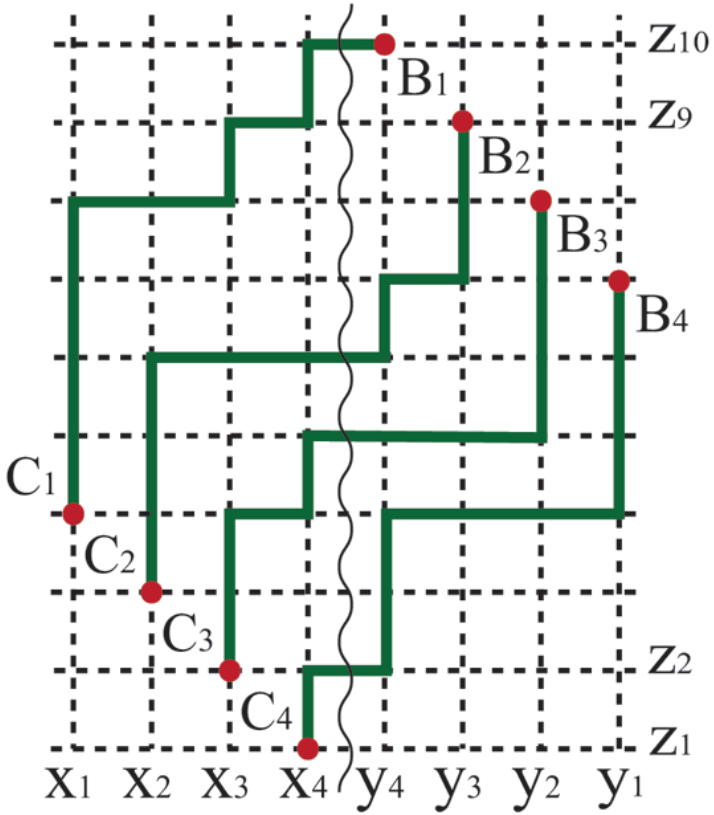
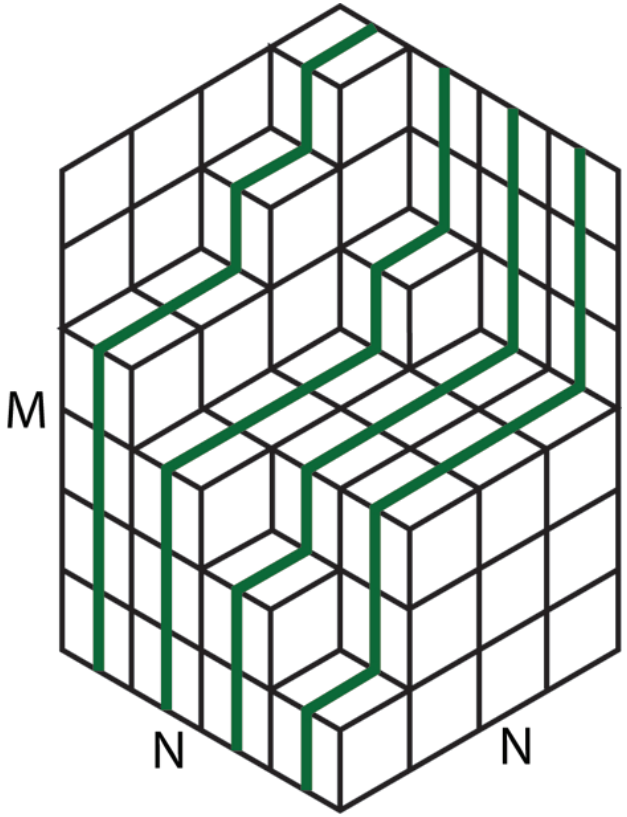


Рис.: A nest of lattice paths C, corresponding to semistandard tableau with weight $x_6^3 x_5^2 x_4^2 x_3^4 x_2^3 x_1^3$ on the diagram $\boldsymbol{\lambda} = (5,5,3,2,2,0)$ for $N=6$.

The scalar product, being the product of two Schur functions, may be graphically expressed as a nest of N self-avoiding lattice paths starting at the equidistant points C_i and terminating at the equidistant points B_i ($1 \leq i \leq N$). This configuration, known as *watermelon*. The scalar product is given by the sum of all such watermelons. Rotating Fig. by $\frac{\pi}{4}$ counter-clockwise we see that the watermelon is a particular case of configuration for lock step random walkers.





The partition function of watermelons is equal to the q-parameterized (In the q-parametrization $\mathbf{v}^{-2} = \mathbf{q} \equiv (q, q^2, \ldots, q^N)$, $\mathbf{u}^2 = \mathbf{q}/q = (1, q, \ldots, q^{N-1})$) scalar product:

$$\langle \Psi_N(\mathbf{q}^{-\frac{1}{2}}) | \Psi_N((\mathbf{q}/q)^{\frac{1}{2}}) \rangle = \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{q}) S_{\boldsymbol{\lambda}}(\mathbf{q}/q)$$

$$= \sum_{\{\mathcal{W}\}} q^{|\xi|_{\mathcal{C}} + |\zeta|_{\mathcal{B}}} = Z(N, N, \mathcal{M}) \,,$$

where the sum is taken over all watermelons with the fixed endpoints C_i, B_i, $1 \leq i \leq N$.

Further, we obtain the following transition amplitude:

$$\begin{aligned}
&\langle \Psi_N(\mathbf{v}_N) \mid \bar{\Pi}_n \, e^{-t\mathcal{H}} \, \bar{\Pi}_n \mid \Psi_N(\mathbf{u}) \rangle \\
&= \sum_{\boldsymbol{\lambda}^L, \boldsymbol{\lambda}^R \subseteq \{(M-N-n)^N\}} G(\boldsymbol{\lambda}^L + \boldsymbol{\delta}_N; \boldsymbol{\lambda}^R + \boldsymbol{\delta}_N \mid t) S_{\boldsymbol{\lambda}^L}(\mathbf{v}^{-2}) \, S_{\boldsymbol{\lambda}^R}(\mathbf{u}^2) \\
&\qquad = \sum_{K=0}^{\infty} \frac{t^K}{K!} \sum_{\boldsymbol{\lambda}^{L,R} \setminus \mathbf{n} \subseteq \{(\mathcal{M}-n)^N\}} S_{\boldsymbol{\lambda}^L}(\mathbf{v}^{-2}) \, S_{\boldsymbol{\lambda}^R}(\mathbf{u}^2) \\
&\qquad\qquad \times \langle \Uparrow | \left(\prod_{l=1}^{N} \sigma^+_{\mu^L_l} \right) (-\mathcal{H})^K \left(\prod_{p=1}^{N} \sigma^-_{\mu^R_p} \right) | \Uparrow \rangle \, .
\end{aligned}$$

Here $\mathbf{u}$ and $\mathbf{v}$ stand for an arbitrary parametrization.
The elements of the expansion in right-hand side may be interpreted in terms of nests of self-avoiding lattice paths as follows. For the first N steps the particles are moving to the west (from the right to the left) according the lock step rules. They are doing next K steps according to the random turns rules. The final N steps to the west are again according the lock step rules.

The example of the described nest of lattice paths is schematically depicted in Figure.

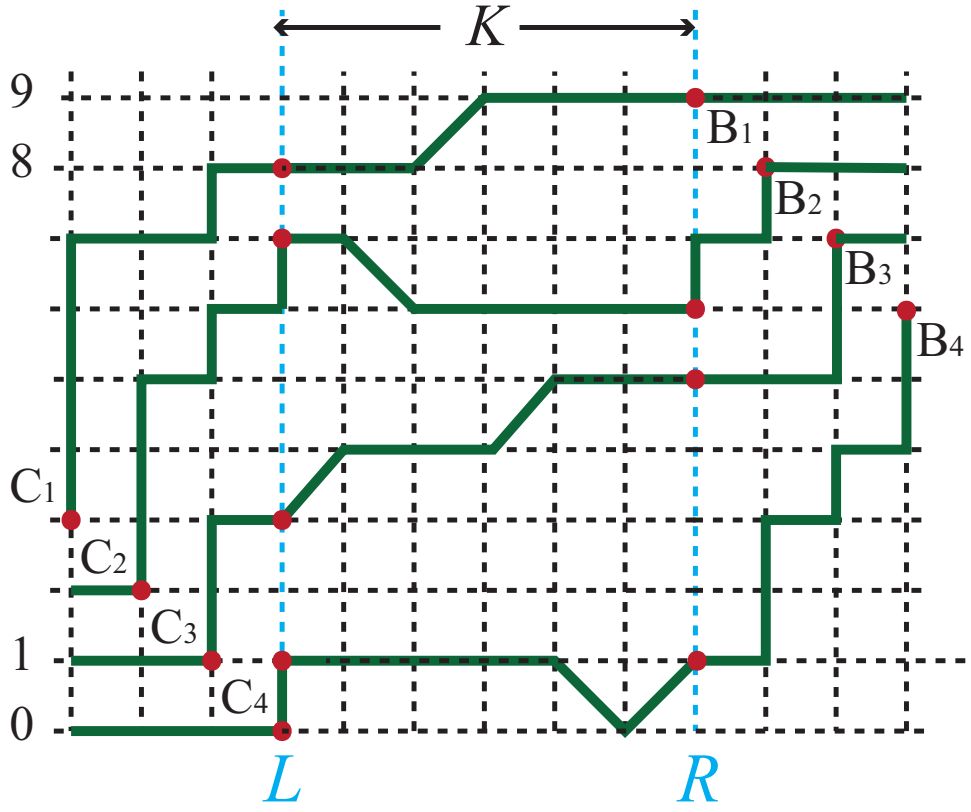


Рис.: One of the nest of lattice paths that corresponds to the amplitude $\langle \Psi(\mathbf{v}_N) \mid \bar{\Pi}_0 \, (-\mathcal{H})^K \, \bar{\Pi}_0 \mid \Psi(\mathbf{u}_N) \rangle$

II.1 WALKS ON SIMPLICIAL LATTICES

Starting from $(M+1)$-dimensional hypercubical lattice with unit spacing $\mathbb{Z}^{M+1} \ni \mathbf{m} \equiv (m_0, m_1, \ldots, m_M)$, let us consider a set of points with coordinates constrained by the requirement $m_0 + m_1 + \ldots + m_M = N$:

$$\mathrm{Symp}_{(N)}(\mathbb{Z}^{M+1}) \equiv \left\{ \mathbf{m} \in \mathbb{Z}^{M+1} \middle| 0 \leq m_i, i \in \mathcal{M}, \sum_{i \in \mathcal{M}} m_i = N \right\}$$

(hereafter $\mathcal{M} \equiv \{0, 1, \ldots M\}$). We call $\mathrm{Symp}_{(N)}(\mathbb{Z}^{M+1})$ *simplicial lattice*.

Random walks of a walker over sites of $\mathrm{Symp}_{(N)}(\mathbb{Z}^{M+1})$ are defined by a set of admissible steps Ω_M such that at each step an ith coordinate m_i increases by unity while a nearest neighboring coordinate decreases by unity. Namely, Ω_M is the set of steps with coordinates $(e_0, e_2, \ldots, e_M)$ such that for all pairs $(i, i+1)$ with $0 \leq i \leq M$ and $M + 1 = 0(\mathrm{mod}\ 2)$, $e_i = \pm 1$, $e_{i+1} = \mp 1$ and $e_j = 0$ for all $j \in \mathcal{M}$ and $j \neq i, i+1$. The step-set $\Omega_M \equiv \Omega_M(\mathbf{m}_0)$ ensures that trajectory of a random walk determined by the starting point $\mathbf{m}_0$ lies in M-dimensional set $\mathrm{Symp}_{(N)}(\mathbb{Z}^{M+1})$.

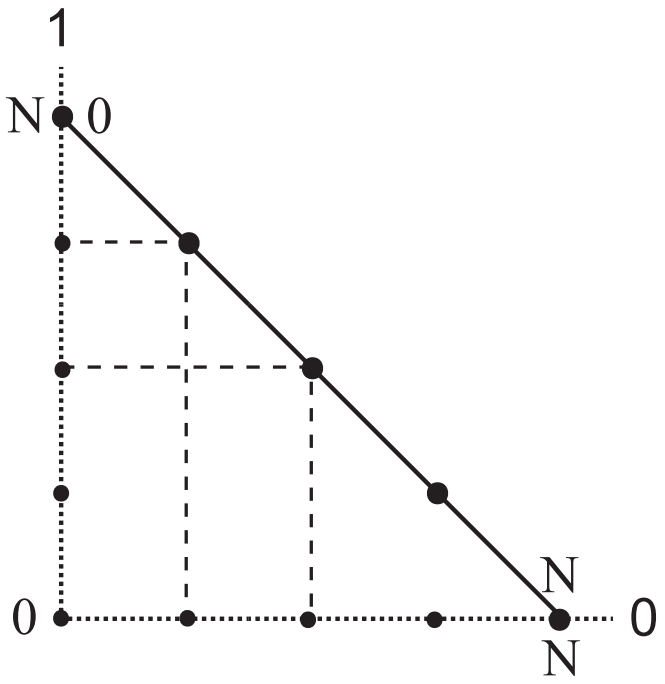


Рис.: The hopping processes for the one-dimensional nearest-neighbor random walk on a segment [0,N].

As an example, the walks on $\mathbb{Z}^2$ are defined by a step-set $\Omega_1 = \{(1,-1),(-1,1)\}$ that ensures that the walks lie on lines $\{(n_0,n_1) \in \mathbb{Z}^2 \mid n_0 + n_1 = N\}$.

The step-set $\Omega_2 = \{(-1,1,0),(1,-1,0),(0,-1,1),(0,1,-1),(1,0,-1),(-1,0,1)\}$ of the random walks, or $\Omega_2 = \{(-1,1,0),(0,-1,1),(1,0,-1)\}$ of the directed walks, ensures that trajectories of random walks belong to $\mathrm{Symp}_{(N)}(\mathbb{Z}^3)$.

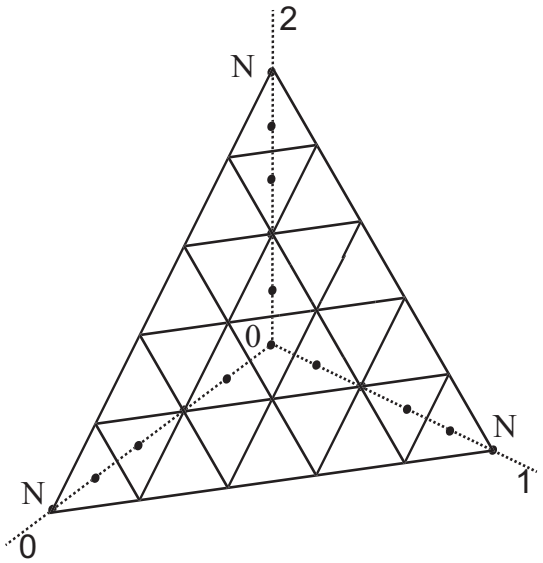


Рис.: A two-dimensional triangular simplicial lattice.

Directed random walks on M-dimensional *orientated simplicial lattice* are defined by a step-set $\Gamma_M = (k_0, k_1, \ldots, k_M)$ such that for all pairs $(i, i+1)$ with $i \in \mathcal{M}$ and $M + 1 = 0(\mod 2)$, $k_i = -1$, $k_{i+1} = 1$, and $k_j = 0$ for all $j \in \mathcal{M} \backslash \{i, i+1\}$.

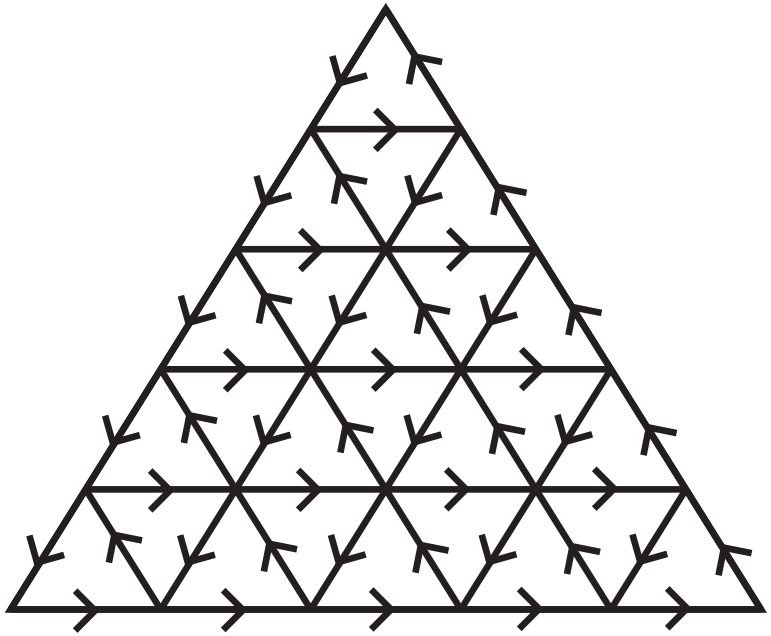


Рис.: The orientated two-dimensional bounded simplicial lattice defined by the step-set Γ_2. The walks are allowed only in the direction of arrows.

The walker's movements should be supplied with appropriate boundary conditions. The *reflecting boundary conditions* imply that when trajectory and boundary are intersecting, the corresponding admissible steps are still taken from Ω_M or Γ_M. The *retaining boundary conditions* imply that the walker on the boundary continues to move in accordance with the elements of Ω_M or Γ_M either stays on the boundary. The boundary of the simplicial lattice consists of $M+1$ faces of highest dimensionality $M-1$. To each component of the boundary a weight g_s, $s = 0, 1, \ldots, M$, is assigned.

Consider the retaining boundary conditions. The exponential generating function of lattice walks is defined as a series

$$F^{(N)}(\mathbf{l},\mathbf{j}|t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} G_k^{(N)}(\mathbf{l},\mathbf{j}) \,,$$

where $G_k^{(N)}(\mathbf{l},\mathbf{j})$ characterize the k-step walks at a node $\mathbf{l} = (l_0, l_1, \ldots, l_M) \in \mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ when starting at $\mathbf{j} = (j_0, j_1, \ldots, j_M) \in \mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$. The master equation is:

$$\begin{aligned}
\partial_t F^{(N)}(\mathbf{l},\mathbf{j}|t) = &\sum_{s=0}^{M} F^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_s - 1, j_{s+1} + 1, \ldots j_M) \\
&+ \sum_{s=0}^{M} F^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_s + 1, j_{s+1} - 1, \ldots j_M) \\
+ \sum_{s=0}^{M} g_s F^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_{s-1}, 0, j_{s+1}, \ldots j_M)\, &\delta(N, \sum_{0 \le k \le M}{}' j_k) \,,
\end{aligned}$$

where $\delta(n,m)$ is the Kronecker symbol, and $\sum'$ implies that $k = s$ is omitted. The equation for the directed walks is obtained by removing the second sum in right-hand side.

The system of equations for $G_k^{(N)}(\mathbf{l},\mathbf{j})$:

$$
\begin{aligned}
G_k^{(N)}(\mathbf{l},\mathbf{j}) \;=\; & \sum_{s=0}^{M} G_{k-1}^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_s - 1, j_{s+1} + 1, \ldots j_M) \\
& + \sum_{s=0}^{M} G_{k-1}^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_s + 1, j_{s+1} - 1, \ldots j_M) \\
+ \sum_{s=0}^{M} g_s G_{k-1}^{(N)}(\mathbf{l}, j_0, j_1, \ldots, j_{s-1}, 0, j_{s+1}, \ldots j_M)\, & \delta(N, \sum_{0 \le k \le M}{}' j_k)\,,
\end{aligned}
$$

where $k \geq 1$, while it is natural to impose the condition $G_0^{(N)}(\mathbf{l},\mathbf{j}) = \delta_{l_0 j_0}\delta_{l_1 j_1} \cdots \delta_{l_M j_M}$.

II.2 GENERALIZED PHASE MODEL AND ITS SOLUTION

• • • Generalized Phase Model describes N bosonic particles on a cyclic chain of $M+1$ nodes. Each configuration of the particles is characterized by a collection of the occupation numbers $(n_M, \ldots, n_1, n_0)$, $\sum_{l\in\mathcal{M}} n_l = N$. The phase operators $\phi_n, \phi_n^\dagger$ respect the algebra

$$[\hat{N}_i, \phi_j] = -\phi_i \delta_{ij}\,, \quad [\hat{N}_i, \phi_j^\dagger] = \phi_i^\dagger \delta_{ij}\,, \quad [\phi_i, \phi_j^\dagger] = \pi_i \delta_{ij}\,,$$

where $\hat{N}_j$ is the number operator, and $\pi_i = 1 - \phi_i^\dagger \phi_i$ is the vacuum projector: $\phi_j \pi_j = \pi_j \phi_j^\dagger = 0$. The Fock states $|n_l\rangle_l = (\phi_l^\dagger)^{n_l}|0\rangle_l$, where $|0\rangle_l$ is the vacuum state $|0\rangle$ at l^{th} site defined by $\phi_l|0\rangle = 0$:

$$\phi_l^\dagger\,|n_l\rangle_l = |n_l+1\rangle_l\,, \qquad \phi_l\,|n_l\rangle_l = |n_l-1\rangle_l\,, \qquad \hat{N}_l\,|n_l\rangle_l = n_l\,|n_l\rangle_l\,.$$

The Fock states $|\mathbf{n}\rangle$ are generated from $|\mathbf{0}\rangle$ by the rising operators $\phi_j^\dagger$:

$$|\mathbf{n}\rangle \equiv \bigotimes_{l=0}^{M} |n_l\rangle_l = \Big(\prod_{j=0}^{M} (\phi_j^\dagger)^{n_j}\Big)|\mathbf{0}\rangle\,, \quad |\mathbf{0}\rangle \equiv \bigotimes_{l=0}^{M} |0\rangle_l\,,$$

where $\mathbf{n} \in \mathrm{Symp}_{(N)}(\mathbb{Z}^{M+1})$.

The operators act on the state-vectors:

$$
\begin{aligned}
\phi_j \mid n_0, \ldots, 0_j, \ldots, n_M\rangle &= 0, \\
\phi_j^\dagger \mid n_0, \ldots, n_j, \ldots, n_M\rangle &= \mid n_0, \ldots, n_j + 1, \ldots, n_M\rangle, \\
\phi_j| \mid n_0, \ldots, n_j, \ldots, n_M\rangle &= |n_0, \ldots, n_j - 1, \ldots, n_M\rangle, \\
\hat{N}_j \mid n_0, \ldots, n_j, \ldots, n_M\rangle &= n_j|n_0, \ldots, n_j, \ldots, n_M\rangle \\
\pi_j \mid n_0, \ldots, 0_j, \ldots, n_M\rangle &= \mid n_0, \ldots, 0_j, \ldots, n_M\rangle.
\end{aligned}
$$

The states $|n_0, \ldots, n_M\rangle$ are orthogonal.

As generator of directed walks with the retaining boundary conditions we consider the non-Hermitian Hamiltonian [N. M. Bogoliubov, T. Nassar, *On the spectrum of the non-Hermitian phase-difference model*, Phys. Lett. A **234**, 345 (1997)]:

$$H = \sum_{m=0}^{M} \left(\phi_m \phi_{m+1}^{\dagger} + g_m \pi_m \right) ,$$

where $M+1$ is even and periodicity is imposed. The Hamiltonian commutes with the number operator $\hat{N}$.
The exponential generating function of the directed walks is expressed as the correlation function:

$$F^{(N)}(\mathbf{l}, \mathbf{j}|t) = \langle \mathbf{l} \mid e^{tH} \mid \mathbf{j} \rangle ,$$

where H is the Hamiltonian. Expanding the correlator we obtain the coefficients $G_k^{(N)}(\mathbf{l}, \mathbf{j})$ characterizing the lattice walks from the node $\mathbf{j}$ to the node $\mathbf{l}$ in k steps:

$$G_k^{(N)}(\mathbf{l}, \mathbf{j}) = \langle \mathbf{l} \mid H^k \mid \mathbf{j} \rangle .$$

• • • Define L-operator which is 2×2 matrix with operator entries:

$$L(n|u) \equiv \begin{pmatrix} u^{-1} + u g_n \pi_n & \phi_n^\dagger \\ \phi_n & u \end{pmatrix} ,$$

where $u \in \mathbb{C}$ is a parameter and $g_n \in \mathbb{R}$. The intertwining relation

$$R(u,v)\,(L(n|u) \otimes L(n|v)) = (L(n|v) \otimes L(n|u))\,R(u,v)\,,$$

in which $R(u,v)$ is the R-matrix

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix} ,$$

where

$$f(v,u) = \frac{u^2}{u^2 - v^2}, \qquad g(v,u) = \frac{uv}{u^2 - v^2}\,, \qquad u, v \in \mathbb{C}\,.$$

The monodromy matrix:

$$T(u) = L(M|u)L(M-1|u)\cdots L(0|u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} .$$

The commutation relations of the matrix elements are given:

$$R(u,v)\,(T(u)\otimes T(v)) = (T(v)\otimes T(u))\,R(u,v)\,.$$

The transfer matrix $\tau(u)$ is: $\tau(u) = \operatorname{tr} T(u) = A(u) + D(u)$. The relation means $[\tau(u), \tau(v)] = 0$, $u, v \in \mathbb{C}$. The entries of $T(u)$:

$$\begin{aligned} u^{M+1}A(u) &= 1 + u^2\Big(\sum_{m=0}^{M-1}\phi_m\phi_{m+1}^\dagger + \sum_{m=0}^{M} g_m\pi_m\Big) + u^{2(M+1)}\prod_{m=0}^{M} g_m\pi_m\,, \\ u^{M+1}D(u) &= u^2\phi_0^\dagger\phi_M + \ldots + u^{2(M+1)}\,, \end{aligned}$$

$$\begin{aligned} u^M B(u) &= \phi_0^\dagger + \ldots + u^{2M}\mathcal{P}_R \equiv \widetilde{B}(u)\,, \\ u^M C(u) &= \phi_M + \ldots + u^{2M}\mathcal{P}_L \equiv \widetilde{C}(u)\,, \end{aligned}$$

where $\mathcal{P}_R = \sum_{k=0}^{M}\phi_k^\dagger g_{k+1}\pi_{k+1}\ldots g_M\pi_M$, $\mathcal{P}_L = \sum_{k=0}^{M} g_0\pi_0\ldots g_{k-1}\pi_{k-1}\phi_k$.

The Hamiltonian is expressed:

$$H = \frac{\partial}{\partial u^2} u^{M+1}\tau(u)\Big|_{u=0} = \frac{\partial}{\partial u^2} u^{M+1}(A(u) + D(u))\Big|_{u=0}.$$

The N-particle state-vectors are taken in the form

$$|\Psi_N(\mathbf{u})\rangle = \Big(\prod_{j=1}^{N} \widetilde{B}(u_j)\Big)|\mathbf{0}\rangle,$$

where $\mathbf{u} = (u_0, u_1, \ldots, u_N)$, and $\widetilde{B}(u)$ is defined above. The vacuum $|\,\mathbf{0}\rangle$ is an eigen-vector of $A(u)$ and $D(u)$,

$$A(u)|\mathbf{0}\rangle = \alpha(u)|\mathbf{0}\rangle, \qquad D(u)|\mathbf{0}\rangle = \delta(u)|\mathbf{0}\rangle$$

with the eigen-values $\alpha(u) = \prod_{j=0}^{M}(u^{-1} + g_j u)$, $\delta(u) = u^{M+1}$.
The state-vector is the eigen-vectors both of H and $\tau(u)$, if and only if:

$$u_n^{-2N} \prod_{j=0}^{M} \left(g_j + u_n^{-2}\right) = (-1)^{N-1} \prod_{j=1}^{N} u_j^{-2}.$$

From the eigen-value $\Theta_N(v)$ of $\tau(v)$ obtain the spectrum:

$$E_N(\mathbf{u}) = \frac{\partial}{\partial v^2} v^M \Theta_N(v)\big|_{v=0} = \sum_{m=0}^{M} g_m + \sum_{m=1}^{N} u_m^{-2}.$$

For the model associated with the R-matrix the scalar product of the state-vectors is [N. Bogoliubov, *Boxed plane partitions as an exactly solvable boson model*, J. Phys. A: Math. Gen. 38, 9415 (2005)]:

$$\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle = \mathcal{V}_N^{-1}(\mathbf{v}^2) \mathcal{V}_N^{-1}(\mathbf{u}^{-2}) \prod_{j=1}^{N} \left(\frac{v_j}{u_j} \right)^{M+N-1} \det Q \,,$$

where $\mathcal{V}_N(\mathbf{x}) \equiv \prod_{1 \le i < k \le N} (x_k - x_i)$, and Q_{jk}, $1 \le j, k \le N$ are:

$$Q_{jk} = \frac{\alpha(v_j)\delta(u_k)(u_k/v_j)^{N-1} - \alpha(u_k)\delta(v_j)(u_k/v_j)^{-N+1}}{u_k/v_j - v_j/u_k} \,,$$

with $\alpha(u)$ and $\delta(u)$ as above. The resolution of the identity operator

$$I = \sum_{\{\mathbf{u}\}} \frac{|\Psi_N(\mathbf{u})\rangle\langle\Psi_N(\mathbf{u})|}{\mathcal{N}^2(\mathbf{u})},$$

where summation $\sum_{\{\mathbf{u}\}}$ is over all independent solutions of the Bethe equations

Let us consider the walker on $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$. His starting point at the node $(N, 0, \ldots, 0)$, and the walk terminates at $(0, 0, \ldots, N)$. The generating function of the walks is:

$$\mathcal{F}^{(N)}(t) \equiv \langle 0, 0, \ldots, N \mid e^{tH} \mid N, 0, \ldots, 0 \rangle = \langle \mathbf{0} \mid (\phi_M)^N e^{tH} (\phi_M^\dagger)^N \mid \mathbf{0} \rangle \,.$$

Inserting the identity operator obtain:

$$\mathcal{F}^{(N)}(t) = \sum_{\{\mathbf{u}\}} \frac{e^{tE_N(\mathbf{u})}}{\mathcal{N}^2(\mathbf{u})} \langle \mathbf{0} \mid (\phi_M)^N \mid \Psi_N(\mathbf{u}) \rangle \langle \Psi_N(\mathbf{u}) \mid (\phi_M^\dagger)^N \mid \mathbf{0} \rangle \,,$$

where the summation is over all independent solutions of Bethe equations. From $B(u)$ and $C(u)$ obtain:

$$\langle \mathbf{0} \mid (\phi_M)^N \mid \Psi_N(\mathbf{u}) \rangle = \lim_{\mathbf{v} \to 0} \langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle = 1 \,,$$

$$\langle \Psi_N(\mathbf{u}) \mid (\phi_M^\dagger)^N \mid \mathbf{0} \rangle = \lim_{\mathbf{v} \to 0} \langle \Psi_N(\mathbf{u}) | \Psi_N(\mathbf{v}) \rangle = 1 \,,$$

[N. Bogoliubov, *Calculation of correlation functions in totally asymmetric exactly solvable models on a ring*, Theoretical and Mathematical Physics **175**, 755 (2013)].

Eventually we obtain:

$$\mathcal{F}^{(N)}(t) = \sum_{\{\mathbf{u}\}} \frac{e^{2t\sum_{k=1}^{N}(g_k + u_k^{-2})}}{\mathcal{N}^2(\mathbf{u})} \,.$$

For instance, at $N = 2$ and $M = 1$ we obtain for $g_k = \gamma = 1 \ \forall k$:

$$\mathcal{F}^{(2)}(t) = e^{4t\gamma} \sum_{\{u_1^2, u_2^2\}} \frac{e^{2t(u_1^{-2} + u_2^{-2})}}{\mathcal{N}^2(u_1^2, u_2^2)} \,,$$

$$\mathcal{N}^2(u_1^2, u_2^2) = 4 + \frac{u_1^2}{u_2^2} + \frac{u_2^2}{u_1^2} + 4\gamma(u_2^2 + u_1^2) + 3\gamma^2 u_2^2 u_1^2 \,.$$

II.3 TOTALLY ASYMMETRIC ZERO RANGE MODEL

When all g_i are equal to $\gamma = 1$, we obtain the Hamiltonian of Totally Asymmetric Zero Range Model:

$$H_{\rm zr} = \sum_{m=0}^{M} \left(\phi_m \phi_{m+1}^\dagger + \pi_m \right) .$$

The state-vector enables the representation in the form

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\lambda} \subseteq \{M^N\}} \chi_{\boldsymbol{\lambda}}^R(\mathbf{u}) \Big(\prod_{j=0}^{M} (\phi_j^\dagger)^{n_j} \Big) |\mathbf{0}\rangle \, ,$$

where the function $\chi_{\boldsymbol{\lambda}}^R$ is equal, up to a multiplicative pre-factor, to

$$\chi_{\boldsymbol{\lambda}}^R(\mathbf{x}) = \chi_{\boldsymbol{\lambda}}^R(x_1, x_2, \ldots, x_N) = \mathcal{V}_N^{-1}(\mathbf{x}) \det \left\{ \left(1 + x_i^{-2}\right)^{\lambda_j} x_i^{2(N-j)} \right\} .$$

Here $\boldsymbol{\lambda}$ denotes the partition $(\lambda_1, \ldots, \lambda_N)$ of non-increasing non-negative integers,

$$M \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0,$$

and $\mathcal{V}_N(\mathbf{x})$ is the Vandermonde determinant.

There is a one-to-one correspondence between a sequence of the occupation numbers $(n_M, \ldots, n_1, n_0)$, $\sum_{j\in\mathcal{M}} n_j = N$, and the partition

$$\boldsymbol{\lambda} = (M^{n_M}, (M-1)^{n_{M-1}}, \ldots, 1^{n_1}, 0^{n_0}),$$

where each number S appears n_S times (see Fig. 4). The summation goes over all partitions $\boldsymbol{\lambda}$ into at most N parts with $N \leq M$.

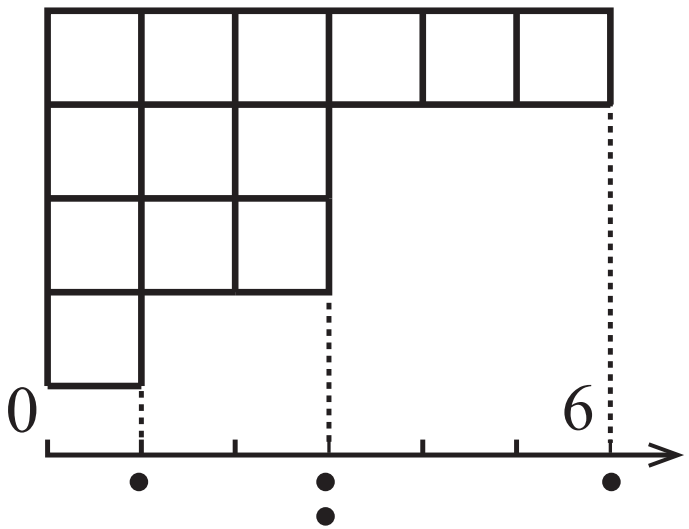


Рис.: A configuration of particles ($N = 4$) on a lattice ($M = 6$), the corresponding partition $\boldsymbol{\lambda} = (6^1, 5^0, 4^0, 3^2, 2^0, 1^1, 0^0) \equiv (6, 3, 3, 1)$ and its Young diagram.

Acting by the Hamiltonian on the state-vector, we find:

$$\sum_{k=1}^{N} \chi^{R}_{\lambda_1,\ldots,\lambda_k+1,\ldots,\lambda_N}(\mathbf{u}) = E_N^{\mathrm{zr}}(\mathbf{u})\chi^{R}_{\lambda_1,\ldots,\lambda_N}(\mathbf{u})\,,$$

together with the exclusion condition

$$\chi^{R}_{\lambda_1,\ldots,\lambda_{l-1}=\lambda_l+1,\lambda_l,\ldots,\lambda_N}(\mathbf{u}) = \chi^{R}_{\lambda_1,\ldots,\lambda_{l-1}=\lambda_l,\lambda_l,\ldots,\lambda_N}(\mathbf{u})\,, \qquad 1 \leq l \leq N.$$

The energy E_N^{zr} is given with all $g_i = 1$. The state-vector is the eigen-vector of the Hamiltonian with the periodic boundary conditions if the parameters u_j satisfy the appropriate Bethe equations.

Expanding the left state-vector, we obtain:

$$\langle \Psi_N(\mathbf{u}) \mid = \mathcal{V}_N^{-1}(\mathbf{x}) \sum_{\boldsymbol{\lambda} \subseteq \{M^N\}} \det \left\{ \left(\frac{x_i}{x_i + x_i^{-1}} \right)^{\lambda_j} x_i^{2(N-j)} \right\} \langle \mathbf{0} \mid \left(\prod_{i=0}^{M} \phi_i^{n_i} \right).$$

Equations take place:

$$\sum_{k=1}^{N} \chi^L_{\lambda_1,\ldots,\lambda_k-1,\ldots,\lambda_N}(\mathbf{u}) = E_N^{\mathrm{zr}}(\mathbf{u}) \chi^L_{\lambda_1,\ldots,\lambda_N}(\mathbf{u}),$$

$$\chi^L_{\lambda_1,\ldots,\lambda_{l-1}=\lambda_l-1,\lambda_l,\ldots,\lambda_N}(\mathbf{u}) = \chi^L_{\lambda_1,\ldots,\lambda_{l-1}=\lambda_l,\lambda_l,\ldots,\lambda_N}(\mathbf{u}), \qquad 1 \le l \le N.$$

Further one obtains:

$$\langle l_0, l_1, \ldots, l_M \mid \Psi_N(\mathbf{u}) \rangle = \chi^R_{\boldsymbol{\lambda}_R}(\mathbf{u})$$

$$\langle \Psi_N(\mathbf{u}) \mid j_0, j_1, \ldots, j_M \rangle = \chi^L_{\boldsymbol{\lambda}_L}(\mathbf{u}),$$

where $\boldsymbol{\lambda}_R = \left(M^{l_M}, (M-1)^{l_{M-1}}, \ldots, 1^{l_1}, 0^{l_0}\right)$, and $\boldsymbol{\lambda}_L = \left(M^{j_M}, (M-1)^{j_{M-1}}, \ldots, 1^{j_1}, 0^{j_0}\right)$.

The exponential generating function in the considered special case is:

$$F_{\rm zr}^{(N)}(\mathbf{l},\mathbf{j}|t) = \sum_{\{\mathbf{u}\}} \frac{e^{tE_N^{\rm zr}(\mathbf{u})}}{\mathcal{N}_{\rm zr}^2(\mathbf{u})} \chi_{\boldsymbol{\lambda}_R}^R(\mathbf{u}) \chi_{\boldsymbol{\lambda}_L}^L(\mathbf{u})\,.$$

Here parameters u_j satisfy Bethe equations with $g_i = 1$, and $\mathcal{N}_{\rm zr}^2(\mathbf{u})$ is the squared norm in the same limit:

$$\mathcal{N}^2(\mathbf{u}) = \langle \Psi_N(\mathbf{u}) | \Psi_N(\mathbf{u}) \rangle = \frac{\det \widetilde{Q}}{\mathcal{V}_N(\mathbf{u}^2)\mathcal{V}_N(\mathbf{u}^{-2})}\,.$$

II.4 THE PHASE MODEL

We shall consider the random walks on $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ which are defined by a step-set Ω_M. The Hermitian Hamiltonian H_{ph} of the phase model is the generator of the walks :

$$H_{\mathrm{ph}} = \sum_{m=0}^{M} \left(\phi_m \phi_{m+1}^{\dagger} + \phi_m^{\dagger} \phi_{m+1} \right) .$$

The L-operator is

$$L(n|u) \equiv \begin{pmatrix} u^{-1} & \phi_n^{\dagger} \\ \phi_n & u \end{pmatrix} ,$$

and the Bethe equations are:

$$u_n^{-2(N+M+1)} = (-1)^{N-1} \prod_{j=1}^{N} u_j^{-2} .$$

The state-vectors are:

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\lambda} \subseteq \{M^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \left(\prod_{l=0}^{M} (\phi_l^{\dagger})^{n_l} \right) |\mathbf{0}\rangle ,$$

where $S_{\boldsymbol{\lambda}}(\mathbf{u}^2)$ is the Schur function. The non-strict partition $\boldsymbol{\lambda}$ is connected with the coordinates of the Bose particles, as it was explained above. On the solutions of the Bethe equations the state-vectors are the eigen-vectors of the Hamiltonian with the eigen-energies

$$E_N^{\mathrm{ph}}(\mathbf{u}) = \sum_{k=1}^{N} \left(u_k^2 + u_k^{-2}\right) .$$

The conjugate state vector is of the form:

$$\langle \Psi_N(\mathbf{v})| = \sum_{\boldsymbol{\lambda} \subseteq \{M^N\}} S_{\boldsymbol{\lambda}}(\mathbf{v}^{-2}) \, \langle 0|(\phi_M)^{n_M} \ldots (\phi_1)^{n_1} (\phi_0)^{n_0} .$$

For arbitrary $\mathbf{u}$ and $\mathbf{v}$ the scalar product of N-particle state vectors is given by the Cauchy-Binet formula:

$$\langle \Psi_N(\mathbf{v}) | \Psi_N(\mathbf{u}) \rangle = \sum_{\boldsymbol{\lambda} \subseteq \{M^N\}} S_{\boldsymbol{\lambda}}(\mathbf{v}^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}^2) = \frac{\det(T_{kj})_{1 \le k,j \le N}}{\mathcal{V}_N(\mathbf{u}^2)\mathcal{V}_N(\mathbf{v}^{-2})} ,$$

$$T_{kj} = \frac{1 - (u_k^2/v_j^2)^{M+N}}{1 - (u_k^2/v_j^2)} .$$

In the q-parametrization $\mathbf{v}^{-2} = \mathbf{q} \equiv (q, q^2, \ldots, q^N)$, $\mathbf{u}^2 = \mathbf{q}/q = (1, q, \ldots, q^{N-1})$ the scalar product takes the form:

$$\langle \Psi_N(\mathbf{q}^{-\frac{1}{2}}) | \Psi_N((\mathbf{q}/q)^{\frac{1}{2}}) \rangle = \frac{\det\left(\frac{1-q^{(M+N)(j+k-1)}}{1-q^{(j+k-1)}}\right)_{1\le j,k\le N}}{\mathcal{V}_N(\mathbf{q})\mathcal{V}_N(\mathbf{q}/q)}$$

$$= \prod_{k=1}^{N}\prod_{j=1}^{N} \frac{1-q^{M-1+j+k}}{1-q^{j+k-1}} = Z_q(N,N,M)\,.$$

As it follows from PROPOSITION, the sum of the Schur functions in q-parametrization may be expressed through the q-*binomial determinant*:

$$\sum_{\boldsymbol{\lambda}\subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{q}) S_{\boldsymbol{\lambda}}(\mathbf{q}/q) = q^{\frac{NM}{2}(1-M)} \det\left(\begin{bmatrix} 2N+i-1 \\ N+j-1 \end{bmatrix}\right)_{1\le i,j\le M}.$$

The entries are the q-binomial coefficients defined as

$$\begin{bmatrix} R \\ r \end{bmatrix} \equiv \frac{[R]!}{[r]!\,[R-r]!}\,, \quad [n]! \equiv [1]\,[2]\,\ldots\,[n], \quad [0]! \equiv 1$$

where $[n]$ is the q-number being q-analogue of a positive integer $n \in \mathbb{Z}^+$,

$$[n] \equiv (1-q^n)/(1-q)\,.$$

The generating function $Z_q(L,N,P)$ takes the form:

$$Z_q(L,N,P) = \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \prod_{j=1}^{L}\prod_{k=1}^{N}\frac{1-q^{P+j+k-1}}{1-q^{j+k-1}}\,.$$

R.H.S. tends to $A(L,N,P)$ at $q \to 1$:

$$A(L,N,P) = \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P}\frac{i+j+k-1}{i+j+k-2} = \prod_{j=1}^{L}\prod_{k=1}^{N}\frac{P+j+k-1}{j+k-1}\,,$$

i.e. there are $A(L,N,P)$ plane partitions in box $\mathcal{B}(L,N,P)$ (the McMahon's formula).

In other words we obtain that the scalar product $\langle\Psi_N(\mathbf{q}^{-\frac{1}{2}})|\Psi_N((\mathbf{q}/q)^{\frac{1}{2}})\rangle$ for the phase model in question is equal to the generating function $Z_q(N,N,M)$ of plane partitions in the box $\mathcal{B}(N,N,M)$.

The generating function of random walks on a $\mathrm{Simp}_{(N)}(\mathbb{Z}^{M+1})$ is similar to that of the Totally Asymmetric Zero Range Model:

$$F_{\mathrm{ph}}^{(N)}(\mathbf{l},\mathbf{j}|t) = \sum_{\{\mathbf{u}\}} \frac{e^{tE_N^{\mathrm{ph}}(\mathbf{u})}}{\mathcal{N}_{\mathrm{ph}}^2(\mathbf{u})} S_{\boldsymbol{\lambda}_R}(\mathbf{u}^2) S_{\boldsymbol{\lambda}_L}(\mathbf{u}^2)$$

with the squared norm $\mathcal{N}_{\mathrm{ph}}^2(\mathbf{u})$:

$$\mathcal{N}_{\mathrm{ph}}^2(\mathbf{u}) = \sum_{\boldsymbol{\lambda}\subseteq\{M^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2) S_{\boldsymbol{\lambda}}(\mathbf{u}^2)\,.$$

III. CONCLUDING REMARKS

We have presented an approach to the calculation of thermal correlation functions of quantum integrable models based on the theory of symmetric functions. We considered $XX0$ model and two modifications of the phase model: the modification described by non-Hermitian Hamiltonian and retaining boundary conditions and the modification described by Hermitian Hamiltonain and by reflecting boundary conditions. The approach is based on representation of the state-vectors with the use of symmetric functions (apart from the symmetric functions for the phase model with retaining conditions, we should mention the Shur functions in the case of XXZ model for infinite anisotropy, as well as for four-vertex model).

Such notions of enumerative combinatorics as partitions, plane partitions, self-avoiding lattice paths (the limiting cases of the XXZ model) appear naturally in the investigation of form factors and in the study of the asymptotic behaviour of the correlation functions. The problem of computing the formfactors of operators in the special q-parametrization is reduced to the computation of q-binomial determinants which are the generating functions for plane partitions in a box of finite size.

Binomial determinants appear in the limit $q \to 1$, thus giving an interpretation of the resulting form factors in terms of self-avoiding lattice paths.

THANKS !