Tetrahedron equation and matrix product method

Atsuo Kuniba (Univ. Tokyo)

Joint work with S. Maruyama and M. Okado

Reference

Multispecies totally asymmetric zero range process I, II
Journal of Integrable Systems (2016)

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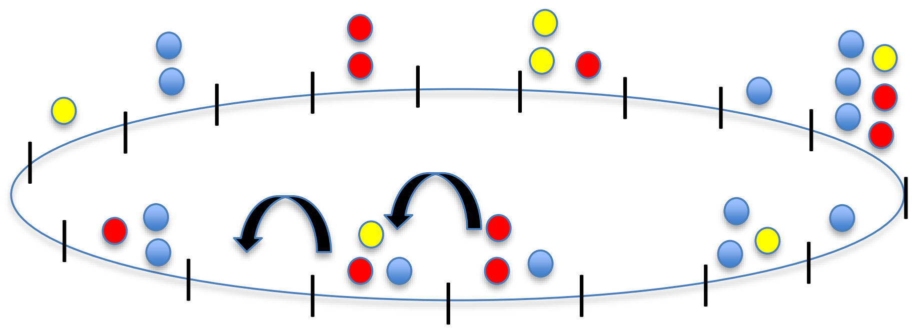
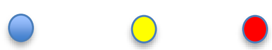
n-species Totally Asymmetric Zero-Rang Process (n-TAZRP)

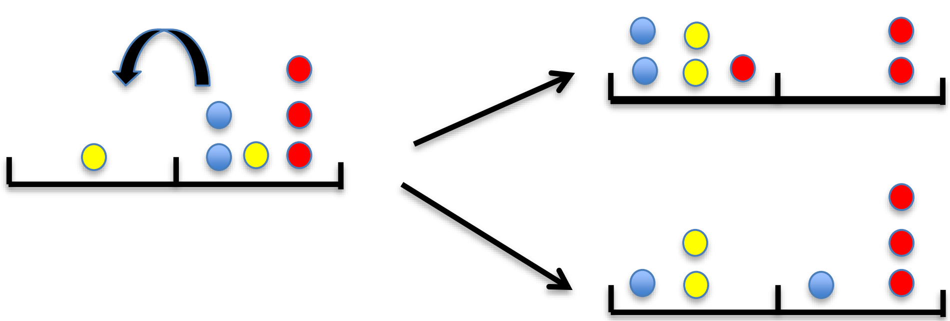
n=3 example



Species 1 2 3



Smaller species particles have *priority* to hop to the left neighbor site



allowed

forbidden

Local state (site variable)

$$\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n, \quad \alpha_a = \#(\text{species } a \text{ particles})$$

$$(\gamma, \delta) > (\alpha, \beta) \overset{\text{def}}{\Longleftrightarrow} \text{local transition } (\gamma, \delta) \to (\alpha, \beta) \text{ is allowed (priority rule obeyed)}$$

Local Markov matrix : $$h|\gamma, \delta\rangle = \sum_{(\gamma,\delta)>(\alpha,\beta)} \big(|\alpha, \beta\rangle - |\gamma, \delta\rangle\big)$$

Markov matrix : $$H = \sum_{i \in \mathbb{Z}_L} h_{i,i+1}$$

We consider n-TAZRP on 1D periodic chain, which is a Markov process governed by the master equation

$$\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle$$

$$|P(t)\rangle = \sum_{(\sigma_1,\ldots,\sigma_L)\in S(m)} \overbrace{\mathbb{P}(\sigma_1,\ldots,\sigma_L;t)}^{\text{Probability of the configuration at time t}} |\sigma_1,\ldots,\sigma_L\rangle$$

Set of configurations in the *sector* m=(m_1,. . ., m_n)

m_a = #(species *a* particles)

Probability of the configuration at time t

$$|P(t)\rangle = \sum_{(\sigma_1,\ldots,\sigma_L)\in S(m)} \underbrace{\mathbb{P}(\sigma_1,\ldots,\sigma_L;t)}_{\text{Probability of the configuration at time t}}|\sigma_1,\ldots,\sigma_L\rangle$$

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- Problem in non-equilibrium statistical mechanics
- Stochastic dynamics of n-species particles with priority constraint within the same departure site (zero-range interaction)

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- Example of *Integrable Probability*:

Today's main topic

$\boxed{\text{Matrix product construction of Steady State} \leftarrow \text{tetrahedron equation}}$

Associated with Stochastic R matrix for $U_q(A^{(1)}_n)$ (arXiv:1604:08304)

These features become most manifest for multispecies setting n>1

Steady states: H |P>=0

Each sector m has the unique steady state

$$|\bar{P}_L(\mathbf{m})\rangle = |\xi_L(\mathbf{m})\rangle + C|\xi_L(\mathbf{m})\rangle + \cdots + C^{L-1}|\xi_L(\mathbf{m})\rangle$$

(L = chain length, m = sector, C = cyclic shift)

Example from 3-TAZRP

$$\begin{aligned}
|\xi_2(1,1,1)\rangle &= 2|1,23\rangle + |2,13\rangle + 3|3,12\rangle + 6|\emptyset,123\rangle, \\
|\xi_3(1,1,1)\rangle &= 5|1,2,3\rangle + |1,3,2\rangle + 9|\emptyset,1,23\rangle + 3|\emptyset,2,13\rangle + 6|\emptyset,3,12\rangle + 12|\emptyset,12,3\rangle \\
&\quad + 3|\emptyset,13,2\rangle + 3|\emptyset,23,1\rangle + 18|\emptyset,\emptyset,123\rangle, \\
|\xi_4(1,1,1)\rangle &= 17|\emptyset,1,2,3\rangle + 3|\emptyset,1,3,2\rangle + 12|\emptyset,1,\emptyset,23\rangle + 3|\emptyset,2,1,3\rangle + 7|\emptyset,2,3,1\rangle + 8|\emptyset,2,\emptyset,13\rangle \\
&\quad + 9|\emptyset,3,1,2\rangle + |\emptyset,3,2,1\rangle + 20|\emptyset,3,\emptyset,12\rangle + 24|\emptyset,\emptyset,1,23\rangle + 6|\emptyset,\emptyset,2,13\rangle + 10|\emptyset,\emptyset,3,12\rangle \\
&\quad + 30|\emptyset,\emptyset,12,3\rangle + 6|\emptyset,\emptyset,13,2\rangle + 4|\emptyset,\emptyset,23,1\rangle + 40|\emptyset,\emptyset,\emptyset,123\rangle, \\
|\xi_2(2,1,1)\rangle &= 2|1,123\rangle + |2,113\rangle + 3|3,112\rangle + 2|11,23\rangle + |12,13\rangle + 6|\emptyset,1123\rangle, \\
|\xi_3(2,1,1)\rangle &= 3|1,1,23\rangle + 2|1,2,13\rangle + |1,3,12\rangle + 5|1,12,3\rangle + |1,13,2\rangle + 5|2,3,11\rangle + |2,11,3\rangle \\
&\quad + 9|\emptyset,1,123\rangle + 3|\emptyset,2,113\rangle + 6|\emptyset,3,112\rangle + 9|\emptyset,11,23\rangle + 3|\emptyset,12,13\rangle + 3|\emptyset,13,12\rangle \\
&\quad + 3|\emptyset,23,11\rangle + 12|\emptyset,112,3\rangle + 3|\emptyset,113,2\rangle + 3|\emptyset,123,1\rangle + 18|\emptyset,\emptyset,1123\rangle,
\end{aligned}$$

n=1-TAZRP has trivial (uniform) steady states

Result: Matrix product formula

Steady state probability

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}_{F^{\otimes n(n-1)/2}}(\underbrace{X_{\sigma_1} \cdots X_{\sigma_L}})$$

F = Fock space of *q*-boson at q=0

TAZRP operators

Piece of *layer transfer matrices* of 3D lattice model satisfying Tetrahedron equation

q-boson and Fock space

$\mathcal{A}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k} \rangle$ acts on $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$ by

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle, \quad \mathbf{k}|m\rangle = q^m|m\rangle$$

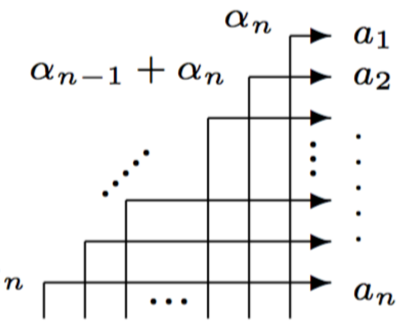
TAZRP operators

= $\mathcal{A}_0^{\otimes n(n-1)/2}$ valued partition function of ◢ region

$$X_\alpha = X_\alpha(z=1)$$

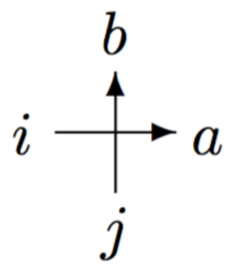
$$X_\alpha(z) = \sum z^{a_1+\cdots+a_n}$$

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$



$$\in \mathcal{A}_0^{\otimes n(n-1)/2}$$

Edges summed over $\mathbb{Z}_{\geq 0}$

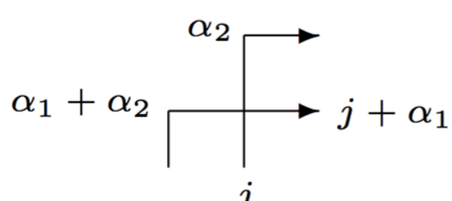


$$= \delta_{i+j}^{a+b}\theta(a \geq j)(\mathbf{a}^+)^j\mathbf{k}^{a-j}(\mathbf{a}^-)^b \in \mathcal{A}_{q=0}$$

$$\theta(\text{true}) = 1,\ \theta(\text{false}) = 0, \quad \delta_{i+j}^{a+b} = \theta(a+b=i+j)$$

n=2 Example

$$X_{\alpha_1,\alpha_2}(z) = \sum_{j\geq 0} z^{\alpha_1+\alpha_2+j}$$



$$= z^{\alpha_1+\alpha_2}\sum_{j\geq 0} z^j(\mathbf{a}^+)^j\mathbf{k}^{\alpha_1}(\mathbf{a}^-)^{\alpha_2}$$

Derivation

To show the steady state condition for TAZRP operators

$$x^{|\beta|} \sum_{(\gamma,\delta)\geq(\alpha,\beta)} X_\gamma(x) X_\delta(y) = (x \leftrightarrow y) \qquad (|\beta| = \beta_1 + \cdots + \beta_n)$$

. . . Highly non-local relation

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. . . Highly non-local relation

$q \longrightarrow 0$

Strategy of the proof

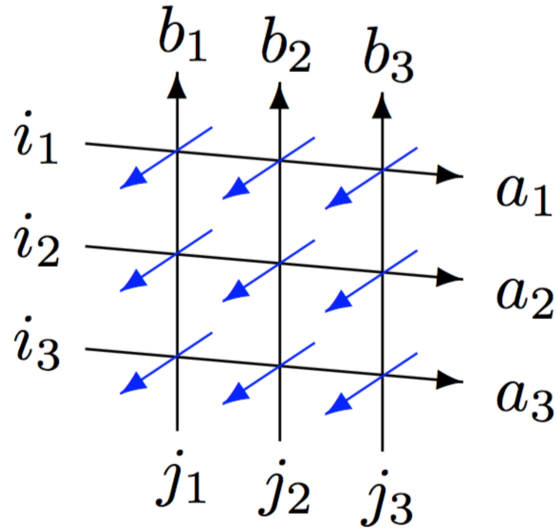
Bilinear relations of
layer transfer matrices

Tetrahedron equation
(Single local relation)

Layer transfer matrix with boundary condition **b**, **i**

n = 3 Example

$$\mathbb{T}(z)^{\mathbf{b}}_{\mathbf{i}} = \chi'_{\mathbf{b}}\,\chi_{\mathbf{i}} \sum_{\mathbf{a},\,\mathbf{j}} \chi'_{\mathbf{a}}\,\chi_{\mathbf{j}}$$



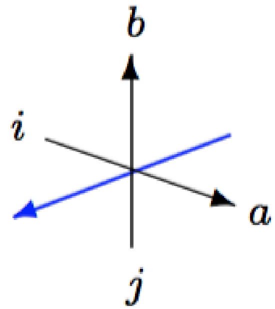
$$\in \mathcal{A}_q^{\otimes n^2}$$

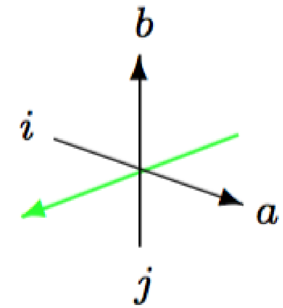
$$\mathbf{a} = (a_1, \ldots, a_n), \quad \chi'_{\mathbf{a}} = \prod_{l=1}^{n} (-q;q)_{a_l}, \quad \chi_{\mathbf{j}} = \prod_{l=1}^{n} (q;q)^{-1}_{j_l} \quad \text{etc}$$

All black edges except **b**, **i** are summed over $\mathbb{Z}_{\geq 0}$

Each 3D vertex is a *q*-boson acting on the Fock space on the blue lines

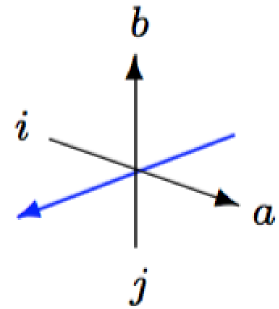
3D R-operators

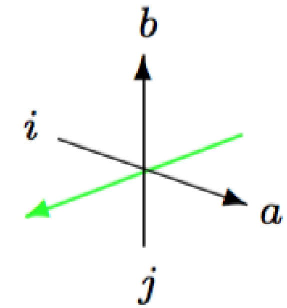
$\hat{\mathcal{R}}^{ab}_{ij}(z) \;=$ 

$\hat{\mathcal{S}}^{ab}_{ij}(z) \;=$ 

$$\hat{\mathcal{R}}^{ab}_{ij}(z) = \hat{\mathcal{S}}^{ba}_{ji}(z^{-1}) = \delta^{a+b}_{i+j}\, z^{j-b} \sum_{\lambda+\mu=b} (-1)^{\lambda} q^{\lambda+\mu^2-ib} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2} (\mathbf{a}^-)^{\mu} (\mathbf{a}^+)^{j-\lambda} \mathbf{k}^{i+\lambda-\mu}$$

3D R-operators

$\hat{\mathcal{R}}^{ab}_{ij}(z) =$ 

$\hat{\mathcal{S}}^{ab}_{ij}(z) =$ 

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Define 3D R-operators $\mathcal{R}(z)$, $\mathcal{S}(z) : F \otimes F \otimes F \to F \otimes F \otimes F$ by

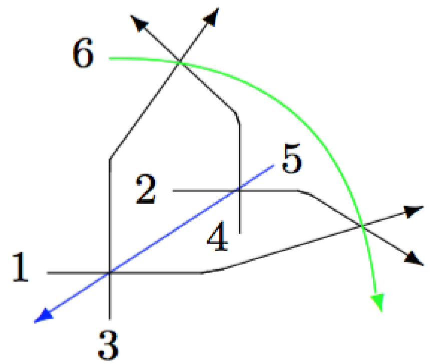
$$\mathcal{R}(z)\big(|i\rangle \otimes |j\rangle \otimes |k\rangle\big) = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes \hat{\mathcal{R}}^{ab}_{ij}(z)|k\rangle$$

$$\mathcal{S}(z)\big(|i\rangle \otimes |j\rangle \otimes |k\rangle\big) = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes \hat{\mathcal{S}}^{ab}_{ij}(z)|k\rangle$$

Fact (reducible to Kapranov-Voevodsky 1994)

As operators on $F^{\otimes 6}$ the tetrahedron eq. holds: $(z_{ij} = z_i/z_j)$

$$\mathcal{S}(z_{12})_{126}\mathcal{S}(z_{34})_{346}\mathcal{R}(z_{13})_{135}\mathcal{R}(z_{24})_{245} = \mathcal{R}(z_{24})_{245}\mathcal{R}(z_{13})_{135}\mathcal{S}(z_{34})_{346}\mathcal{S}(z_{12})_{126}$$



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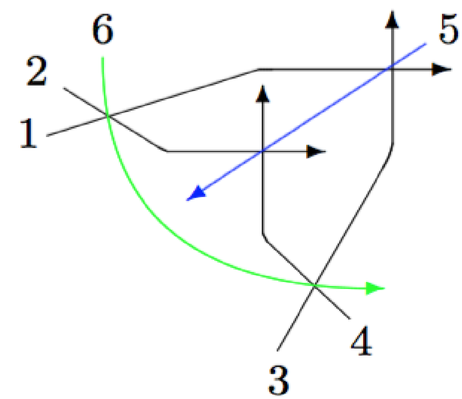

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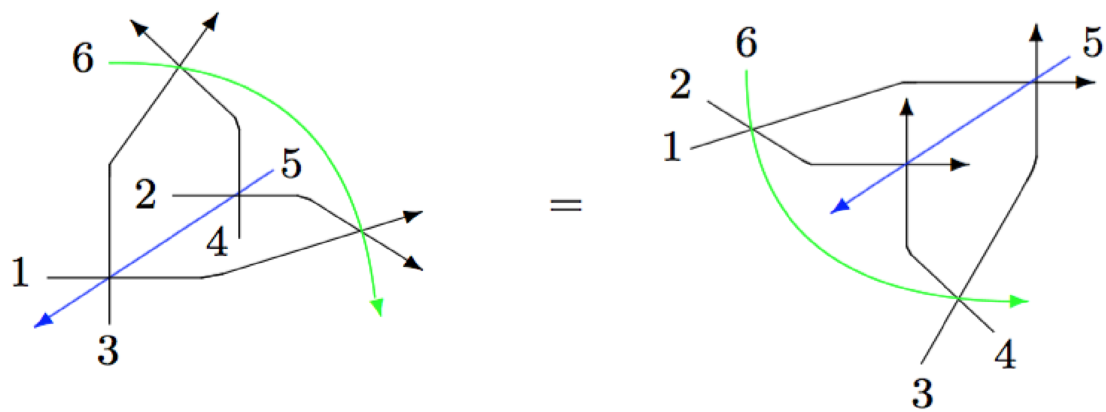

Background and relevant topics:

- Intertwiner of Soibelman's represetations of quantized coordinate ring of SL4
- Transition coefficients of the PBW bases of $U_q^+(sl_3)$
- Quantum geometry interpretation (Bazhanov-Mangazeev-Sergeev 2008)

Theorem (Bilinear relation of layer transfer matrices)

$$\sum_{\substack{\mathbf{b},\mathbf{b}',\mathbf{i},\mathbf{i}' \\ \mathbf{b}+\mathbf{b}'=\mathbf{s},\ \mathbf{i}+\mathbf{i}'=\mathbf{r}}} x^{|\mathbf{b}|+|\mathbf{i}|} y^{|\mathbf{b}'|+|\mathbf{i}'|}\, \mathbb{T}(x)^{\mathbf{b}}_{\mathbf{i}}\, \mathbb{T}(y)^{\mathbf{b}'}_{\mathbf{i}'} = (x \leftrightarrow y)$$

$$\forall \mathbf{s},\mathbf{r} \in (\mathbb{Z}_{\geq 0})^n$$

Generalizes the commutativity corresponding to $\mathbf{s} = \mathbf{r} = (0,\ldots,0)$

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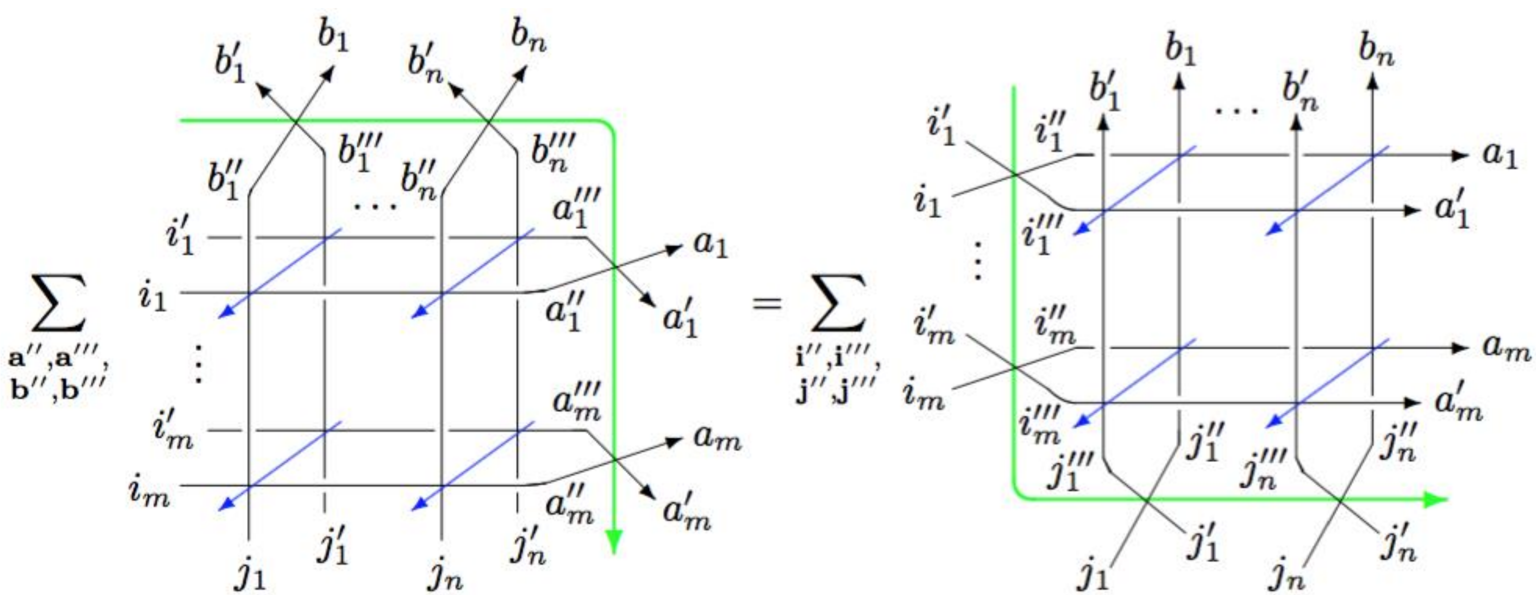
$$\forall \mathbf{s}, \mathbf{r} \in (\mathbb{Z}_{\geq 0})^n$$

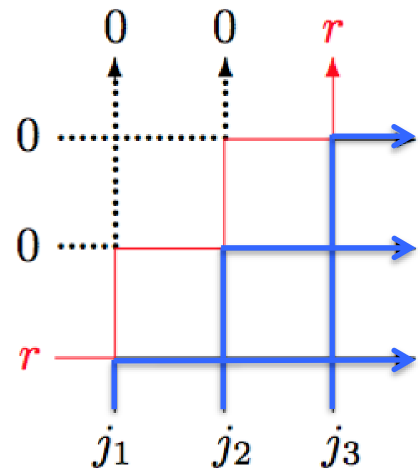
Generalizes the commutativity corresponding to **s** = **r** = (0,…,0)

Follows from repeated applications of the tetrahedron eq.



At q=0, Layer transfer matrix is frozen to TAZRP operators

$$\mathbb{T}(z)^{0,\dots,0,r}_{0,\dots,0,r} = z^{-r}\sum_{\mathbf{a},\mathbf{j}} z^{|\mathbf{j}|}$$



(n=3 example)

$$= z^{-r}\sum_{\alpha\in(\mathbb{Z}_{\geq 0})^n} \overbrace{X_\alpha(z)}^{\mathcal{A}_0^{\otimes n(n-1)/2}} \otimes \overbrace{\text{simply described operators}}^{\mathcal{A}_0^{\otimes n(n+1)/2}}$$

TAZRP operator

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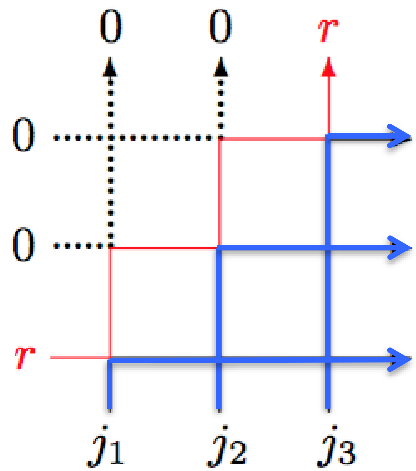

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TAZRP operator

Stationary condition for TAZRP operators ← (q=0) Bilinear relation of layer transfer matrices ← Tetrahedron eq.

Concluding remarks

A parallel story holds for
n-species *Totally Asymmetric Simple Exclusion Process* (n-TASEP)

TAZRP and TASEP correspond to the *two* situations in which type A quantum R matrices are factorized into solutions of the tetrahedron equation as follows:

Tetrahedron:	3D R operator	3D L operator
Yang-Baxter:	R matrix for symm. tensor rep.	R matrix for anti-symm. tensor rep.
Markov process:	n-TAZRP (today's talk)	n-TASEP (arXiv:1506.04490, 1509.09018)

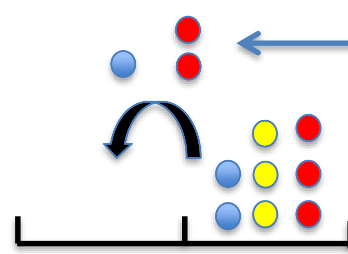
Integrable origin of n-TAZRP

n-TAZRP $\xleftarrow[\mu = 0]{q=0}$ $U_q(A^{(1)}_n)$-Zero Range Process
associated with *Stochastic R matrix*

(K-Mangazeev-Maruyama-Okado, arXiv:1604.08304)

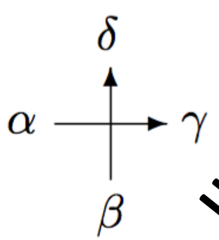
Quantum R matrix in a special gauge
satisfying the axioms of Markov matrix

Matrix elements generalize Povolotsky's transition rate for *q-Hahn process* (n=1)



γ =(1,0,2)

β=(2,3,3)



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$$q^{\sum_{1\le i<j\le n}(\beta_i-\gamma_i)\gamma_j}\left(\frac{\mu}{\lambda}\right)^{\gamma_1+\cdots+\gamma_n}\frac{(\lambda;q)_{\gamma_1+\cdots+\gamma_n}(\frac{\mu}{\lambda};q)_{\beta_1+\cdots+\beta_n-\gamma_1-\cdots-\gamma_n}}{(\mu;q)_{\beta_1+\cdots+\beta_n}}\prod_{i=1}^{n}\frac{(q;q)_{\beta_i}}{(q;q)_{\gamma_i}(q;q)_{\beta_i-\gamma_i}}$$

Matrix product formula for $U_q(A^{(1)}_2)$-ZRP (K-Okado, arXiv:1608.02779)

(Discrete time Markov process with inhomogeneity $\mu_1, \ldots, \mu_L$)

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}(X_{\sigma_1}(\mu_1) \cdots X_{\sigma_L}(\mu_L)),$$

$$X_\alpha(\mu) = \mu^{-\alpha_1-\alpha_2} \frac{(\mu; q)_{\alpha_1+\alpha_2}}{(q;q)_{\alpha_1}(q;q)_{\alpha_2}} \frac{(\mathbf{a}^+; q)_\infty}{(\mu^{-1}\mathbf{a}^+; q)_\infty} \frac{\mathbf{k}^{\alpha_2}}{(-q\mathbf{k}; q)_{\alpha_1}} (\mathbf{a}^-)^{\alpha_1}$$

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$$X_\alpha(\mu) = \mu^{-\alpha_1-\alpha_2} \frac{(\mu; q)_{\alpha_1+\alpha_2}}{(q;q)_{\alpha_1}(q;q)_{\alpha_2}} \frac{(\mathbf{a}^+; q)_\infty}{(\mu^{-1}\mathbf{a}^+; q)_\infty} \frac{\mathbf{k}^{\alpha_2}}{(-q\mathbf{k}; q)_{\alpha_1}} (\mathbf{a}^-)^{\alpha_1}$$

↑

A q-boson representation of *Zamolodchikov-Faddeev algebra*

$$X(\mu) \otimes X(\lambda) = \underbrace{\check{S}(\lambda, \mu)}_{U_q(A^{(1)}_2)\text{ Stochastic R matrix}} [X(\lambda) \otimes X(\mu)]$$

satisfying an ***auxiliary condition***

q=0, μ_i =0 case agrees with the tetrahedron result for 2-TAZRP