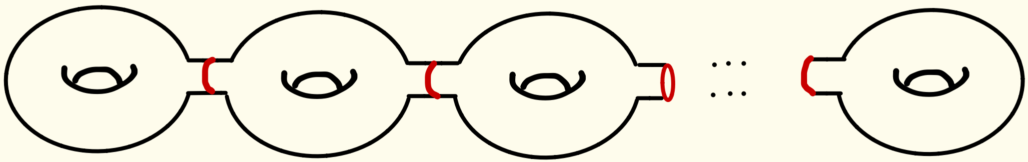
Discrete Time Evolution and Baxter's Q-operator

Christian.Korff@glasgow.ac.uk



1) Cylindric Macdonald functions and a deformed Verlinde algebra,
CMP 318 (2013) 173-246

2) From quantum Bäcklund transformations to TQFT,
JPA 49 (2016) 104001

RAQIS 16

Road map: quantisation of the Ablowitz-Ladik chain

Classical integrable system [Ablowitz-Ladik '76]

$$\partial_t \psi_j = \{H, \psi_j\}, \quad \partial_t \psi_j^* = \{H, \psi_j^*\}$$

$$H = -\sum_j \left(\psi_j^* \psi_{j+1} + \psi_j \psi_{j+1}^* - 2\ln(1-\psi_j^*\psi_j)\right)$$

Quantum integrable system

$$H = -\sum_j \left(\frac{\beta_j \beta_{j+1}^* + \beta_{j+1}\beta_j^*}{1-q} - 2N_j\right)$$

[Kulish 1981]

Poisson algebra

$$\{\psi_i, \psi_j^*\} = \delta_{ij}(1-\psi_j^*\psi_j)$$

$$\{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0$$

$$[\cdot,\cdot] = -i\hbar\{\cdot,\cdot\} + \mathcal{O}(\hbar^2)$$

$$\xrightarrow[\text{quantisation}]{}$$

$$t \to it, \quad \psi^* \to \pm\bar{\psi}$$

q-boson algebra

$$[\beta_i, \beta_j^*] = \delta_{ij}(1-q)(1-\beta_i^*\beta_i)$$

$$0 < q = e^{-\hbar c} < 1$$

integrals of motion

$$L_j(u) = \begin{pmatrix} 1 & u\psi_j^* \\ \psi_j & u \end{pmatrix}$$

⇝ monodromy matrix, spectral invariants

'Bethe algebra' ≃ 2D TQFT

$$L_j(u) = \begin{pmatrix} 1 & u\beta_j^* \\ \beta_j & u \end{pmatrix}$$

⇝ monodromy matrix, YB-algebra, Baxter's commuting transfer matrices

Ablowitz-Ladik chain : separation of time flow

Equations of motion

$$\begin{cases} \partial_t \psi_j = \psi_{j+1} - 2\psi_j + \psi_{j-1} - \psi_j^* \psi_j (\psi_{j+1} + \psi_{j-1}) \\ \partial_t \psi_j^* = -\psi_{j+1}^* + 2\psi_j^* - \psi_{j-1}^* + \psi_j^* \psi_j (\psi_{j+1}^* + \psi_{j-1}^*) \end{cases}$$

Decomposition of Hamiltonian into left- and right movers

$$H = H_R + H_L + H_0, \quad H_R = \sum_j \psi_j^* \psi_{j+1}, \quad H_L = \sum_j \psi_j \psi_{j+1}^*, \quad \{H_L, H_R\} = 0$$

'Auxiliary time flow'

$$\partial_{t_1} \psi_j = -\{H_L, \psi_j\} = \psi_{j-1}(1 - \psi_j^* \psi_j)$$

Since all 3 flows commute, we can consider them separately.

Darboux matrices : $D_{j+1}(u,v)\,L_j(u) = \tilde{L}_j(u)\,D_j(u,v)$, discrete zero-curvature equation

$$\det D_j(v,v) = 0 \quad \text{and} \quad D_j(u,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Bäcklund transform : $(\psi_j, \psi_j^*) \xrightarrow{\mathcal{B}(v)} (\tilde{\psi}_j, \tilde{\psi}_j^*)$, $v = \Delta t$, $\tilde{\psi}_j = \psi_j(\Delta t)$

1. Canonical map which preserves the Poisson structure $\{\cdot,\cdot\}$

2. Commutativity : $\mathcal{B}(v_1)\circ\mathcal{B}(v_2) = \mathcal{B}(v_2)\circ\mathcal{B}(v_1)$ [Veselov '91]

'time' discretisation : [Suris 1997]

$$\boxed{\frac{\psi_j(\Delta t) - \psi_j(0)}{\Delta t} = \left(1 - \psi_j^*\psi_j(\Delta t)\right)\psi_{j-1}(\Delta t)}$$

What is the quantum analogue of this evolution eqn?

q-boson Fock space

vacuum: $\beta\,|0\rangle = 0$

m q-boson state: $|m\rangle = \frac{(\beta^*)^m}{(q)_m}\,|0\rangle$

multi-particle state: $|\lambda\rangle = \bigotimes_{i=1}^{n} |m_i(\lambda)\rangle$

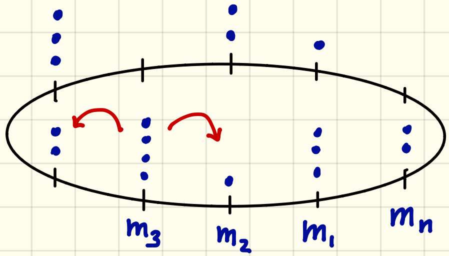
$$\beta_j^*\,|\lambda\rangle = (1-q^{m_j+1})\,|m_1,\dots,m_j+1,\dots,m_n\rangle\,,$$

$$\beta_j\,|\lambda\rangle = |m_1,\dots,m_j-1,\dots,m_n\rangle$$

→ canonical quantisation of the Poisson algebra:

$$[\beta_i,\beta_j^*] = \delta_{ij}\,(1-q)\,(1-\beta_i^*\beta_i)$$

periodic boundary conditions

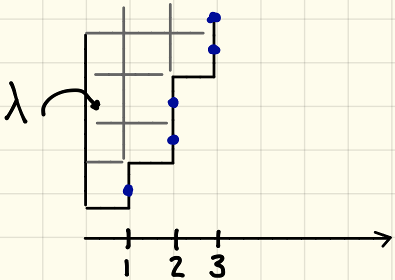


partition $\lambda = (\lambda_1,\dots,\lambda_k) = (1^{m_1}2^{m_2}\cdots n^{m_n})$

Example shown above: $n = 10$

$\lambda = (10,10,8,7,7,5,5,5,4,4,3,3,3,3,2,1,1,1)$

$k = 2+1+2+3+2+4+1+3 = 18$



Quantum Bäcklund transform $\longrightarrow$ Baxter's Q-operator [Pasquier-Gaudin 1992] (Toda chain)

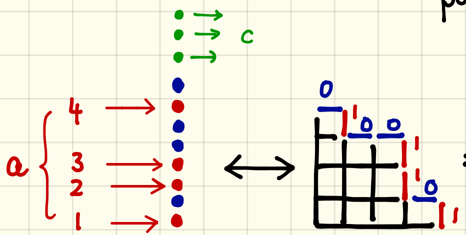
$$(\beta_j, \beta_j^*) \longmapsto (\tilde{\beta}_j, \tilde{\beta}_j^*)$$

$$\tilde{\beta}_j = Q(v)\,\beta_j\,Q(v)^{-1}$$
$$\tilde{\beta}_j^* = Q(v)\,\beta_j^*\,Q(v)^{-1}$$

Problem: find Q

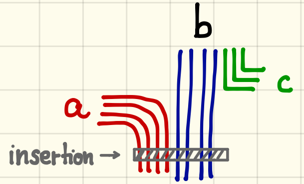
Define $Q(v)$ as the transfer matrix of an exactly solvable vertex model:

q-insertion: insert a particles into a pile of $b-c$ particles inside a gravitational potential with $q = e^{-\beta E}$, E energy to lift one particle.



$$= \lambda$$

$$\sum_{\lambda} q^{|\lambda|} = \begin{bmatrix} d \\ a \end{bmatrix}_q$$



$$d = a + b - c$$

$$a, b, c, d \in \mathbb{Z}_{\geq 0}$$

$$q^{4+3+3+1} = q^{|\lambda|}$$

particle picture

Boltzmann weight

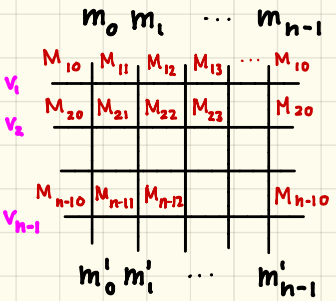
vertex configuration

Lattice configurations & q-Whittaker polynomials

Let $M \in \mathrm{Mat}_{(n-1)\times n}(\mathbb{N}_0)$ and set $a_i = \beta_i \beta^*_{i+1}$

INPUT

$$\mu = (1^{m_1} 2^{m_2} \cdots (n-1)^{m_{n-1}})$$



$$\lambda = (1^{m'_1} \cdots (n-1)^{m'_{n-1}})$$

OUTPUT

LEMMA

Define $$a^{(M)} = \overrightarrow{\prod_{1\leq i\leq n-1}} \frac{\beta_0^{*\,M_{i0}} a_1^{M_{i1}} \cdots a_{n-1}^{M_{in-1}} \beta_{n-1}^{M_{i0}}}{(q)_{M_{i0}} (q)_{M_{i1}} \cdots (q)_{M_{in-1}}}$$ then

the matrix elements of

$$Z(v) = \sum_M v^M a^{(M)}, \quad v^M = \prod_{\langle i,j\rangle} v_i^{M_{ij}}$$

are the partition function.

(cylindrical) skew q-Whittaker functions

THM [CK'13] open boundaries $M_{i0} = 0$: $\langle \lambda | Z(v) | \mu \rangle = P_{\lambda'/\mu'}(v; q, 0)$

periodic boundaries $M_{i0} > 0$: $\langle \lambda | Z(v) | \mu \rangle = \sum_d z^d P_{\lambda'/d/\mu'}(v; q, 0)$

Quantum Bäcklund transform

$\tilde{\beta}_j(v) = Q(v)\,\beta_j\,Q(v)^{-1}$ where

$Q(v)$ is the row-to-row transfer matrix of the q-insertion model.

<u>THM</u> [CK'16] discrete quantum "time flow"

$$\boxed{\frac{\tilde{\beta}_j(v)-\beta_j}{v} = \left(1-\beta_j^*\,\tilde{\beta}_j(v)\right)\tilde{\beta}_{j-1}(v) \qquad \frac{\tilde{\beta}_j^*(v)-\beta_j^*}{v} = \beta_{j+1}^*\left(\beta_j^*\,\tilde{\beta}_j(v)-1\right)}$$

Discrete time evolution

$[Q(u), Q^*(v)] = 0$

$\bar{\beta}_j = Q^*(v)^{-1}\beta_j\,Q^*(v)$ $\qquad$ $\bar{\beta}_j - \beta_j = -v(1-\bar{\beta}_j^*\beta_j)\beta_{j+1}$ $\qquad$ functional equation

$$\beta_j(\Delta t) = \underbrace{Q^*(-i\Delta t)^{-1}Q(i\Delta t)}_{U(\Delta t)^{-1}}\,\beta_j\,\underbrace{Q(i\Delta t)^{-1}Q^*(-i\Delta t)}_{U(\Delta t)}, \qquad U(\Delta t)^{-1} = U(\Delta t)^*$$

time evolution operator for time step Δt

$$\frac{\beta_j(\Delta t)-\beta_j}{i\Delta t} = \left(1-\bar{\beta}_j^*(-i\Delta t)\beta_j\right)\beta_{j+1} + \left(1-\bar{\beta}_j^*(-i\Delta t)\beta_j(\Delta t)\right)\beta_{j-1}(\Delta t)$$

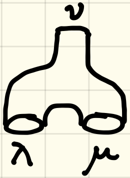
Multivariate Bäcklund transforms & TQFT fusion matrices

$$\mathcal{B}^+(v_1)\circ\cdots\circ\mathcal{B}^+(v_{n-1}) \rightsquigarrow Z(v) = Q(v_1)\cdots Q(v_{n-1}) = \sum_\lambda \underline{Q}_\lambda \, P_\lambda(v;q,0)$$

Fusion matrix   q-Whittaker function

<u>THM</u> [CK'13] Fusion matrices of 2D TQFT for fixed particle number k.

$$N^{\nu,k}_{\lambda\mu}(q) = \langle \nu | \underline{Q}_\lambda | \mu \rangle =$$



'pair of pants' 2-cobordism

Recurrence relations for fusion coefficients

$$\boxed{(1-q^{m_j(\mu)+1})\, N^{\lambda,k+1}_{r\,\beta_j^*\mu} = (1-q^{m_j(\lambda)})\, N^{\beta_j\lambda,k}_{r\mu} - (1-q^{m_{j+1}(\lambda)})\, N^{\beta_{j+1}\lambda,k}_{r-1\,\mu} + (1-q^{m_j(\lambda)})(1-q^{m_{j+1}(\lambda)})\, N^{\beta_j\beta_{j+1}\lambda,k-1}_{r-1\,\beta_j\mu}}$$

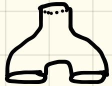
$$q\to 0: \quad N^{\lambda,k+1}_{r\,\beta_j^*\mu} = N^{\beta_j\lambda,k}_{r\mu} - N^{\beta_{j+1}\lambda,k}_{r-1\,\mu} + N^{\beta_j\beta_{j+1}\lambda,k-1}_{r-1\,\beta_j\mu}$$

$\widehat{su}(n)_k$ - WZW fusion ring 'phase model' [C.K., C. Stroppel Adv. Math. 2009]

2D TQFT $\hat{=}$ Symmetric Frobenius algebras [Atiyah 1988]

Functor $\quad Z: 2\mathrm{Cob} \to \mathrm{Vec}_{\mathbb{F}}$

$\mathbb{F}$ - vector spaces with $\dim V < \infty$ $\qquad$ $Z($  $) = V$ S^1 $\qquad$ $Z($  $) = V^*$

multiplication $\; m: Z(S^1) \otimes Z(S^1) \to Z(S^1)$ $\qquad$ $Z($  $) \in \mathrm{Hom}(Z(S^1) \otimes Z(S^1), Z(S^1))$
(commutative)

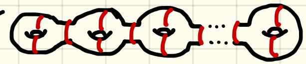
$a \otimes b \mapsto ab$

invariant bilinear form $\; \langle \cdot , \cdot \rangle : Z(S^1) \otimes Z(S^1) \to \mathbb{F}$ $\qquad$ $Z($  $) \in \mathrm{Hom}(Z(S^1) \otimes Z(S^1), \underbrace{Z(\cdot)}_{\mathbb{F}})$

$\langle ab, c \rangle = \langle a, bc \rangle$

unit element $\; e: \mathbb{F} \to Z(S^1)$ $\qquad$ $Z($  $) \in \mathrm{Hom}(\mathbb{F}, Z(S^1))$

$1 \mapsto \mathbb{1}$

TQFT partition function $\qquad$ $Z($  $)$

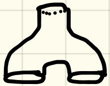
genus g surface

2D TQFT $\xleftarrow{\quad}$ operator version $\xrightarrow{\quad}$ q-bosons

$Z($  $) \in \mathrm{Vec}_{\mathbb{F}}$, $\quad \mathbb{F} = \mathbb{Z}[q, q^{-1}]$

S^1

Bethe algebra $\mathcal{B}_{n,k} \subset \mathrm{End}(\mathcal{F}_k)$

$Z($  $) \in \mathrm{Hom}(Z(S^1) \otimes Z(S^1), Z(S^1))$

$$\underline{Q}_\lambda \underline{Q}_\mu = \sum_\nu N_{\lambda\mu}^{\nu,k}(q)\, \underline{Q}_\nu$$

$N_{\lambda\mu}^{\nu,k}(q) \in \mathbb{Z}[q]$

$Z($  $) \in \mathrm{Hom}(Z(S^1) \otimes Z(S^1), Z(\cdot))$, $\quad Z(\cdot) = \mathbb{F}$

invariant bilinear form

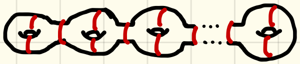
$$\langle \underline{Q}_\lambda, \underline{Q}_\mu \rangle = \delta_{\lambda\mu^*} \prod_{i \geq 1} \frac{1}{[m_i(\lambda)]_q!}$$

$$\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$$

$Z($  $) \in \mathrm{Hom}(\mathbb{F}, Z(S^1))$

unit element

$$\underline{Q}_{(n,\dots,n)} \underline{Q}_\lambda = \underline{Q}_\lambda$$

$Z($  $)$ TQFT partition function $\quad \mathrm{Tr}\left(\sum_\lambda \underline{Q}^*_\lambda \underline{Q}_{\lambda^*}\right)^{g-1}$

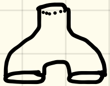
genus g surface

2D TQFT $\longleftarrow$ operator version $\longrightarrow$ q-bosons

$Z($$) \in \mathrm{Vec}_{\mathbb{F}}$, $\mathbb{F} = \mathbb{Z}[q, q^{-1}]$

S^1

Bethe algebra $\mathcal{B}_{n,k} \subset \mathrm{End}(\mathcal{F}_k)$

$Z($$) \in \mathrm{Hom}(Z(S^1) \otimes Z(S^1), Z(S^1))$

$$\underline{Q}_\lambda \underline{Q}_\mu = \sum_\nu \underbrace{N^{\nu,k}_{\lambda\mu}(q)}_{\in\, \mathbb{Z}[q]} \underline{Q}_\nu$$

$Z($$) \in \mathrm{Hom}(Z(S^1) \otimes Z(S^1), \overset{\mathbb{F}}{Z(\cdot)})$

invariant bilinear form

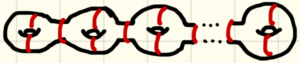
$$\langle \underline{Q}_\lambda, \underline{Q}_\mu \rangle = \delta_{\lambda\mu^*} \prod_{i \geq 1} \frac{1}{[m_i(\lambda)]_q!}$$

$$\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$$

$Z($$) \in \mathrm{Hom}(\mathbb{F}, Z(S^1))$

unit element

$$\underline{Q}_{(n,\dots,n)} \underline{Q}_\lambda = \underline{Q}_\lambda$$

$Z($$)$ TQFT partition function $\mathrm{Tr}\left(\sum_\lambda \underline{Q}^*_\lambda \underline{Q}_{\lambda^*}\right)^{g-1}$

genus g surface

AIM: describe the discrete time dynamics in terms of the TQFT

Two Q-operators

$$Q^{\pm}(v) = \sum_{r\geq 0} v^r Q_r^{\pm}$$

$$Q_r^{+} = \sum_{\alpha\vdash r} \beta_n^{\alpha_n} \frac{(\beta_{n-1}\beta_n^*)^{\alpha_{n-1}} \cdots (\beta_1\beta_2^*)^{\alpha_1}}{(q)_{\alpha_1} \cdots (q)_{\alpha_n}} \beta_1^{*\,\alpha_n}, \qquad Q_r^{-} = (-1)^r \sum_{\alpha\vdash r} \beta_1^{*\,\alpha_n} \frac{(\beta_1\beta_2^*)^{\alpha_1} \cdots (\beta_{n-1}\beta_n^*)^{\alpha_{n-1}}}{(q)_{\alpha_1} \cdots (q)_{\alpha_n}} \beta_n^{\alpha_n} \prod_i q^{\frac{\alpha_i(\alpha_i+1)}{2}}$$

THM ① Functional identities $\Delta(u) = q^{2N} u^n$ $N = \#$ of q-bosons

CK'13,16

$$T(u)Q^+(u) = Q^+(uq^2) + \Delta(u) Q^+(uq^{-2}),$$ TQ^+ equation

$$Q^-(u)T(u) = Q^-(uq^{-2}) + \Delta(uq^2) Q^-(uq^2),$$ Q^-T equation

$$Q^+(u)Q^-(uq^{-2}) - u^n q^{2N} Q^+(uq^{-2}) Q^-(u) = 1$$ 'quantum Wronskian'

The functional relations also imply the quantum analogue of $\mathcal{B}(v_1)\circ\mathcal{B}(v_2) = \mathcal{B}(v_2)\circ\mathcal{B}(v_1)$

COR $[Q_r^{\pm}, T_s] = [Q_r^{+}, Q_s^{-}] = [Q_r^{\pm}, Q_s^{\pm}] = 0$

'Bethe algebra' $\subset$ q-boson algebra

commutative non-commutative

Because we are dealing with non-commutative variables in the quantum case, the equation defining the Darboux transformation is now replaced with the Yang-Baxter eqn:

$$\boxed{D^{\pm}_{12}(u,v)\, L_{1j}(u)\, L^{\pm}_{2j}(v) = L^{\pm}_{2j}(v)\, L_{1j}(u)\, D^{\pm}_{12}(u,v)}$$

This allows one to define $Q^{\pm}$ in a similar way as $T(u) = \mathrm{Tr}\, L_n(u) \cdots L_1(u)$

$Q^{\pm}$-operators for the q-boson model [CK 2013, 2016]

$$L^{+}_{j}(v) = \left((-v)^m \frac{\beta_j^{*m} \beta_j^{m'}}{(q)_m} \right)_{m,m' \geq 0} \qquad L^{-}_{j}(v) = \left(v^m q^{\frac{m(m+1)}{2}} \frac{\beta_j^{m'} \beta_j^{*m}}{(q)_m} \right)_{m,m' \geq 0}$$

Current operators (formal power series in v with coefficients in q-boson algebra)

$$Q^{\pm}(v) = \mathrm{Tr}\, L^{\pm}_{n}(v) \cdots L^{\pm}_{1}(v) = \sum_{r \geq 0} v^r Q^{\pm}_{r}$$ explicitly known

Omitted from the discussion

▷ combinatorial approach to compute $N^{\nu}_{\lambda\mu}(q) \in \mathbb{Z}[q]$

Hall polynomial

Recall skew Macdonald functions : $P_{\lambda/\mu}(x;q,t) = \sum_{\nu} f^{\lambda}_{\mu\nu}(q,t)\, P_{\nu}(x;q,t)$

⇝ cylindric q-Whittaker functions $P_{\lambda/d/\mu}(x;q,0) = \sum_{\nu} N^{\lambda}_{\mu\nu}(q)\, P_{\nu}(x;q,0)$

▷ $N^{\nu}_{\lambda\mu}(0) \in \mathbb{Z}_{\geq 0}$ are the $\widehat{su}(n)_k$-WZW-fusion coefficients $k = \#$ of q-bosons

$\mathrm{TQFT}_{q=0} \cong K_0(\mathcal{C})$, $\mathcal{C}$ tensor category of $U_\varepsilon\, su(n)$ tilting modules

with $\varepsilon = e^{i\pi/k+n}$ (QFT: Chern-Simons)

⇝ Geometric interpretation at $q \neq 0$?