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Lax representation of the hyperbolic van Diejen system with two coupling parameters

A joint work with B.G. PUSZTAI

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Introduction

Our principal goal is to study the dynamics generated by the smooth *Hamiltonian function*

$$H=\sum_{a=1}^{n}\cosh(\theta_a)\left(1+\frac{\sin(\nu)^2}{\sinh(2\lambda_a)^2}\right)^{\frac{1}{2}}\prod_{\substack{c=1\\(c\neq a)}}^{n}\left(1+\frac{\sin(\mu)^2}{\sinh(\lambda_a-\lambda_c)^2}\right)^{\frac{1}{2}}\left(1+\frac{\sin(\mu)^2}{\sinh(\lambda_a+\lambda_c)^2}\right)^{\frac{1}{2}},$$

where $\mu,\nu\in\mathbb{R}$ are arbitrary coupling constants satisfying the conditions $\sin(\mu)\neq 0\neq\sin(\nu)$.

The van Diejen systems are multi-parametric integrable deformations of the translation invariant Ruijsenaars-Schneider (RS) models. In the so-called 'non-relativistic' limit, they reproduce the Calogero-Moser-Sutherland (CMS) models associated with the BC-type root systems. The *configuration space* of the hyperbolic n-particle van Diejen model is the open subset

$$Q=\{\lambda=(\lambda_1,\dots,\lambda_n)\,|\,\lambda_1>\ldots>\lambda_n>0\}\subseteq\mathbb{R}^n,$$

that can be seen as an open Weyl chamber of type BC_n. The cotangent bundle of Q is trivial, and it can be naturally identified with the open subset

$$P=Q\times\mathbb{R}^n=\{(\lambda,\theta)=(\lambda_1,\dots,\lambda_n,\theta_1,\dots,\theta_n)\,|\,\lambda_1>\ldots>\lambda_n>0\}\subseteq\mathbb{R}^{2n}.$$

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Abstract. In his 1994 thesis, Jan Felipe van Diejen proved the quantum integrability of the hyperbolic Ruijsenaars-Schneider model attached to the BC_n root system. This led to explicit formulas for a complete set of Poisson commuting functions in the classical limit, but a Lax matrix generating these Hamiltonians as its spectral invariants was lacking ever since*. In a recent joint work with B.G. Pusztai, we constructed a Lax pair for the classical hyperbolic BC_n system with two independent couplings. We showed that the dynamics can be solved by a projection method and worked out the asymptotic form of the solutions. The equivalence of the first integrals provided by the eigenvalues of our Lax matrix and van Diejen's commuting Hamiltonians was also demonstrated.

*Except for the 1-coupling cases obtained from the standard A_m models by `folding'.

Physical interpretation of the model

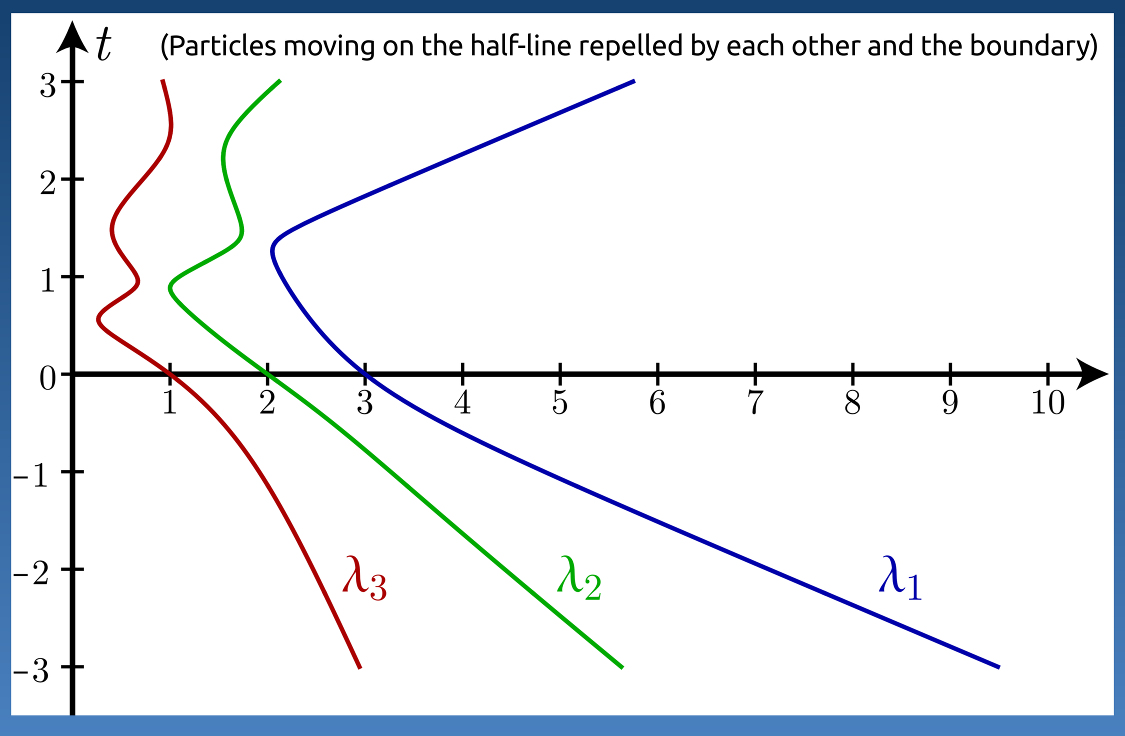

Figure. The space-time diagram of 3 particles with couplings

$$\mu=\frac{1}{2},\quad \nu=\frac{7}{5}.$$

The initial conditions are $\lambda_a(0)=4-a$ and $\theta_a(0)=-1$, $a=1,2,3$.

Group-theoretic background

With the aid of the $2n\times 2n$ matrix

$$C=\begin{bmatrix}0_n & \mathbf{1}_n\\ \mathbf{1}_n & 0_n\end{bmatrix}$$

we define the non-compact real reductive matrix Lie group

$$G=U(n,n)=\{y\in GL(2n,\mathbb{C})\,|\,y^*Cy=C\},$$

in which the set of unitary elements

$$K=\{y\in G\,|\,y^*y=\mathbf{1}_{2n}\}\cong U(n)\times U(n)$$

forms a maximal compact subgroup.

The Lie algebra of G takes the form

$$\mathfrak{g}=\mathfrak{u}(n,n)=\{Y\in\mathfrak{gl}(2n,\mathbb{C})\,|\,Y^*C+CY=0\},$$

whereas for the Lie subalgebra corresponding to K we have

$$\mathfrak{k}=\{Y\in\mathfrak{g}\,|\,Y^*+Y=0\}\cong\mathfrak{u}(n)\oplus\mathfrak{u}(n).$$

Upon introducing the subspace

$$\mathfrak{p}=\{Y\in\mathfrak{g}\,|\,Y^*=Y\},$$

we get $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$.

The set of the positive definite elements of the group $U(n,n)$ is

$$\exp(\mathfrak{p})=\{y\in U(n,n)\,|\,y>0\}.$$

We introduce the maximal Abelian subspace

$$\mathfrak{a}=\{X=\operatorname{diag}(x,-x)\,|\,x\in\mathbb{R}^n\}.$$

The centralizer of $\mathfrak{a}$ inside K is an Abelian Lie group with Lie algebra

$$\mathfrak{m}=\{\mathrm{i}\operatorname{diag}(\chi,\chi)\,|\,\chi\in\mathbb{R}^n\}.$$

If $\mathfrak{m}^\perp$ and $\mathfrak{a}^\perp$ denote the sets of the off-diagonal elements, then we can write the refined orthogonal decomposition

$$\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{m}^\perp\oplus\mathfrak{a}\oplus\mathfrak{a}^\perp.$$

Consider the commuting family of linear operators

$$\mathrm{ad}_X:\mathfrak{gl}(2n,\mathbb{C})\to\mathfrak{gl}(2n,\mathbb{C}),\quad Y\mapsto[X,Y]$$

The subspace $\mathfrak{m}^\perp\oplus\mathfrak{a}^\perp$ is invariant under ad_X, i.e.,

$$\widetilde{\mathrm{ad}}_X=\mathrm{ad}_X|_{\mathfrak{m}^\perp\oplus\mathfrak{a}^\perp}\in\mathfrak{gl}(\mathfrak{m}^\perp\oplus\mathfrak{a}^\perp)$$

makes sense. The regular part of $\mathfrak{a}$ is

$$\mathfrak{a}_{\mathrm{reg}}=\{X\in\mathfrak{a}\,|\,\widetilde{\mathrm{ad}}_X\text{ is invertible}\},$$

whereas the regular part of $\mathfrak{p}$ is

$$\mathfrak{p}_{\mathrm{reg}}=\{kXk^{-1}\in\mathfrak{p}\,|\,X\in\mathfrak{a}_{\mathrm{reg}}\text{ and }k\in K\}.$$

Some notations

We use the following shorthand notations:

- $n\in\mathbb{N}$ for the number of particles
- $\mathbb{N}_n=\{1,\dots,n\}$, $\mathbb{N}_{2n}=\{1,\dots,2n\}$
- $\mathbf{1}_n, 0_n$ for the $n\times n$ unit and zero matrix
- $\Lambda=(\lambda_1,\dots,\lambda_n,-\lambda_1,\dots,-\lambda_n)$
- $\mathbf{\Lambda}=\operatorname{diag}(\lambda_1,\dots,\lambda_n,-\lambda_1,\dots,-\lambda_n)$
- M^* for the conjugate transpose of the matrix M
- $\operatorname{Spec}(M)$ for the set of eigenvalues of the matrix M

Main result

We constructed a Lax pair (L,B), where L is the Lax matrix

$$L_{j,k}=\frac{\mathrm{i}\sin(\mu)F_j\bar{F}_k+\mathrm{i}\sin(\mu-\nu)C_{j,k}}{\sinh(\mathrm{i}\mu+\Lambda_j-\Lambda_k)}$$

for the hyperbolic van Diejen system with two independent coupling parameters!

Algebraic properties of the Lax matrix

The matrix L and the Lie group U(n,n)

Now we define our Lax matrix $L\in\mathfrak{gl}(2n,\mathbb{C})$ with the entries

$$L_{j,k}=\frac{\mathrm{i}\sin(\mu)F_j\bar{F}_k+\mathrm{i}\sin(\mu-\nu)C_{j,k}}{\sinh(\mathrm{i}\mu+\Lambda_j-\Lambda_k)}\quad(j,k\in\mathbb{N}_{2n}).$$

Here $F\in\mathbb{C}^{2n}$ is the column vector with components

$$F_a=e^{\frac{\theta_a}{2}}u_a^{\frac{1}{2}}\quad\text{and}\quad F_{n+a}=e^{-\frac{\theta_a}{2}}\bar{z}_a u_a^{-\frac{1}{2}}\quad(a\in\mathbb{N}_n)$$

with $z_j\in\mathbb{C}$ $(j\in\mathbb{N}_{2n})$ defined by

$$z_j=-\frac{\sinh(\mathrm{i}\nu+2\Lambda_j)}{\sinh(2\Lambda_j)}\prod_{\substack{c=1\\(c\neq j,j-n)}}^{n}\frac{\sinh(\mathrm{i}\mu+\Lambda_j-\lambda_c)}{\sinh(\Lambda_j-\lambda_c)}\frac{\sinh(\mathrm{i}\mu+\Lambda_j+\lambda_c)}{\sinh(\Lambda_j+\lambda_c)}.$$

Proposition. *The matrix L is Hermitian and obeys the quadratic equation $LCL=C$, i.e., L takes values in the Lie group $U(n,n)$.*

Lemma. *At each point of the phase space we have $L\in\exp(\mathfrak{p})$.*

Commutation relation and regularity

Lemma. *The matrix L and the diagonal matrix $e^{\mathbf{\Lambda}}$ obey the Ruijsenaars type commutation relation*

$$e^{\mathrm{i}\mu}e^{\mathrm{ad}_{\mathbf{\Lambda}}}L-e^{-\mathrm{i}\mu}e^{-\mathrm{ad}_{\mathbf{\Lambda}}}L=2\mathrm{i}\sin(\mu)FF^*+2\mathrm{i}\sin(\mu-\nu)C.$$

Lemma. *Under the additional assumption on the coupling parameters*

$$\sin(2\mu-\nu)\neq 0,$$

the spectrum of the matrix L is simple, and of the form

$$\operatorname{Spec}(L)=\{e^{\pm 2\hat{\theta}_a}\,|\,a\in\mathbb{N}_n\},$$

where $\hat{\theta}_1>\cdots>\hat{\theta}_n>0$. In other words, L is regular in the sense that $L\in\exp(\mathfrak{p}_{\mathrm{reg}})$.

Spectral invariants of the Lax matrix

Poisson brackets of the eigenvalues of L

Theorem. *The eigenvalues of the Lax matrix L are in involution.*

Link to van Diejen's Hamiltonians

The complete set of Poisson commuting functions found by van Diejen can be defined by introducing the following constituents:

$$v(x)=\frac{\sinh(\mathrm{i}\mu+x)}{\sinh(x)},\quad w(x)=\frac{\sinh(\mathrm{i}\nu+2x)}{\sinh(2x)},$$

$$\theta_{\varepsilon J}=\sum_{j\in J}\varepsilon_j\theta_j,\quad V_{\varepsilon J;J^c}=\prod_{j\in J}w(\varepsilon_j\lambda_j)\prod_{\substack{j,j'\in J\\(j<j')}}v(\varepsilon_j\lambda_j+\varepsilon_{j'}\lambda_{j'})^2\prod_{\substack{j\in J\\k\in J^c}}v(\varepsilon_j\lambda_j+\lambda_k)v(\varepsilon_j\lambda_j-\lambda_k),$$

$$U_{J^c,\ell-|J|}=(-1)^{\ell-|J|}\sum_{\substack{I\subseteq J^c,\ |I|=\ell-|J|\\ \varepsilon_i=\pm1,\ i\in I}}\prod_{i\in I}w(\varepsilon_i\lambda_i)\prod_{\substack{i,i'\in I\\(i<i')}}|v(\varepsilon_i\lambda_i+\varepsilon_{i'}\lambda_{i'})|^2\prod_{\substack{i\in I\\k\in J^c\setminus I}}v(\varepsilon_i\lambda_i+\lambda_k)v(\varepsilon_i\lambda_i-\lambda_k).$$

$$H_\ell=\sum_{\substack{J\subseteq\mathbb{N}_n,\ |J|\leq\ell\\ \varepsilon_j=\pm1,\ j\in J}}\cosh(\theta_{\varepsilon J})|V_{\varepsilon J;J^c}|U_{J^c,\ell-|J|}$$

Our Hamiltonians are the coefficients of the characteristic polynomial:

$$\det(L-y\mathbf{1}_N)=\sum_{m=0}^{N}K_m y^{N-m}$$

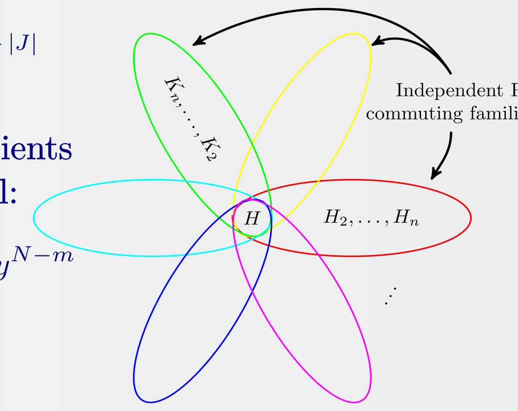

Figure. A possible, yet undesired scenario.

Lemma. *The two distinguished families of first integrals $\{H_\ell\}_{\ell=0}^n$ and $\{K_m\}_{m=0}^n$ are connected by an invertible linear relation with purely numerical coefficients depending only on the paramteres μ,ν.*

Analyzing the dynamics

Completeness of the Hamiltonian vector field

Theorem. *The Hamiltonian vector field $\boldsymbol{X}_H$ generated by the van Diejen type Hamiltonian function H is complete. That is, the maximum interval of existence of each integral curve is the whole real axis $\mathbb{R}$.*

Lax representation of the dynamics

Let $B\in\mathfrak{k}$ be defined by

$$B_{a,a}=B_{n+a,n+a}=\frac{\mathrm{i}}{F_a}\mathrm{Im}\left(\sum_{\substack{k=1\\(k\neq a)}}^{2n}\frac{L_{a,k}-(L^{-1})_{a,k}}{2\sinh(\Lambda_a-\Lambda_k)}F_k\right),\quad(a\in\mathbb{N}_n)$$

$$B_{j,k}=-\coth(\Lambda_j-\Lambda_k)\frac{L_{j,k}-(L^{-1})_{j,k}}{2}.\quad(j,k\in\mathbb{N}_{2n},\ j\neq k)$$

Theorem. *The derivative of L along the Hamiltonian vector field $\boldsymbol{X}_H$ takes the Lax form*

$$\boldsymbol{X}_H[L]=[L,B].$$

Thus (L,B) is a Lax pair for the dynamics generated by the Hamiltonian H.

Temporal asymptotics

Lemma. *For an arbitrary maximal solution of the hyperbolic n-particle van Diejen system H the particles move asymptotically freely as $|t|\to\infty$. More precisely, for all $a\in\mathbb{N}_n$ we have the asymptotics*

$$\lambda_a(t)\sim t\sinh(\theta_a^\pm)+\lambda_a^\pm\quad\text{and}\quad\theta_a(t)\sim\theta_a^\pm,$$

where the asymptotic momenta obey

$$\theta_a^-=-\theta_a^+\quad\text{and}\quad\theta_1^+>\ldots>\theta_n^+>0.$$

CMS and RSvD models

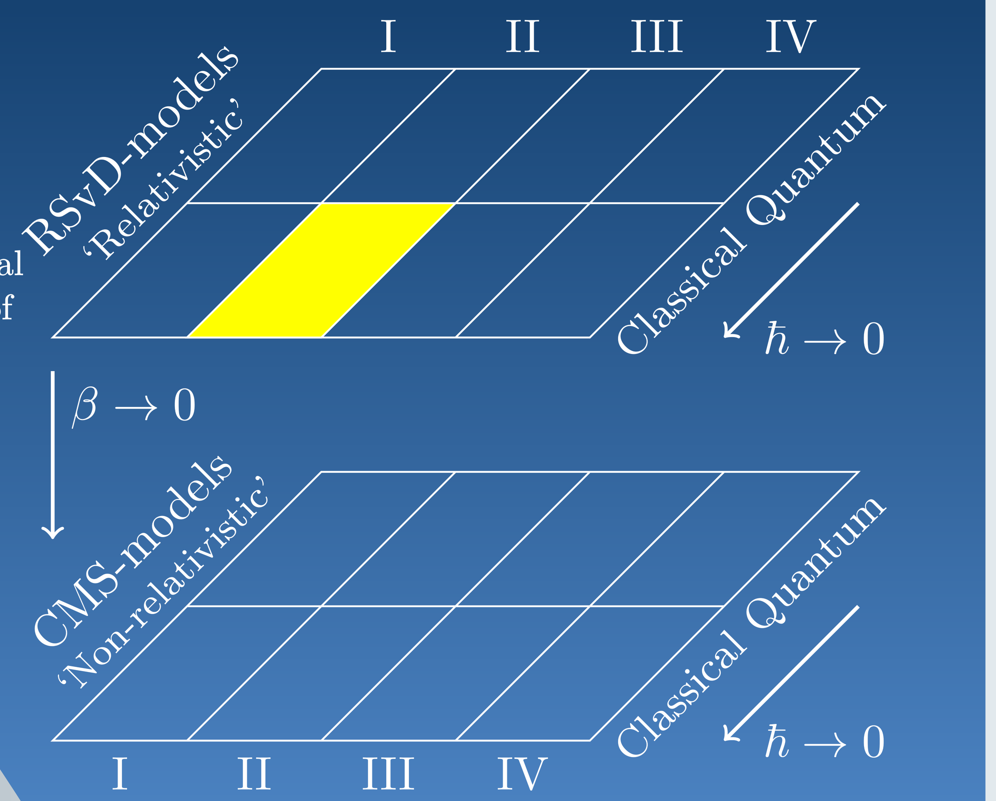

Figure. The roman numerals label the type of particle interaction:
I. Rational
II. Hyperbolic
III. Trigonometric
IV. Elliptic
The yellow field is the classical hyperbolic RSvD model of type A.

Generalizations

Lax matrix with spectral parameter:

$$\mathcal{L}_{j,k}=\big(\mathrm{i}\sin(\mu)F_j\bar{F}_k+\mathrm{i}\sin(\mu-\nu)C_{j,k})\big)\times\Phi(\mathrm{i}\mu+\Lambda_j-\Lambda_k\,|\,\eta)$$

with $\Phi(x\,|\,\eta)=e^{x\coth(\eta)}\left(\coth(x)-\coth(\eta)\right)$.

Lax matrix containing 3 independent coupling parameters:

$$\tilde{L}=h^{-1}Lh^{-1},$$

where h is a certain Hermitian matrix in the Lie group $U(n,n)$. It has the form

$$h(\lambda)=\begin{bmatrix}\operatorname{diag}(\alpha(\lambda)) & \operatorname{diag}(\beta(\lambda))\\ -\operatorname{diag}(\beta(\lambda)) & \operatorname{diag}(\alpha(\lambda))\end{bmatrix}.$$

Applications

+ particle-soliton picture in the context of boundary field theories

+ new integrable tops based on the Lax matrices of CMS and RSvD systems

+ classical/quantum duality relating the spectra of certain quantum spin chains to Lax matrices of the classical CMS and RSvD systems

+ action-angle duality for the (self-dual) hyperbolic RSvD systems (Hamiltonian reduction)

+ integrable random matrix ensembles

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