The space of Gaudin subalgebras

Giovanni Felder

Geneva, 26 August 2016

Gaudin models

- $\mathfrak{g}$ complex Lie algebra with invariant non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$.
- Tensor Casimir $\Omega = \sum_{a=1}^{\dim \mathfrak{g}} e_a \otimes e^a \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$, $\langle e_a, e^b \rangle = \delta_{ab}$
- $\Omega_{ij} = \sum_{a=1}^{\dim \mathfrak{g}} 1 \otimes \cdots \otimes 1 \otimes e_a \otimes \cdots \otimes e^a \otimes 1 \otimes \cdots \otimes 1 \in (U\mathfrak{g}^{\otimes n})^{\mathfrak{g}}$

Follows from the relations $[\Omega_{ij}, \Omega_{kl}] = 0$, $[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0$, i, j, k, l distinct.

Gaudin models

- $\mathfrak{g}$ complex Lie algebra with invariant non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$.
- Tensor Casimir $\Omega = \sum_{a=1}^{\dim \mathfrak{g}} e_a \otimes e^a \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$, $\langle e_a, e^b \rangle = \delta_{ab}$
- $\Omega_{ij} = \sum_{a=1}^{\dim \mathfrak{g}} 1 \otimes \cdots \otimes 1 \otimes e_a \otimes \cdots \otimes e^a \otimes 1 \otimes \cdots \otimes 1 \in (U\mathfrak{g}^{\otimes n})^{\mathfrak{g}}$

Theorem (Gaudin)

The *Gaudin Hamiltonians*

$$H_i = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \in (U\mathfrak{g})^{\otimes n}$$

commute for any $z \in \mathbb{C}^n \setminus \cup_{i<j}\{z, z_i = z_j\}$

Proof. *Follows from the relations* $[\Omega_{ij}, \Omega_{kl}] = 0$, $[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0$, i, j, k, l *distinct.*

Kohno–Drinfeld Lie algebra

- $\mathfrak{t}_n$ Lie algebra generated by $t_{ij} = t_{ji}$, $i \neq j = 1, \ldots, n$ subject to the "infinitesimal braid" relations

$$[t_{ij}, t_{kl}] = 0, \quad i, j, k, l \text{ distinct},$$
$$[t_{ij}, t_{ik} + t_{jk}] = 0, \quad i, j, k \text{ distinct}.$$

- $\mathfrak{t}_n = \oplus_{i=1}^{\infty} \mathfrak{t}_n^i$ is a graded Lie algebra

The *configuration space* of n points on X is $\mathrm{Conf}_n(X) = \{z \in X^n, z_i \neq z_j \text{ if } i \neq j\}$.

Lemma

The *Gaudin Hamiltonians*

$$H_i(z) = \sum_{j \neq i} \frac{t_{ij}}{z_i - z_j} \in \mathfrak{t}_n^1$$

commute for any $z \in \mathrm{Conf}_n(\mathbb{C})$: $[H_i(z), H_j(z)] = 0$.

Gaudin subalgebras

The Gaudin Hamiltonians span an $(n-1)$-dimensional abelian Lie subalgebra of $\mathfrak{t}_n$ contained in $\mathfrak{t}_n^1$. ($\sum H_i = 0$).

Definition

A Gaudin subalgebra (for A_{n-1}) is an abelian Lie subalgebra of $\mathfrak{t}_n$ in $\mathfrak{t}_n^1$ of maximal dimension $n-1$.

Examples:

- $G_n(z) = \mathrm{span}(H_1(z), \ldots, H_n(z))$ for $z_i \neq z_j$

Gaudin subalgebras

The Gaudin Hamiltonians span an $(n-1)$-dimensional abelian Lie subalgebra of $\mathfrak{t}_n$ contained in $\mathfrak{t}_n^1$. ($\sum H_i = 0$).

Definition

A Gaudin subalgebra (for A_{n-1}) is an abelian Lie subalgebra of $\mathfrak{t}_n$ in $\mathfrak{t}_n^1$ of maximal dimension $n-1$.

Examples:

- $G_n(z) = \mathrm{span}(H_1(z), \ldots, H_n(z))$ for $z_i \neq z_j$
- Limits as $z_i \to z_j$, e.g. $\mathrm{span}(J_1, \ldots, J_n)$,

$$J_i = \lim_{t \to 0} t^{n-i} H_i(t^{n-1}, \ldots, t^2, t, 1) = t_{1i} + \cdots + t_{i-1,i}$$

Gaudin subalgebras

The Gaudin Hamiltonians span an $(n-1)$-dimensional abelian Lie subalgebra of $\mathfrak{t}_n$ contained in $\mathfrak{t}_n^1$. ($\sum H_i = 0$).

Definition

A Gaudin subalgebra (for A_{n-1}) is an abelian Lie subalgebra of $\mathfrak{t}_n$ in $\mathfrak{t}_n^1$ of maximal dimension $n-1$.

Examples:

- $G_n(z) = \mathrm{span}(H_1(z), \ldots, H_n(z))$ for $z_i \neq z_j$
- Limits as $z_i \to z_j$, e.g. $\mathrm{span}(J_1, \ldots, J_n)$,

$$J_i = \lim_{t \to 0} t^{n-i} H_i(t^{n-1}, \ldots, t^2, t, 1) = t_{1i} + \cdots + t_{i-1,i}$$

Remark. The transpositions $s_{ij} \in \mathbb{C}S_n$ obey the Kohno–Drinfeld relations. The image of the commuting elements J_i in $\mathbb{C}S_n$ are called Jucys–Murphy elements. Their spectrum in irreducible representations is simple.

Questions

Gaudin subalgebras are points in the Grassmannian of $(n-1)$-planes in the $n(n-1)/2$-dimensional space $\mathfrak{t}_n^1$ spanned by t_{ij}.

- What is the subset $\mathcal{G}_n$ of $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$ of Gaudin subalgebras?

Questions

Gaudin subalgebras are points in the Grassmannian of $(n-1)$-planes in the $n(n-1)/2$-dimensional space $\mathfrak{t}_n^1$ spanned by t_{ij}.

- What is the subset $\mathcal{G}_n$ of $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$ of Gaudin subalgebras?
- What is the subset $\mathcal{G}_n^{\mathrm{prin}}$ of *principal* Gaudin subalgebras, the closure of the subset of $G_n(z)$, $z \in \mathrm{Conf}_n(\mathbb{C})$ in the Grassmannian?

Questions

Gaudin subalgebras are points in the Grassmannian of $(n-1)$-planes in the $n(n-1)/2$-dimensional space $\mathfrak{t}_n^1$ spanned by t_{ij}.

- What is the subset $\mathcal{G}_n$ of $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$ of Gaudin subalgebras?
- What is the subset $\mathcal{G}_n^{\mathrm{prin}}$ of *principal* Gaudin subalgebras, the closure of the subset of $G_n(z)$, $z \in \mathrm{Conf}_n(\mathbb{C})$ in the Grassmannian?
- What is the tautological bundle on $\mathcal{G}_n$, whose fibres are the Gaudin subalgebras?

Questions

Gaudin subalgebras are points in the Grassmannian of $(n-1)$-planes in the $n(n-1)/2$-dimensional space $\mathfrak{t}_n^1$ spanned by t_{ij}.

- What is the subset $\mathcal{G}_n$ of $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$ of Gaudin subalgebras?
- What is the subset $\mathcal{G}_n^{\mathrm{prin}}$ of *principal* Gaudin subalgebras, the closure of the subset of $G_n(z)$, $z \in \mathrm{Conf}_n(\mathbb{C})$ in the Grassmannian?
- What is the tautological bundle on $\mathcal{G}_n$, whose fibres are the Gaudin subalgebras?
- For each representation $\mathfrak{t}_n \to \mathrm{End}(V)$ we get commuting operators acting on V. What is their common spectrum?

1. Gaudin models
2. Gaudin subalgebras and moduli of stable curves
3. Tautological bundle and spectrum
4. Gaudin subalgebra for Coxeter systems
5. Non-principal families
6. General arrangements of hyperplanes

Moduli space of rational curves with marked points

- The affine group $\mathrm{Aff} = \{z \mapsto az + b, a \neq 0\}$ acts diagonally on $\mathrm{Conf}_n(\mathbb{C})$. Then

$$H_i(\gamma z) = a_\gamma^{-1} H_i(z), \quad \text{for all } \gamma \in \mathrm{Aff},$$

Moduli space of rational curves with marked points

- The affine group $\mathrm{Aff} = \{z \mapsto az + b, a \neq 0\}$ acts diagonally on $\mathrm{Conf}_n(\mathbb{C})$. Then

$$H_i(\gamma z) = a_\gamma^{-1} H_i(z), \quad \text{for all } \gamma \in \mathrm{Aff},$$

- Thus $G_n(z) = \mathrm{span}(H_i(z))$ is parametrized by

$$[z] \in \mathrm{Conf}_n(\mathbb{C})/\mathrm{Aff} = \mathrm{Conf}_{n+1}(\mathbb{P}^1)/SL_2(\mathbb{C}) =: M_{0,n+1}.$$

Moduli space of rational curves with marked points

- The affine group $\mathrm{Aff} = \{z \mapsto az + b, a \neq 0\}$ acts diagonally on $\mathrm{Conf}_n(\mathbb{C})$. Then

$$H_i(\gamma z) = a_\gamma^{-1} H_i(z), \quad \text{for all } \gamma \in \mathrm{Aff},$$

- Thus $G_n(z) = \mathrm{span}(H_i(z))$ is parametrized by

$$[z] \in \mathrm{Conf}_n(\mathbb{C})/\mathrm{Aff} = \mathrm{Conf}_{n+1}(\mathbb{P}^1)/SL_2(\mathbb{C}) =: M_{0,n+1}.$$

- $M_{0,n+1}$ is the moduli space classifying smooth genus zero connected projective curves with $n+1$ distinct marked points.

Moduli space of rational stable curves

- The moduli space $M_{0,n+1}$ has a Deligne–Mumford compactification $\bar{M}_{0,n+1}$. It is a smooth projective variety.

Moduli space of rational stable curves

- The moduli space $M_{0,n+1}$ has a Deligne–Mumford compactification $\bar{M}_{0,n+1}$. It is a smooth projective variety.
- Its points are in one-to-one correspondence with *stable curves*, curves with ordinary double points and no continuous automorphisms.

Moduli space of rational stable curves

- The moduli space $M_{0,n+1}$ has a Deligne–Mumford compactification $\bar{M}_{0,n+1}$. It is a smooth projective variety.
- Its points are in one-to-one correspondence with *stable curves*, curves with ordinary double points and no continuous automorphisms.
- One definition is as the closure of the image of the crossratio map

$$M_{0,n+1} \to \prod_{i,j,k,l \text{ distinct}} \mathbb{P}^1.$$

Gaudin subalgebras and stable rational curves

Theorem (L. Aguirre, G. F., A. Veselov 2011)

All Gaudin subalgebras of $\mathfrak{t}_n$ are principal. They form a smooth subvariety of $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$ isomorphic to $\bar{M}_{0,n+1}$.

Gaudin subalgebras and stable rational curves

Theorem (L. Aguirre, G. F., A. Veselov 2011)

All Gaudin subalgebras of $\mathfrak{t}_n$ are principal. They form a smooth subvariety of $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$ isomorphic to $\bar{M}_{0,n+1}$.

- Thus Gaudin subalgebras provide an embedding of $\bar{M}_{0,n+1}$ into a Grassmannian.

Gaudin subalgebras and stable rational curves

Theorem (L. Aguirre, G. F., A. Veselov 2011)

All Gaudin subalgebras of $\mathfrak{t}_n$ are principal. They form a smooth subvariety of $\mathrm{Gr}(n-1, \mathfrak{t}_n^1)$ isomorphic to $\bar{M}_{0,n+1}$.

- Thus Gaudin subalgebras provide an embedding of $\bar{M}_{0,n+1}$ into a Grassmannian.
- The proof uses a modification of a description of the image of $\bar{M}_{0,n+1}$ by explicit equations in $\prod_{ijkl} \mathbb{P}^1$, due to Gerritzen, Herrlich and van der Put.

Modified Gerritzen-Herrlich-van der Put relations

Theorem

The moduli space $\bar{M}_{0,n+1}$ is the subvariety of $\prod_{(i,j,k)} \mathbb{P}^1$ with homogeneous coordinates $(x_{ijk} : y_{ijk})$ labeled by distinct triples in $\{1, \ldots, n\}$ defined by the following equations.

- $x_{ikj} x_{ijk} = y_{ikj} y_{ijk}$ for all (i, j, k).
- $x_{jik} y_{ijk} = y_{ijk} y_{jik} - x_{ijk} y_{jik}$ for all (i, j, k) .
- $x_{ijk} x_{ikl} y_{ijl} = y_{ijk} y_{ikl} x_{ijl}$ for all (i, j, k, l) distinct .

The open subvariety $M_{0,n+1}$ is embedded via the cross ratios

$$(x_{ijk} : y_{ijk}) = ((z_i - z_k)(z_{n+1} - z_j) : (z_i - z_j)(z_{n+1} - z_k)).$$

Modified Gerritzen-Herrlich-van der Put relations

Theorem

The moduli space $\bar{M}_{0,n+1}$ is the subvariety of $\prod_{(i,j,k)} \mathbb{P}^1$ with homogeneous coordinates $(x_{ijk} : y_{ijk})$ labeled by distinct triples in $\{1, \ldots, n\}$ defined by the following equations.

- $x_{ikj}x_{ijk} = y_{ikj}y_{ijk}$ for all (i, j, k).
- $x_{jik}y_{ijk} = y_{ijk}y_{jik} - x_{ijk}y_{jik}$ for all (i, j, k) .
- $x_{ijk}x_{ikl}y_{ijl} = y_{ijk}y_{ikl}x_{ijl}$ for all (i, j, k, l) distinct .

The open subvariety $M_{0,n+1}$ is embedded via the cross ratios

$$(x_{ijk} : y_{ijk}) = ((z_i - z_k)(z_{n+1} - z_j) : (z_i - z_j)(z_{n+1} - z_k)).$$

Key observation: the projection $\mathfrak{t}_n^1 \to \mathbb{C}^3$ onto the coefficients of t_{ij}, t_{ik}, t_{jl} maps a Gaudin subalgebra onto a two-dimensional vector subspace.

Logarithmic tangent bundle

Let $D \subset X$ be a divisor in a variety X. A vector field on X is called *logarithmic* if its restriction to D is tangent to D.
Logarithmic vector fields are sections of a vector bundle $T_X(-\log D)$.
Let now X be $\bar{M}_{0,n+1}$ and D the compactification divisor $D = \bar{M}_{0,n+1} \setminus M_{0,n+1}$.

The tautological bundle

Theorem (AFV 2011)

Let G_n be the vector bundle on $X = \bar{M}_{0,n}$ whose fibre at x is the corresponding Gaudin subalgebra $G_n(x)$. Then we have an exact sequence of vector bundles

$$0 \to \mathcal{O}_X \to G_n \to T_X(-\log D) \to 0,$$

where $\mathcal{O}_X$ is the trivial bundle.

The tautological vector bundle

- The subbundle O_X is spanned by the central element

$$c = \sum_{i<j} t_{ij}$$

- The isomorphism $T_X(-\log D) \to G_n/\mathcal{O}_X$ sends a logarithmic vector field v to $\omega(v)$, where $\omega|_{M_{0,n+1}}$ is the $\mathfrak{t}_n^1/\mathbb{C}c$-valued 1-form

$$\omega = \sum_{i<j} t_{ij} \frac{dz_i - dz_j}{z_i - z_j}.$$

- G_n is isomorphic to the sheaf of first order differential operators with logarithmic symbol, twisted by a line bundle with divisor class $\sum_{S\subset\{1,2\}}[D_S]$ (the compactification divisor has irreducible components labeled by subsets $S \subset \{1, \dots, n+1\}$).

Representations, spectrum

Let $\rho\colon \mathfrak{t}_n \to \mathrm{End}(V)$ be a finite dimensional representation of the Kohno–Drinfeld Lie algebra. Assume for simplicity that $\rho(c) = 0$.
Then the eigenvalues of the commuting operators are linear forms on the fibres of the logarithmic tangent bundle = sections of the logarithmic cotangent bundle $T^*_{\log}\bar{M}_{0,n+1}$, which is a Poisson manifold.

Theorem

The spectrum is a coisotropic subscheme of $T^*_{\log}\bar{M}_{0,n+1}$, lagrangian over $M_{0,n+1}$.

Representations, spectrum

Let $\rho\colon \mathfrak{t}_n \to \mathrm{End}(V)$ be a finite dimensional representation of the Kohno–Drinfeld Lie algebra. Assume for simplicity that $\rho(c) = 0$. Then the eigenvalues of the commuting operators are linear forms on the fibres of the logarithmic tangent bundle = sections of the logarithmic cotangent bundle $T^*_{\log}\bar{M}_{0,n+1}$, which is a Poisson manifold.

Theorem

The spectrum is a coisotropic subscheme of $T^*_{\log}\bar{M}_{0,n+1}$, lagrangian over $M_{0,n+1}$.

If $\rho(c) = \alpha\,\mathrm{Id}$ then the logarithmic cotangent bundle is replaced by another Poisson manifold, the α-twisted logarithmic cotangent bundle.

Coxeter arrangements

The Kohno–Drinfeld Lie algebra is the special case of a *holonomy Lie algebra* $\mathfrak{t}_\Delta$ associated to a Coxeter systems (in fact any arrangements of hyperplanes).
A finite Coxeter group is a finite group generated by orthogonal reflections in a Euclidean space V. It is uniquely defined by the arrangement $(H_\alpha)_{\alpha\in\Delta}$ of complexified reflection hyperplanes, defined by linear forms $\alpha \in V^*$.

Holonomy Lie algebras and Gaudin hamiltonians

Kohno's holonomy Lie algebra $\mathfrak{t}_\Delta$ associated to an arrangement Δ is generated by t_α, $\alpha \in \Delta$ with a relation

$$[t_\alpha, \sum_{\beta \in W \cap \Delta} t_\beta] = 0$$

for each $\alpha \in W \subset V^*$ with $\dim W = 2$.
These relations ensure the commutativity of the Gaudin Hamiltonians

$$H(z, v) = \sum_{\alpha \in \Delta} \frac{\alpha(v)}{\alpha(z)} t_\alpha, \quad v \in V,$$

for fixed $z \in V \setminus \cup_{\alpha \in \Delta} H_\alpha$.
$G_\Delta(z) = \text{span}(H(z, v), v \in V)$, abelian Lie subalgebra of $\mathfrak{t}_\Delta$ in the span of generators.
Assume $V^* = \text{span}(\Delta)$. Then $\dim G_\Delta(z) = \dim V = r$.

Occurences of Kohno relations

- $\Delta =$ positive roots of a simple Lie algebra $\mathfrak{g}$ with root generators e_α such that $\langle e_{-\alpha}, e_\alpha \rangle = 1$. Then

$$t_\alpha = e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha$$

  obey the Kohno relations. The algebras $G_\Delta(z) \in U\mathfrak{g}$ were studied by Vinberg.
- The reflections s_α of a finite Coxeter group obey the Kohno relations. More generally $t_\alpha = k_\alpha s_\alpha$, $k \colon \Delta \to \mathbb{C}$ invariant function.
- For $\Delta = A_{n-1}, B_n$ there is a map $\mathfrak{t}_\Delta \to (Ug)^{\otimes n}$ and any Lie algebra $\mathfrak{g}$. Also for $\Delta = D_n$ and $\mathfrak{g} = \mathfrak{sl}_n$.

Gaudin subalgebras

Let $(H_\alpha)_{\alpha\in\Delta}$ be a Coxeter arrangement of rank $r = \dim V$. Assume that Δ spans V^*.

Definition

- A *Gaudin subalgebra* is an r-dimensional abelian subalgebra of $\mathfrak{t}_\Delta$ contained in the span t^1_Δ of generators.
- A *principal Gaudin subalgebra* is a Gaudin subalgebra in the closure of the family $(G_\Delta(z))_{z\in P(V\smallsetminus\cup_\alpha H_\alpha)}$ in the Grassmannian $\mathrm{Gr}(r, \mathfrak{t}^1_\Delta)$.

De Concini–Procesi compactification

The projectivized hyperplane complement $M_\Delta = P(V \setminus \cup_\alpha H_\alpha)$ has a compactification $\bar{M}_\Delta$ with normal crossing compactification divisor. It is a smooth projective variety.

Proposition

The map $i \colon M_\Delta \to \mathrm{Gr}(r, \mathfrak{t}^1_\Delta)$ $z \mapsto G_\Delta(z)$ is an embedding and extends uniquely to a map

$$\bar{i} \colon \bar{M}_\Delta \to \mathrm{Gr}(r, \mathfrak{t}^1_\Delta).$$

By definition $\bar{i}(\bar{M}_\Delta)$ is the set of principal Gaudin subalgebras.
Remark: The proposition holds for any arrangement of hyperplanes. In general $\bar{i}$ is not injective.

Results for Coxeter systems

Theorem (APV 2016)

Let Δ be a Coxeter system. Then $\bar{i}\colon \bar{M}_\Delta \to \mathrm{Gr}(r, \mathfrak{t}^1_\Delta)$ is a closed embedding. In particular the principal Gaudin subalgebras form a smooth subvariety of the Grassmannian.

Results for Coxeter systems

Theorem (APV 2016)

Let Δ be a Coxeter system. Then $\bar{i}\colon \bar{M}_\Delta \to \mathrm{Gr}(r, \mathfrak{t}^1_\Delta)$ is a closed embedding. In particular the principal Gaudin subalgebras form a smooth subvariety of the Grassmannian.

The statements about the tautological bundle and the spectrum also hold in this case.

Non-principal families: the B_n-case

- The Kohno algebra has generators r_i, t_{ij}, s_{ij} corresponding to the positive roots $\epsilon_i, \epsilon_i \pm \epsilon_j$.

Non-principal families: the B_n-case

- The Kohno algebra has generators r_i, t_{ij}, s_{ij} corresponding to the positive roots $\epsilon_i, \epsilon_i \pm \epsilon_j$.
- Additionally to the principal family, parametrized by $\bar{M}_{B_n}$, we have (at least) a second family, parametrized by $\bar{M}_{0,n+2}$.

Non-principal families: the B_n-case

- The Kohno algebra has generators r_i, t_{ij}, s_{ij} corresponding to the positive roots $\epsilon_i, \epsilon_i \pm \epsilon_j$.
- Additionally to the principal family, parametrized by $\bar{M}_{B_n}$, we have (at least) a second family, parametrized by $\bar{M}_{0,n+2}$.
- The principal family is

$$G_{B_n}(z) = \{\sum_{i=1}^{n} \frac{a_i}{z_i} + \sum_{i<j} \frac{a_i - a_j}{z_i - z_j} t_{ij} + \sum_{i<j} \frac{a_i + a_j}{z_i + z_j} s_{ij}, \quad a \in \mathbb{C}^n\}$$

The A_n-type family is

$$G_{A_n}(z) = \{\sum_{i=1}^{n} \frac{a_i}{z_i} + \sum_{i<j} \frac{a_i - a_j}{z_i - z_j}(t_{ij} + s_{ij}), \quad a \in \mathbb{C}^n\}$$

Non-principal families: the B_n-case

- The Kohno algebra has generators r_i, t_{ij}, s_{ij} corresponding to the positive roots $\epsilon_i, \epsilon_i \pm \epsilon_j$.
- Additionally to the principal family, parametrized by $\bar{M}_{B_n}$, we have (at least) a second family, parametrized by $\bar{M}_{0,n+2}$.
- The principal family is

$$G_{B_n}(z) = \{\sum_{i=1}^{n} \frac{a_i}{z_i} + \sum_{i<j} \frac{a_i - a_j}{z_i - z_j} t_{ij} + \sum_{i<j} \frac{a_i + a_j}{z_i + z_j} s_{ij}, \quad a \in \mathbb{C}^n\}$$

The A_n-type family is

$$G_{A_n}(z) = \{\sum_{i=1}^{n} \frac{a_i}{z_i} + \sum_{i<j} \frac{a_i - a_j}{z_i - z_j} (t_{ij} + s_{ij}), \quad a \in \mathbb{C}^n\}$$

- For B_2, Gaudin subalgebras form a $\mathbb{P}^2 \subset \mathrm{Gr}(2,4)$, and the two families are curves of degree 1 and 2.
- For B_3 Gaudin subalgebras form a singular variety with eight non-singular irreducible components of dimension 2,2,2,2,2,1,1,1.

General arrangements: maximality

An arrangement of hyperplanes $H = \cup_{\alpha \in \Delta} H_\alpha \subset V$ is called fibre-type if either

$H = \{0\} \subset V$, $\dim(V) = 1$.

or

There is a linear projection $V \setminus H \to V' \setminus H'$ onto a fibre-type arrangement, which is a fibre bundle with fibre $\cong \mathbb{C} \setminus$ finitely many points.

Theorem (Aguirre 2014)

Let $\Delta \subset V$ span V and be fibre-type. Then the maximal dimension of abelian subalgebras of $\mathfrak{t}_\Delta$ contained in $\mathfrak{t}^1_\Delta$ is $\dim(V)$.

In the case of Coxeter systems this happen only for $A_n, B_n, I_2(n)$.

Principal Gaudin subalgebras for general arrangements

- Recall: principal Gaudin subalgebras form the image of a regular map

$$\bar{i}_\Delta \colon \bar{M}_\Delta \to \mathrm{Gr}(r, \mathfrak{t}_\Delta).$$

When is $\bar{i}$ a closed embedding?

Principal Gaudin subalgebras for general arrangements

- Recall: principal Gaudin subalgebras form the image of a regular map

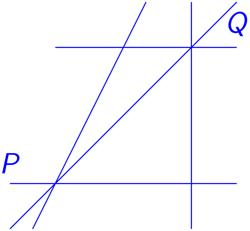
$$\bar{i}_\Delta \colon \bar{M}_\Delta \to \mathrm{Gr}(r, \mathfrak{t}_\Delta).$$

  When is $\bar{i}$ a closed embedding?

- There is a geometric necessary and sufficient condition. To formulate it it is convenient to use the language of matroid.

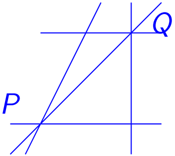
A counterexample

- Let V be a three dimensional vector space with basis $\alpha_1, \alpha_2, \alpha_3$, $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3\}$, with $\beta_2 = \alpha_1 + \alpha_2$, $\beta_3 = \alpha_1 + \alpha_3$.



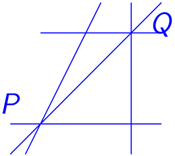
The projectivized arrangement of hyperplanes in three-dimensional space

A counterexample



- Principal Gaudin subalgebras are parametrized by $\mathbb{P}^1 \times \mathbb{P}^1$
  $(x : y, u : v) \mapsto \mathrm{span}(c_\Delta, xt_{\alpha_2} + yt_{\beta_2}, ut_{\alpha_3} + vt_{\beta_3})$.

A counterexample



- Principal Gaudin subalgebras are parametrized by $\mathbb{P}^1 \times \mathbb{P}^1$ $(x : y, u : v) \mapsto \mathrm{span}(c_\Delta, xt_{\alpha_2} + yt_{\beta_2}, ut_{\alpha_3} + vt_{\beta_3})$.
- The wonderful compactification is the blow-up $\widehat{\mathbb{P}^2}_{P,Q} \to \mathbb{P}^2$ at the two points P, Q where three lines meet. Then

$$\bar{i}_\Delta \colon \bar{M}_\Delta \to \mathcal{G}_\Delta \subset \mathrm{Gr}(3, \mathfrak{t}^1_\Delta)$$

is the blow-down map

$$\widehat{\mathbb{P}^2}_{P,Q} \to \mathbb{P}^1 \times \mathbb{P}^1$$

of the proper transform of the line PQ.

Aguirre's criterion

Let $\Delta \subset V$ a finite set of nonzero non pairwise collinear vectors spanning V.

Aguirre's criterion

Let $\Delta \subset V$ a finite set of nonzero non pairwise collinear vectors spanning V.

- A subset A of Δ is called a *flat* if it contains all the vectors of Δ in its linear span. The *rank* of a flat is the dimension of its span.

Aguirre's criterion

Let $\Delta \subset V$ a finite set of nonzero non pairwise collinear vectors spanning V.

- A subset A of Δ is called a *flat* if it contains all the vectors of Δ in its linear span. The *rank* of a flat is the dimension of its span.
- A flat is called *irreducible* if its span is not a nontrivial direct sum of the span of subflats.

Aguirre's criterion

Let $\Delta \subset V$ a finite set of nonzero non pairwise collinear vectors spanning V.

- A subset A of Δ is called a *flat* if it contains all the vectors of Δ in its linear span. The *rank* of a flat is the dimension of its span.
- A flat is called *irreducible* if its span is not a nontrivial direct sum of the span of subflats.
- A subspace W of the span of a flat F is called "dense" if it is not contained in the span of a proper subflat.

Aguirre's criterion

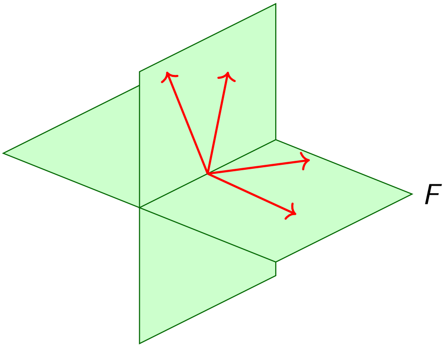
Let $\Delta \subset V$ a finite set of nonzero non pairwise collinear vectors spanning V.

- A subset A of Δ is called a *flat* if it contains all the vectors of Δ in its linear span. The *rank* of a flat is the dimension of its span.
- A flat is called *irreducible* if its span is not a nontrivial direct sum of the span of subflats.
- A subspace W of the span of a flat F is called "dense" if it is not contained in the span of a proper subflat.

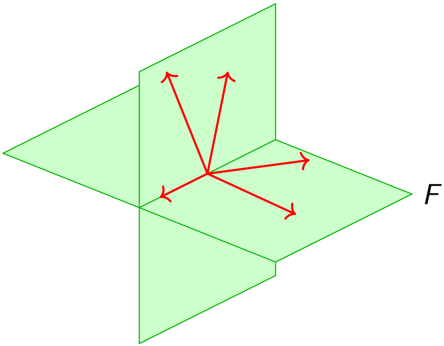
Theorem (Aguirre)

$\bar{i}_\Delta$ is a closed embedding if and only if for every irreducible flat A and every subflat $F \subset A$ of corank 1, $\mathrm{span}(F) \cap \mathrm{span}(A \setminus F)$ is dense in F.

Aguirre's criterion



Four vectors in generic position in 3D, $\bar{i}$ injective



Five vectors, two sets of three coplanar, $\bar{i}$ not injective

Summary

- Principal Gaudin subalgebras for Coxeter arrangements form smooth projective varieties isomorphic to wonderful compactifications of the projectivised hyperplane complement.
- The only Coxeter system for which all Gaudin subalgebras are principal is A_n.

Summary

- Principal Gaudin subalgebras for Coxeter arrangements form smooth projective varieties isomorphic to wonderful compactifications of the projectivised hyperplane complement.
- The only Coxeter system for which all Gaudin subalgebras are principal is A_n.
- For $A_r, B_r, I_2(n)$ (fibre-type Coxeter arrangements) there are no abelian subalgebras of dimension $> r$.

Summary

- Principal Gaudin subalgebras for Coxeter arrangements form smooth projective varieties isomorphic to wonderful compactifications of the projectivised hyperplane complement.
- The only Coxeter system for which all Gaudin subalgebras are principal is A_n.
- For $A_r, B_r, I_2(n)$ (fibre-type Coxeter arrangements) there are no abelian subalgebras of dimension $> r$.
- One can characterize arrangements so that $\bar{i}$ is a closed embedding. They include the Coxeter arrangements and the generic arrangements.

The end