DUAL HAMILTONIAN STRUCTURES IN AN INTEGRABLE HIERARCHY

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Somes references and collaborators

Based on joint work with J. Avan, A. Doikou and A. Kundu

- *Lagrangian and Hamiltonian structures in an integrable hierarchy and space-time duality*, Nucl. Phys. B902 (2016), 415-439.
- *Multisymplectic approach to integrable defects in the sine-Gordon model*, J. Phys. A 48 (2015) 195203.
- *A multisymplectic approach to defects in integrable classical field theory*, JHEP 02 (2015), 088

and some ongoing work with A. Fordy.

Plan

1. The fundamentals: classical and quantum R matrix
   - The general scheme
   - Tracing the origin of the classical r-matrix
2. Some new observations on the classical r-matrix
   - New input from covariant field theory
   - Poisson brackets for the "time" Lax matrix
3. Why the dual picture?
   - Motivation: integrable defects
   - The bigger picture: initial-boundary value problems
4. Speculations, outlook, quantum case

1.1 Classical and quantum R matrix: general scheme

Quantum		*Classical*
YBE		*YBE*
$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$	$\xrightarrow{R=1+\hbar r}$	$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$

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Lax matrix		Lax matrix
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ultralocality

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Monodromy matrix		Monodromy matrix
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Classical discrete		*Classical continuous*
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- With all explicit dependences, it reads

$$\{U_1(x,\lambda), U_2(y,\mu)\} = \delta(x-y)[r_{12}(\lambda-\mu), U_1(x,\lambda)+U_2(y,\mu)]$$

where $U_1 = U \otimes \mathbb{1}$, $U_2 = \mathbb{1} \otimes U$ and (rational case)

$$r_{12}(\lambda) = g\frac{P_{12}}{\lambda}\,, \quad P_{12} \text{ permutation}\,, \quad g \text{ constant}$$

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- Ultralocal Poisson algebra for the entries of the matrix U.

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Example: nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} - 2g|q|^2 q = 0$$

one imposes *equal time* Poisson brackets (at $t = 0$)

$$\{q(x), q^*(y)\} = i\delta(x - y)$$

so that one can write NLS as a Hamiltonian system

$$q_t = \{H_{NLS}, q\}\,, \quad H_{NLS} = \int \left(|q_x|^2 + g|q|^4\right)\, dx$$

[Zakharov, Manakov '74]

1.2 Tracing the origin of the classical r-matrix

On the other hand, NLS as a PDE is obtained as the compatibility condition of the auxiliary problem

$$\begin{cases} \Psi_x = U\,\Psi \\ \Psi_t = V\,\Psi \end{cases}$$

i.e. the zero curvature condition

$$U_t - V_x + [U, V] = 0$$

with Lax pair

$$U(x,\lambda) = \begin{pmatrix} -i\lambda & q(x) \\ g\,q^*(x) & i\lambda \end{pmatrix}, \quad V = \begin{pmatrix} -2i\lambda^2 + i|q|^2 & 2\lambda q + iq_x \\ 2\lambda g q^* - igq^*_x & 2i\lambda^2 - i|q|^2 \end{pmatrix}$$

[Zakharov, Shabat '71]

1.2 Tracing the origin of the classical r-matrix

Then, the canonical Poisson brackets on the fields

$$\{q(x), q^*(y)\} = i\delta(x - y)$$

are equivalent to the ultralocal Poisson algebra for U

$$\{U_1(x,\lambda), U_2(y,\lambda)\} = \delta(x-y)[r_{12}(\lambda-\mu), U_1(x,\lambda)+U_2(y,\mu)]$$

[Sklyanin '79]

1.2 Tracing the origin of the classical r-matrix

Continue the reasoning: but then, what is the origin of the canonical PB for the fields

$$\{q(x), q^*(y)\} = i\delta(x - y),$$

source of all the rest of the approach?

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For NLS

$$\mathcal{L}_{NLS} = \frac{i}{2}(q^* q_t - q q_t^*) - q_x^* q_x - g(q^* q)^2$$

Then

$$\pi = \frac{\partial \mathcal{L}_{NLS}}{\partial q_t} = \frac{i}{2} q^* \,, \quad \pi^* = \frac{\partial \mathcal{L}_{NLS}}{\partial q_t^*} = -\frac{i}{2} q$$

and one requires

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- This yields the known brackets

$$\{q(x), q^*(y)\} = i\delta(x-y)$$

(NB: Dirac procedure must be used).

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Summary: taking a step back

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$$\{q(x), q^*(y)\} = i\delta(x-y)$$

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• Restore symmetry between independent variables by introducing "the other half" of Legendre transform

$$\Pi(t) = \frac{\partial \mathcal{L}_{NLS}}{\partial q_x(t)}\,, \quad \Pi^*(t) = \frac{\partial \mathcal{L}_{NLS}}{\partial q_x^*(t)}$$

[De Donder '35; Weyl '35]

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$$\{\Pi(t), q(\tau)\}_T = \delta(t-\tau)\,, \quad \{\Pi^*(t), q^*(\tau)\}_T = \delta(t-\tau)$$

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- Then proceed as before with canonical prescription for Poisson brackets between fields and momenta.
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$$\{\Pi(t), q(\tau)\}_T = \delta(t-\tau)\,, \quad \{\Pi^*(t), q^*(\tau)\}_T = \delta(t-\tau)$$

- In our case

$$\{q(t), q_x^*(\tau)\}_T = \delta(t-\tau)\,, \quad \{q^*(t), q_x(\tau)\}_T = \delta(t-\tau)$$

Remark: $\{\ ,\ \}_T$ together with standard bracket $\{\ ,\ \}_S$ do NOT form a bi-Hamiltonian structure! [Magri '78]

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• NLS consistently recovered from Hamilton equations with respect to x

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• In geometrical terms, the vector field ∂_x is Hamiltonian with respect to the new PB $\{\ ,\ \}_T$, with Hamiltonian

$$H_T = \int (\frac{i}{2}(qq_t^* - q^* q_t) - q_x^* q_x + g(q^* q)^2)\, dt$$

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$iq_t + q_{xx} - 2g\|q\|^2 q = 0$	$iq_t + q_{xx} - 2g\|q\|^2 q = 0$
$\Updownarrow$	$\Updownarrow$
$q_t = \{H_S, q\}_S$	$q_x = \{H_T, q\}_T \,, \quad (q_x)_x = \{H_T, q_x\}_T$
$H_S = \int \mathcal{H}_S \, dx$	$H_T = \int \mathcal{H}_T \, dt$

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• Can repeat the established procedure w.r.t. $\{\ ,\ \}_T$ with interesting consequences:

1. Time Lax matrix V has same ultralocal Poisson algebra as standard Lax matrix U

$$\{V_1(t,\lambda), V_2(\tau,\mu)\}_T = -\delta(t-\tau)[r_{12}(\lambda-\mu), V_1(t,\lambda)+V_2(\tau,\mu)]$$

2.2 Dual Hamiltonian picture

2. Standard construction:
Lax matrix $\rightarrow$ transition matrix $\rightarrow$ monodromy matrix go over completely into dual formulation:

$$U(x,\lambda) \mapsto \mathcal{T}_S(x,y,\lambda) = \mathcal{P}_S e^{\int_y^x U(z,\lambda)dz} \mapsto \mathcal{T}_S(\lambda)$$

$$V(t,\lambda) \mapsto \mathcal{T}_T(t,\tau,\lambda) = \mathcal{P}_T e^{\int_\tau^t V(s,\lambda)ds} \mapsto \mathcal{T}_T(\lambda)$$

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3. Liouville integrability established in dual picture. Infinite sequences of charges conserved in space and in involution w.r.t. $\{\ ,\ \}_T$.
4. Conclusion: we have two ways of tackling Liouville integrability for classical field theories.

2.2 Dual Hamiltonian picture

First open questions

• How do we fit the new brackets and the associated Poisson algebras into the well established theory of Poisson-Lie group?

• Standard equal-time picture used for canonical quantization. What does the dual equal-space picture translate into at the quantum level?

• Can we devise a covariant quantization of integrable field theories in the sense of multisymplectic field theory?

3.1 Why the dual picture?

• How did we come to these observations and what did they achieve?

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• Original motivation: understand *Liouville* integrability of classical field theories with a defect. [Bowcock, Corrigan, Zambon '03]

• Integrability well understood from a Lax pair/PDE point of view: generating functions of conserved charges with defect are known. [V.C '07]

• Major problem for Liouville integrability: the defect is modeled by internal boundary conditions at some point $x = x_0$.

3.1 Why the dual picture?

• Consequence: any attempt based on the construction of a monodromy matrix with defect of the form

$$\mathcal{T}(\lambda) = \mathcal{T}^{+}(\lambda) D_{x_0}(\lambda) \mathcal{T}^{-}(\lambda)$$

faces the challenge of making sense of

$$\{D_{x_0}(\lambda), D_{x_0}(\mu)\}_S \quad \text{and} \quad \{D_{x_0}(\lambda), \mathcal{T}^{\pm}(\mu)\}_S$$

• Involves PB of canonical fields at the same space point: "$\delta(0)$" divergence!

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2. Turn the argument around: impose that $D_{x_0}(\lambda)$ be in a representation of desired Poisson algebra

$$\{D_{x_0 1}(\lambda), D_{x_0 2}(\mu)\}_S = [r_{12}, D_{x_0 1}(\lambda) D_{x_0 2}(\mu)]$$

with unknown fields sitting at the defect. Extra fields couple dynamically to bulk field at x_0 (gluing conditions). [Avan, Doikou '11]

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with unknown fields sitting at the defect. Extra fields couple dynamically to bulk field at x_0 (gluing conditions). [Avan, Doikou '11]

$\rightarrow$ Solve the particular problem in their own right but very hard to reconcile with Lax pair integrability obtained before.

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• Reasoning:
– Without defect: two different but equivalent ways of establishing Liouville integrability.

– With defect: standard approach fails but the dual approach applies without a problem thanks to swapping of x and t.

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• Reasoning:
– Without defect: two different but equivalent ways of establishing Liouville integrability.

– With defect: standard approach fails but the dual approach applies without a problem thanks to swapping of x and t.

• Bonus: integrable defect conditions (frozen Bäcklund transformations) naturally incorporated as canonical transformations w.r.t to new bracket $\{\ ,\ \}_T$.

• Liouville integrability with certain defects reconciled with Lax pair formulation without gluing conditions.

3.2 Why? The bigger picture

Initial value problem vs initial-boundary value problem

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• Standard approach based on monodromy matrix associated only to space Lax matrix U is optimized for so-called "initial" value problems.

• Completely natural since the original motivation was to perform canonical quantization of classical integrable field theories.

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Initial value problem vs initial-boundary value problem

• Terminology problem: never really an initial-value problem.

• Standard approach based on monodromy matrix associated only to space Lax matrix U is optimized for so-called "initial" value problems.

• Completely natural since the original motivation was to perform canonical quantization of classical integrable field theories.

• Gives the *illusion* that time Lax matrix V plays no role. BUT! Works well only because one chooses *nice* boundary conditions: periodic, fast decay, open (a la Sklyanin).

3.2 Why? The bigger picture

- This is what allows us to "discard" V from general time evolution equation of transition matrix

$$\partial_t \mathcal{T}(x,y,\lambda) = V(x,\lambda)\mathcal{T}(x,y,\lambda) - \mathcal{T}(x,y,\lambda)V(y,\lambda)$$

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- This is what allows us to "discard" V from general time evolution equation of transition matrix

$$\partial_t \mathcal{T}(x,y,\lambda) = V(x,\lambda)\mathcal{T}(x,y,\lambda) - \mathcal{T}(x,y,\lambda)V(y,\lambda)$$

- Example: NLS with fast decay boundary conditions as $|x| \to \infty$, this implies

$$\partial_t \mathcal{T}(\lambda) = i\lambda^2[\sigma_3, \mathcal{T}(\lambda)]$$

Hence, the crucial result:

$$\partial_t \mathrm{Tr}\mathcal{T}(\lambda) = 0$$

- Last result also holds for periodic boundary conditions for instance.

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In short

Speaking of integrability without reference to initial AND boundary data is meaningless, *even* in so-called initial-value problem.

- Hence, one always has to deal with space and time symmetrically. Dual Hamiltonian approach restores the balance at Hamiltonian level.

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• Potential application: can we revisit Sklyanin's prescription for integrable boundary conditions from dual point of view?

$\rightarrow$ gain freedom on allowed boundary conditions by giving away some freedom on initial conditions

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• Ideally, set up a theory of integrable *initial-boundary* conditions: Hamiltonian counterpart of linearizable initial-boundary conditions generalizing Fokas approach to initial-boundary value problems [V.C. '15]

4. Speculations, outlook, quantum case

• Towards time-dependent open integrable systems via dual picture: out-of-equilibrium integrable systems?

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• Towards time-dependent open integrable systems via dual picture: out-of-equilibrium integrable systems?

→ requires an understanding of the combination of the two Poisson structures $\{\ ,\ \}_S$ and $\{\ ,\ \}_T$:

"Covariant Poisson-Lie theory"?

→ Then, understand covariant quantization:
– Role of time Lax matrix V at quantum level?
– Understand how the canonical quantization of r and $\{\ ,\ \}_S$ into R and quantum group structures can incorporate $\{\ ,\ \}_T$

4. Speculations, outlook, quantum case

Questions to the specialists in the audience:

1. Is it meaningful/interesting to consider quantized time Lax matrices (IQFT)? What about spin chains (time-independent)?

2. Anyone aware of works related to covariant integrable systems, classical or quantum?

References

THANK YOU!

- J. Avan, V. Caudrelier, A. Doikou, A. Kundu, *Lagrangian and Hamiltonian structures in an integrable hierarchy and space-time duality*, Nucl. Phys. B902 (2016), 415-439.
- V. Caudrelier, *Multisymplectic approach to integrable defects in the sine-Gordon model*, J. Phys. A48 (2015) 195203.
- V. Caudrelier, A. Kundu, *A multisymplectic approach to defects in integrable classical field theory*, JHEP 02 (2015), 088
- V. Caudrelier, *On the inverse scattering method for integrable PDEs on a star graph*, Comm. Math. Phys 338 (2015), 893