Bethe states of quantum integrable models solved via off-diagonal Bethe ansatz

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Outline

I. Introduction

II. Retrieve the eigenstates of the models solved by off-diagonal Bethe ansatz

- Eigenstates of XXZ spin chain with antiperiodic boundary condition
- Eigenstates of the trigonometric su(n) spin chain with antiperiodic boundary condition

III. Conclusion & perspective

Introduction

Quantum integrable systems:

1. interacting particles with δ-function
2. spin chain and spin ladder
3. Hubbard, supersymmetry t-J, Kondo
4. τ_2, Chiral Potts, vertex,
5. long range interaction ($1/r$, $1/r^2$)

Methods:

1. coordinate Bethe ansatz
2. algebraic Bethe ansatz or quantum inverse scattering method
3. T-Q relation
4. others: functional Bethe ansatz, asymptotic Bethe ansatz,…

Besides the integrable models with U(1) symmetry, there exist some integrable models without U(1) symmetry.

Examples of U(1) symmetry-broken integrable models:

1. non-diagonal boundary problems
2. anti-periodic boundary conditions
3. XYZ model with odd sites number

Due to the U(1) symmetry-broken, there is no obvious reference state.

Traditional Bethe ansatz does not work.

Although the model has been proved to be integrable, the exact solutions are difficult to obtain.

- q-Onsager algebra method

  P. Baseilhac, Nucl. Phys. B 754 (2006) 309;
  P. Baseilhac and K. Koizumi, J. Stat. Mech. (2007) P09006;
  P. Baseilhac and S. Belliard, Lett. Math. Phys. 93 (2010) 213.

- separation of variables (SoV) method

  H. Frahm, A. Seel and T. Wirth, Nucl. Phys. B 802 (2008) 351;
  H. Frahm, J.H. Grelik, A. Seel and T. Wirth, J. Phys. A 44 (2011) 015001;
  S. Niekamp, T. Wirth and H. Frahm, J. Phys. A 42 (2009) 195008;
  G. Niccoli, J. Stat. Mech. (2012) P10025;
  G. Niccoli, Nucl. Phys. B 870 (2013) 397;
  G. Niccoli, J. Phys. A 46 (2013) 075003;
  N. Kitanine, J.-M. Maillet and G. Niccoli, J. Stat. Mech. (2014) P05015.

- modified algebraic Bethe ansatz method

  S. Belliard and N. Cramp’e, SIGMA 9 (2013) 072;
  S. Belliard, Nucl. Phys. B 892 (2015) 1;
  S. Belliard and R.A. Pimenta, Nucl. Phys. B 894 (2015) 527;
  J. Avan, S. Belliard, N. Grosjean and R.A. Pimenta, Nucl. Phys. B 899 (2015) 229.

- off-diagonal Bethe ansatz

The central idea of Off-D Bethe ansatz

Yang-Baxter equation and reflection equation

$$\Lambda^{(p)}(\theta_j)\Lambda^{(p)}(\theta_j-\eta)=a(\theta_j)d(\theta_j-\eta),\quad j=1,\ldots,N.$$

$$\Lambda(u)=2f^N(u)+\cdots$$

Asymptotic behavior

Inhomogeneous T-Q relation

$$\Lambda(u)=a(u)\frac{Q_1(u-\eta)}{Q_2(u)}+d(u)\frac{Q_2(u+\eta)}{Q_1(u)}+c(u)\underline{\frac{a(u)d(u)}{Q_1(u)Q_2(u)}}$$

Regularity → Bethe ansatz equations

→ Completeness

→ Eigenstates

Framework of Off-D Bethe ansatz

Off-diagonal Bethe ansatz	[PRL 111, 137201 (2013); NPB 875, 152 (2013); NPB 877, 152 (2013); NPB 886, 185 (2014)]
Thermodynamics ODBA	[NPB 884, 17 (2014)]
Nested ODBA	[JHEP 04, 143 (2014)]]
Hierarchic ODBA	[JHEP 02, 036 (2015)]
Beyond the A_n Lie algebra	[JHEP 06, 128 (2014)]
Retrieve eigenstates	[NPB 893, 70 (2015); JSTAT P05014 (2015); JPA 49, 014001 (2016); JHEP 05, 119 (2016)]

Eigenstates of spin torus

Hamiltonian

$$H = -\sum_{n=1}^{N}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \cosh\eta\,\sigma_n^z\sigma_{n+1}^z)$$

anti-periodic boundary condition

$$\sigma_{N+1}^x = \sigma_1^x, \quad \sigma_{N+1}^y = -\sigma_1^y, \quad \sigma_{N+1}^z = -\sigma_1^z$$

Transfer matrix

$$t(u) = tr_0(\sigma_0^x T_0(u)) = \bar{B}(u) + \bar{C}(u)$$

$$H = -2\sinh\eta \left\{ \frac{\partial \ln t(u)}{\partial u}\bigg|_{u=0,\{\theta_j=0\}} - \frac{1}{2} N \coth\eta \right\}$$

Monodromy matrix

$$\begin{aligned} T_0(u) &= \bar{R}_{0N}(u-\theta_N)\bar{R}_{0\,N-1}(u-\theta_{N-1})\cdots\bar{R}_{01}(u-\theta_1) \\ &= \begin{pmatrix} \bar{A}(u) & \bar{B}(u) \\ \bar{C}(u) & \bar{D}(u) \end{pmatrix}. \end{aligned}$$

R-matrix

$$\bar{R}(u) = \frac{1}{\sinh\eta} \begin{pmatrix} \sinh(u+\eta) & & & \\ & \sinh u & \sinh\eta & \\ & \sinh\eta & \sinh u & \\ & & & \sinh(u+\eta) \end{pmatrix}$$

Functional relations

$$\Lambda(u+i\pi) = (-1)^{N-1}\Lambda(u),$$

$\Lambda(u)$, as function of u, is a trigonometrical polynomial of degree $N-1$,

$$\Lambda(\theta_j)\,\Lambda(\theta_j-\eta) = -\bar{a}(\theta_j)\bar{d}(\theta_j-\eta), \quad j=1,\cdots,N.$$

$$\bar{a}(u) \;=\; \prod_{l=1}^{N}\sinh(u-\theta_l+\eta)$$

$$\bar{d}(u) \;=\; \bar{a}(u-\eta)$$

T-Q relation

$$\Lambda(u) = \bar{a}(u)e^{u}\frac{Q(u-\eta)}{Q(u)} - e^{-u-\eta}\bar{d}(u)\frac{Q(u+\eta)}{Q(u)} - c(u)\frac{\bar{a}(u)\bar{d}(u)}{Q(u)}$$

$$Q(u) = \prod_{j=1}^{N}\frac{\sinh(u-\lambda_j)}{\sinh\eta}$$

$$c(u) = e^{u-N\eta+\sum_{j=1}^{N}(\theta_j-\lambda_j)} - e^{-u-\eta-\sum_{j=1}^{N}(\theta_j-\lambda_j)}$$

Bethe ansatz equations

$$e^{\lambda_j}\bar{a}(\lambda_j)Q(\lambda_j-\eta) - \bar{d}(\lambda_j)e^{-\lambda_j-\eta}Q(\lambda_j+\eta) - c(\lambda_j)\bar{a}(\lambda_j)\bar{d}(\lambda_j) = 0,$$

$$j = 1, \cdots, N.$$

Orthogonal basis of the Hilbert space:

$$\langle \theta_{p_1}, \cdots, \theta_{p_n}| \;=\; \langle 0| \prod_{j=1}^{n} \bar{C}(\theta_{p_j}), \qquad 1 \le p_1 < p_2 < \cdots < p_n \le N,$$

$$|\theta_{q_1}, \cdots, \theta_{q_n}\rangle \;=\; \prod_{j=1}^{n} \bar{B}(\theta_{q_j})|0\rangle, \qquad 1 \le q_1 < q_2 < \cdots < q_n \le N.$$

$$|0\rangle = \otimes_{j=1}^{N} |\uparrow\rangle_j, \quad \langle 0| = \langle \uparrow |_j \otimes_{j=1}^{N},$$

From the commutation relations, we know that above states are the eigenstates of operator D bar

$$\bar{D}(u)|\theta_{p_1}, \cdots, \theta_{p_n}\rangle = \bar{d}(u) \prod_{j=1}^{n} \frac{\sinh(u - \theta_{p_j} + \eta)}{\sinh(u - \theta_{p_j})} |\theta_{p_1}, \cdots, \theta_{p_n}\rangle,$$

$$\langle \theta_{p_1}, \cdots, \theta_{p_n}|\bar{D}(u) = \bar{d}(u) \prod_{j=1}^{n} \frac{\sinh(u - \theta_{p_j} + \eta)}{\sinh(u - \theta_{p_j})} \langle \theta_{p_1}, \cdots, \theta_{p_n}|.$$

Total number of the right (or left) states & completeness

$$\sum_{n=0}^{N} \frac{N!}{(N-n)!n!} = 2^N.$$

Orthogonal relations

$$\langle \theta_{p_1}, \cdots, \theta_{p_n} | \theta_{q_1}, \cdots, \theta_{q_m} \rangle = f_n(\theta_{p_1}, \cdots, \theta_{p_n})\, \delta_{m,n} \prod_{j=1}^{n} \delta_{p_j, q_j},$$

$$f_n(\theta_{p_1}, \cdots, \theta_{p_n}) = \prod_{l=1}^{n} \left\{ \bar{a}(\theta_{p_l}) \bar{d}_{p_l}(\theta_{p_l}) \prod_{k \neq l}^{n} \frac{\sinh(\theta_{p_l} - \theta_{p_k} + \eta)}{\sinh(\theta_{p_l} - \theta_{p_k})} \right\}$$

$$f_0 = \langle 0|0 \rangle = 1 \qquad \bar{d}_l(u) = \prod_{j \neq l}^{N} \frac{\sinh(u - \theta_j)}{\sinh \eta}$$

Thus these right (or left) states form an orthogonal right (or left) basis of the Hilbert space,

and the eigenstates of the system can be decomposed as a unique linear combination of these basis.

Retrieving the Bethe states

Due to the fact that the left states

$$\{\langle\theta_{p_1},\cdots,\theta_{p_n}|\,|n=0,\cdots,N,\quad 1\leqslant p_1<p_2<\cdots<p_n\leqslant N\}$$

form a basis of the dual Hilbert space, an eigenstate |Ψ> of the transfer matrix is completely determined (up to an overall scalar factor) by the following set of scalar products

$$F_n(\theta_{p_1},\cdots,\theta_{p_n})=\langle\theta_{p_1},\cdots,\theta_{p_n}|\Psi\rangle,\quad n=0,\cdots,N,$$

$$F_0=1$$

Direct calculation shows

$$F_n(\theta_{p_1},\cdots,\theta_{p_n})=\prod_{l=1}^{n}\Lambda(\theta_{p_l}),$$

Write the eigenstates as the form of Bethe state

$$|\lambda_1, \cdots, \lambda_N\rangle = \prod_{j=1}^{N} \frac{\bar{D}(\lambda_j)}{\bar{d}(\lambda_j)} \, |\Omega; \{\theta_j\}\rangle$$

$\{\lambda_j\}$ are the Bethe roots, $|\Omega>$ is the reference state to be determined so that the scalar products between Bethe state and basis satisfy the conditions

$$\langle\theta_{p_1}, \cdots, \theta_{p_n}|\lambda_1, \cdots, \lambda_N\rangle = \prod_{l=1}^{n} \Lambda(\theta_{p_l})$$

From

$$\Lambda(\theta_j) = \bar{a}(\theta_j)\, e^{\theta_j} \frac{Q(\theta_j - \eta)}{Q(\theta_j)}$$

we obtain the following requirements on the reference state

$$\langle\theta_{p_1}, \cdots, \theta_{p_n}|\Omega; \{\theta_j\}\rangle = \prod_{l=1}^{n} a(\theta_{p_l}) e^{\theta_{p_l}}$$

The solution of the reference state: the q-deformed coherence state

$$|\Omega;\{\theta_j\}\rangle=\sum_{l=0}^{\infty}\frac{\left(\tilde{B}^-\right)^l}{[l]_q!}|0\rangle=\sum_{l=0}^{N}\frac{\left(\tilde{B}^-\right)^l}{[l]_q!}|0\rangle$$

where

$$[l]_q=\frac{1-q^{2l}}{1-q^2},\quad [0]_q=1,$$

$$[l]_q!=[l]_q\,[l-1]_q\cdots[1]_q,\quad q=e^{\eta},$$

$$\tilde{B}^-=\lim_{u\to+\infty}\left\{\left(2\sinh\eta\, e^{-u}\right)^{N-1}e^{\sum_{l=1}^{N}\theta_l}\,\bar{B}(u)\right\}$$

$$\tilde{B}^-=\sum_{l=1}^{N}e^{\theta_l+\frac{(N-1)\eta}{2}}\,e^{\frac{\eta}{2}\sum_{k=l+1}^{N}\sigma_k^z}\,\sigma_l^-\,e^{-\frac{\eta}{2}\sum_{k=1}^{l-1}\sigma_k^z}$$

Then we conclude that the Bethe state with the corresponding reference state is an eigenstate of the transfer matrix provided that the Bethe roots satisfy the associated Bethe ansatz equations.

Homogeneous limit

$$|\Omega\rangle = \lim_{\{\theta_j \to 0\}} |\Omega; \{\theta_j\}\rangle = \sum_{l=0}^{\infty} \frac{(B^-)^l}{[l]_q!} |0\rangle = \sum_{l=0}^{N} \frac{(B^-)^l}{[l]_q!} |0\rangle$$

$$B^- = \lim_{\{\theta_j \to 0\}} \tilde{B}^- = \sum_{l=1}^{N} e^{\frac{(N-1)\eta}{2}} e^{\frac{\eta}{2}\sum_{k=l+1}^{N}\sigma_k^z} \sigma_l^- e^{-\frac{\eta}{2}\sum_{k=1}^{l-1}\sigma_k^z}$$

Therefore, the homogeneous limit of the Bethe state gives rise to the eigenstate of the corresponding homogeneous transfer matrix.

In contrast to that used in the algebraic Bethe ansatz scheme, the reference state is no longer a pure product state but a highly entangled state (actually a q-spin coherent state).

Eigenstates of the trigonometric su(3) spin chain with antiperiodic boundary condition

$$R(u)=\left(\begin{array}{ccc|ccc|ccc}
\bar a(u) & & & & & & & & \\
 & \bar b(u) & & \bar c(u) & & & & & \\
 & & \bar b(u) & & & & \bar d(u) & & \\
\hline
 & \bar d(u) & & \bar b(u) & & & & & \\
 & & & & \bar a(u) & & & & \\
 & & & & & \bar b(u) & & \bar c(u) & \\
\hline
 & & \bar c(u) & & & & \bar b(u) & & \\
 & & & & & \bar d(u) & & \bar b(u) & \\
 & & & & & & & & \bar a(u)
\end{array}\right)
\qquad
\begin{aligned}
\bar a(u) &= \sinh(u+\eta)\,,\\
\bar c(u) &= e^{\frac{u}{3}}\sinh\eta\,,\\
\bar b(u) &= \sinh u\,,\\
\bar d(u) &= e^{-\frac{u}{3}}\sinh\eta
\end{aligned}$$

Initial condition : $\qquad R_{12}(0)=\sinh\eta P_{1,2},$

Unitarity : $\qquad R_{12}(u)R_{21}(-u)=\rho_1(u)\times \mathrm{id},\quad \rho_1(u)=-\sinh(u+\eta)\sinh(u-\eta),$

Crossing-unitarity : $R_{12}^{t_1}(u)R_{21}^{t_1}(-u-n\eta)=\rho_2(u)\times \mathrm{id},\ \rho_2(u)=-\sinh u\sinh(u+n\eta),$

Fusion conditions : $\qquad R_{12}(-\eta)=-2\sinh\eta P^{(-)}_{1,2}.$

twist matrix

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and } g^3 = 1$$

the R-matrix is invariant with g, that is to say, the R-matrix and g-matrix satisfy the Yang-Baxter equation

$$g_0\, g_{0'} R_{00'}(u)\, g_0^{-1}\, g_{0'}^{-1} = R_{00'}(u).$$

The anti-periodic boundary conditions of this case is (n=3)

$$E_{N+1}^{k,l} = g_1\, E_1^{k,l}\, g_1^{-1}, \quad k, l = 1, \cdots, n.$$

monodromy matrix

$$T_0(u) = R_{0N}(u-\theta_N)R_{0\,N-1}(u-\theta_{N-1})\cdots R_{01}(u-\theta_1)$$

transfer matrix

$$t(u) = \mathrm{tr}_0\{g_0\,T_0(u)\}$$

The Yang-Baxter equation lead to the fact that the transfer matrices t(u) with different spectral parameters are mutually commuting: [t(u), t(v)] = 0.

The Hamiltonian with the anti-periodic boundary condition can be obtained from the transfer matrix as

$$H = \sinh\eta\,\frac{\partial \ln t(u)}{\partial u}\Big|_{u=0,\{\theta_j\}=0}$$

Now we derive the operator product identities or the functional relation which are crucial to obtain eigenvalues of the transfer matrix.

By using the fusion, we obtain closed operator identities

$$t(\theta_j)t(\theta_j-\eta)=t_2(\theta_j),$$

$$t(\theta_j)t_2(\theta_j-\eta)=t_3(\theta_j)$$

fused transfer matrices

$$t_2(u)=tr_{12}\{g_{<12>}T_{<12>}(u)\},$$

$$t_3(u)=tr_{123}\{g_{<123>}T_{<123>}(u)\}.$$

$$g_{<12>}\equiv P_{21}^{(-)}g_1g_2P_{21}^{(-)},$$

$$T_{<12>}(u)\equiv P_{21}^{(-)}T_1(u)T_2(u-\eta)P_{21}^{(-)},$$

$$g_{<123>}=P_{321}^{(-)}g_1g_2g_3P_{321}^{(-)},$$

$$T_{<123>}(u)=P_{321}^{(-)}T_1(u)T_2(u-\eta)T_3(u-2\eta)P_{321}^{(-)}.$$

$$t_3(u)=\mathrm{Det}_qT(u)\times\mathrm{id},$$

$$\mathrm{Det}_qT(u)=\prod_{l=1}^{N}\sinh(u-\theta_l+\eta)\sinh(u-\theta_l-\eta)\sinh(u-\theta_l-2\eta).$$

Next, we construct the eigenvalues of the transfer matrices.

The commutativity of the transfer matrices t(u) and $t_2(u)$ with different spectral parameters implies that they have common eigenstates. Let |Ψ> be a common eigenstate of $\{t_m(u)\}$, which dose not depend upon u, with the eigenvalues $\Lambda_m(u)$. We have

$$\Lambda(\theta_j)\Lambda_m(\theta_j-\eta)=\Lambda_{m+1}(\theta_j), \qquad m=1,2, \quad j=1,\cdots,N,$$

$$\Lambda_3(u)=\prod_{l=1}^{N}\sinh(u-\theta_l+\eta)\prod_{k=1}^{2}\sinh(u-\theta_l-k\eta),$$

$$\Lambda_2(\theta_j+\eta)=0, \qquad j=1,\cdots,N.$$

$$\Lambda(u+i\pi)=e^{-\frac{2i\pi}{3}}(-1)^N\Lambda(u), \quad \Lambda_2(u+i\pi)=e^{-\frac{4i\pi}{3}}\Lambda_2(u).$$

$$\Lambda(u) \;=\; e^{\frac{u}{3}}\left\{I_1^{(1)}e^{(N-1)u}+I_2^{(1)}e^{(N-3)u}+\cdots+I_N^{(1)}e^{-(N-1)u}\right\},$$

$$\Lambda_2(u) \;=\; e^{-\frac{u}{3}}\left\{I_1^{(2)}e^{(2N-1)u}+I_2^{(2)}e^{(2N-3)u}+\cdots+I_{2N}^{(2)}e^{-(2N-1)u}\right\},$$

where $\{I^{(1)}_j|j=1,\ldots,N\}$ and $\{I^{(2)}_j|j=1,\ldots,2N\}$ are 3N constants which are eigenstate dependent.

The above relations allow us to express the eigenvalues in terms of inhomogeneous T − Q relations.

$$
\begin{aligned}
\Lambda(u) &= Z_1(u) + Z_2(u) + Z_3(u) + X_1(u) + X_2(u) \\
&= e^{\frac{u}{3}} \Bigg\{ e^{\phi_1} e^{u} a(u) \frac{Q^{(1)}(u-\eta)}{Q^{(2)}(u)} + e^{-\phi_1} \omega_3 e^{-u-\frac{2\eta}{3}} d(u) \frac{Q^{(2)}(u+\eta) Q^{(3)}(u-\eta)}{Q^{(1)}(u) Q^{(4)}(u)} \\
&\quad + \omega_3^2 e^{-u-\frac{4\eta}{3}} d(u) \frac{Q^{(4)}(u+\eta)}{Q^{(3)}(u)} + a(u) d(u) \frac{Q^{(3)}(u-\eta) f_1(u)}{Q^{(1)}(u) Q^{(2)}(u)} \\
&\quad + a(u) d(u) \frac{Q^{(2)}(u+\eta) f_2(u)}{Q^{(3)}(u) Q^{(4)}(u)} \Bigg\},
\end{aligned}
$$

where

$$
a(u) = \prod_{l=1}^{N} \sinh(u - \theta_l + \eta), \quad d(u) = \prod_{l=1}^{N} \sinh(u - \theta_l) = a(u - \eta),
$$

$$
Q^{(i)}(u) = \prod_{l=1}^{N_i} \sinh(u - \lambda_l^{(i)}), \quad i = 1, 2, 3, 4,
$$

$$
f_1(u) = f_1^{(+)} e^{u} + f_1^{(-)} e^{-u}, \quad f_2(u) = f_2^{(-)} e^{-u},
$$

The eigenvalues of the fused transfer matrices are

$$\Lambda_2(u) = Z_1(u) Z_2^{(1)}(u) + Z_1(u) Z_3^{(1)}(u) + Z_2(u) Z_3^{(1)}(u) + X_1(u) Z_3^{(1)}(u) + Z_1(u) X_2^{(1)}(u)$$

$$\Lambda_3(u) = Z_1(u)\, Z_2^{(1)}(u)\, Z_3^{(2)}(u)$$

where $\quad Z_i^{(l)}(u) = Z_i(u - l\eta), \quad X_i^{(l)}(u) = X_i(u - l\eta)$

BAEs:

$$\omega_3 e^{-\phi_1} e^{-\lambda_j^{(1)}-\frac{2\eta}{3}} \frac{Q^{(2)}(\lambda_j^{(1)}+\eta)}{Q^{(4)}(\lambda_j^{(1)})} + a(\lambda_j^{(1)}) \frac{f_1(\lambda_j^{(1)})}{Q^{(2)}(\lambda_j^{(1)})} = 0, \quad j = 1, \cdots, N,$$

$$e^{\phi_1} e^{\lambda_j^{(2)}} Q^{(1)}(\lambda_j^{(2)}-\eta) + d(\lambda_j^{(2)}) \frac{Q^{(3)}(\lambda_j^{(2)}-\eta) f_1(\lambda_j^{(2)})}{Q^{(1)}(\lambda_j^{(2)})} = 0, \quad j = 1, \cdots, N,$$

$$\omega_3^2 e^{-\lambda_j^{(3)}-\frac{4\eta}{3}} Q^{(4)}(\lambda_j^{(3)}+\eta) + a(\lambda_j^{(3)}) \frac{Q^{(2)}(\lambda_j^{(3)}+\eta) f_2(\lambda_j^{(3)})}{Q^{(4)}(\lambda_j^{(3)})} = 0, \ j = 1, \cdots, N,$$

$$\omega_3 e^{-\phi_1} e^{-\lambda_j^{(4)}-\frac{2\eta}{3}} \frac{Q^{(3)}(\lambda_j^{(4)}-\eta)}{Q^{(1)}(\lambda_j^{(4)})} + a(\lambda_j^{(4)}) \frac{f_2(\lambda_j^{(4)})}{Q^{(3)}(\lambda_j^{(4)})} = 0, \quad j = 1, \cdots, N,$$

$$e^{\phi_1} e^{-\Theta-\chi^{(1)}+\chi^{(2)}} + e^{-2\Theta+\chi^{(1)}+\chi^{(2)}-\chi^{(3)}} f_1^{(+)} = 0,$$

$$\begin{aligned}&\omega_3 e^{-\phi_1} e^{-\frac{2\eta}{3}+\Theta-\chi^{(1)}+\chi^{(2)}+\chi^{(3)}-\chi^{(4)}} + \omega_3^2 e^{-\frac{4\eta}{3}+\Theta-\chi^{(3)}+\chi^{(4)}-N\eta} \\ &\qquad + e^{2\Theta-N\eta}\left\{e^{-\chi^{(1)}-\chi^{(2)}+\chi^{(3)}+N\eta} f_1^{(-)} + e^{+\chi^{(2)}-\chi^{(3)}-\chi^{(4)}-N\eta} f_2^{(-)}\right\} = 0,\end{aligned}$$

$$\begin{aligned}&\omega_3 e^{-\Theta-\chi^{(3)}+\chi^{(4)}} + \omega_3^2 e^{\phi_1} e^{-\frac{2\eta}{3}-\Theta-\chi^{(1)}+\chi^{(2)}+\chi^{(3)}-\chi^{(4)}+N\eta} \\ &\qquad + e^{-2\Theta+N\eta}\left\{\omega_3^2 e^{-\frac{2\eta}{3}+\chi^{(1)}+\chi^{(2)}-\chi^{(4)}} f_1^{(+)} + e^{\phi_1} e^{\frac{2\eta}{3}-\chi^{(1)}+\chi^{(3)}+\chi^{(4)}+N\eta} f_2^{(-)}\right\} = 0,\end{aligned}$$

$$e^{-\phi_1} e^{-\frac{4\eta}{3}+\Theta-\chi^{(1)}+\chi^{(2)}-N\eta} + \omega_3^2 e^{-\frac{2\eta}{3}+2\Theta-\chi^{(1)}-\chi^{(2)}+\chi^{(4)}-N\eta} f_1^{(-)} = 0,$$

$$\Theta = \sum_{l=1}^{N} \theta_l, \quad \chi^{(i)} = \sum_{l=1}^{N} \lambda_l^{(i)}, \quad i = 1, 2, 3, 4. \qquad N_1 = N_2 = N_3 = N_4 = N,$$

The BEAs ensure that the T-Q relations indeed satisfy the asymptotic behavior, periodicity properties and have no singular points.

Next, we construct the eigenstates of the transfer matrices.

Steps:

1. Find an orthogonal basis of the Hilbert space of the system;
2. Express the eigenstate as the linear combination of these basis;
3. Calculate the coefficients.

First, we construct a nested separation of variables (SoV) basis of the Hilbert space.

monodromy matrix

$$T(u) = \begin{pmatrix} A(u) & B_2(u) & B_3(u) \\ C^2(u) & D_2^2(u) & D_3^2(u) \\ C^3(u) & D_2^3(u) & D_3^3(u) \end{pmatrix}$$

transfer matrix

$$t(u) = B_2(u) + D_3^2(u) + C^3(u)$$

For two non-negative integers m_2 and m such that $m_2 \leq m \leq N$, let us introduce m positive integers P = $\{p_1, \ldots, p_m\}$ such that

$$1 \leq p_1 < p_2 < \cdots < p_{m_2} \leq N\,, \qquad 1 \leq p_{m_2+1} < \cdots < p_m \leq N\,, \quad \text{and} \quad p_j \neq p_l$$

For each P satisfies the above condition, let us introduce left and right states parameterized by N inhomogeneity parameters $\{\theta_j\}$ as

$$\boxed{\begin{aligned} \langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}| &= \langle 0| C^2(\theta_{p_1}) \cdots C^2(\theta_{p_{m_2}})\, C^3(\theta_{p_{m_2+1}}) \cdots C^3(\theta_{p_m})\,, \\ |\theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}\rangle &= B_3(\theta_{p_m}) \cdots B_3(\theta_{p_{m_2+1}})\, B_2(\theta_{p_{m_2}}) \cdots B_2(\theta_{p_1})|0\rangle\,, \end{aligned}}$$

$$\langle 0| = \langle 1, \cdots, 1|\,, \qquad |0\rangle = |1, 1, \cdots, 1\rangle$$

It is easy to check that the above states are eigenstates of the operator $D_3^3(u)$

$$\langle\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}|\,D_3^3(u)=d(u)\prod_{l=m_2+1}^{m}\frac{\sinh(u-\theta_{p_l}+\eta)}{\sinh(u-\theta_{p_l})}\times\langle\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}|\,,$$

$$D_3^3(u)\,|\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}\rangle=d(u)\prod_{l=m_2+1}^{m}\frac{\sinh(u-\theta_{p_l}+\eta)}{\sinh(u-\theta_{p_l})}\times|\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}\rangle\,.$$

Total number of the linear-independent left (right) states is

$$\sum_{m=0}^{N} \frac{N!}{(N-m)!m!} \sum_{m_2=0}^{m} \frac{m!}{(m-m_2)!m_2!} = \sum_{m=0}^{N} \frac{N!}{(N-m)!m!} 2^m = 3^N.$$

Orthogonal relations between the left states and the right states

$$\langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m} | \theta_{q_1}, \cdots, \theta_{q_{m'_2}}; \theta_{q_{m'_2+1}}, \cdots, \theta_{q_{m'}} \rangle = \delta_{m,m'} \delta_{m_2,m'_2} \times \prod_{k=1}^{m} \delta_{p_k,q_k} G_m(\theta_{p_1}, \cdots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}),$$

where the factor G_m is given by

$$G_m(\theta_{p_1}, \cdots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}) = \prod_{k=1}^{m_2} \sinh\eta \, d_{p_k}(\theta_{p_k}) \, a(\theta_{p_k}) \prod_{l=1, l\neq k}^{m_2} \frac{\sinh(\theta_{p_k} - \theta_{p_l} + \eta)}{\sinh(\theta_{p_k} - \theta_{p_l})}$$
$$\times \prod_{k=m_2+1}^{m} \sinh\eta \, d_{p_k}(\theta_{p_k}) \, a(\theta_{p_k}) \left\{ \prod_{l=m_2+1, l\neq k}^{m} \frac{\sinh(\theta_{p_k} - \theta_{p_l} + \eta)}{\sinh(\theta_{p_k} - \theta_{p_l})} \times \prod_{l=1}^{m_2} \frac{\sinh(\theta_{p_k} - \theta_{p_l} - \eta)}{\sinh(\theta_{p_k} - \theta_{p_l})} \right\}.$$

$$d_l(u) = \prod_{k=1, k\neq l}^{N} \sinh(u - \theta_k)$$

Thus these right (left) states form an orthogonal right (left) basis of the Hilbert space, namely,

$$\mathrm{id} = \sum_{m=0}^{N} \sum_{m_2=0}^{m} \sum_{P} \frac{1}{G_m(\theta_{p_1}, \cdots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \cdots, \theta_{p_m})} \times |\theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}\rangle \langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}|,$$

Meanwhile, direct calculation shows that actions of the monodromy matrix elements on this basis become drastically simple. Here we list some of them relevant to construct eigenstates of the transfer matrix,

$$\langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}| D_3^2(u) = \sum_{l=m_2+1}^{m} \frac{\sinh \eta \, e^{\frac{u-\theta_{p_l}}{3}} d(u)}{\sinh(u-\theta_{p_l})} \times \prod_{k=m_2+1, k\neq l}^{m} \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})} \frac{\sinh(\theta_{p_l}-\theta_{p_k}-\eta)}{\sinh(\theta_{p_l}-\theta_{p_k})} \times \langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}, \theta_{p_l}; \theta_{p_{m_2+1}}, \cdots, \underline{\theta_{p_{l-1}}, \theta_{p_{l+1}}}, \cdots, \theta_{p_m}|,$$

$$
\begin{aligned}
\langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m} | \, D_2^3(u) &= \sum_{l=1}^{m_2} \frac{\sinh\eta \, e^{-\frac{u-\theta_{p_l}}{3}} d(u)}{\sinh(u-\theta_{p_l})} \\
&\times \left\{ \prod_{k=1,k\neq l}^{m_2} \frac{\sinh(\theta_{p_l}-\theta_{p_k}+\eta)}{\sinh(\theta_{p_l}-\theta_{p_k})} \prod_{k=m_2+1}^{m} \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})} \right\} \\
&\times \langle \theta_{p_1}, \cdots \underline{, \theta_{p_{l-1}}, \theta_{p_{l+1}}}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}, \theta_{p_l} |,
\end{aligned}
$$

$$
\begin{aligned}
\langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m} | \, B_3(u) = & \sum_{l=m_2+1}^{m} \frac{\sinh\eta \, e^{-\frac{u-\theta_{p_l}}{3}} d(u)}{\sinh(u-\theta_{p_l})} a(\theta_{p_l}) \\
\times & \prod_{k=m_2+1,k\neq l}^{m} \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})} \frac{\sinh(\theta_{p_l}-\theta_{p_k}-\eta)}{\sinh(\theta_{p_l}-\theta_{p_k})} \\
\times & \prod_{\alpha=1}^{m_2} \frac{\sinh(\theta_{p_l}-\theta_{p_\alpha}-\eta)}{\sinh(\theta_{p_l}-\theta_{p_\alpha})} \langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots \underline{, \theta_{p_{l-1}}, \theta_{p_{l+1}}}, \cdots, \theta_{p_m} | \\
+ & \sum_{l=m_2+1}^{m} \frac{\sinh\eta \, e^{-\frac{u-\theta_{p_l}}{3}} d(u)}{\sinh(u-\theta_{p_l})} \prod_{k=m_2+1,k\neq l}^{m} \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})} \frac{\sinh(\theta_{p_l}-\theta_{p_k}-\eta)}{\sinh(\theta_{p_l}-\theta_{p_k})} \\
\times & \sum_{\alpha=1}^{m_2} \frac{\sinh\eta \, e^{-\frac{\theta_{p_\alpha}-\theta_{p_l}}{3}}}{\sinh(\theta_{p_l}-\theta_{p_\alpha})} a(\theta_{p_\alpha}) \prod_{k=1,k\neq\alpha}^{m_2} \frac{\sinh(\theta_{p_\alpha}-\theta_{p_k}-\eta)}{\sinh(\theta_{p_\alpha}-\theta_{p_k})} \\
\times & \langle \theta_{p_1}, \cdots \underline{, \theta_{p_{\alpha-1}}, \theta_{p_l}, \theta_{p_{\alpha+1}}}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots \underline{, \theta_{p_{l-1}}, \theta_{p_{l+1}}}, \cdots, \theta_{p_m} |,
\end{aligned}
$$

$$
\begin{aligned}
\langle\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}|\,C^3(u) &= \sum_{l=m+1}^{N}\frac{e^{\frac{u-\theta_{p_l}}{3}}}{\sinh(u-\theta_{p_l})}\frac{d(u)}{d_{p_l}(\theta_{p_l})}\\
&\times\prod_{k=m_2+1}^{m}\frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})}\frac{\sinh(\theta_{p_l}-\theta_{p_k})}{\sinh(\theta_{p_l}-\theta_{p_k}+\eta)}\\
&\times\langle\theta_{p_1},\cdots\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\underline{\theta_{p_m},\theta_{p_l}}|\\
&+\sum_{l=m+1}^{N}\sum_{\alpha=1}^{m_2}\frac{e^{\frac{u-\theta_{p_\alpha}}{3}}}{\sinh(u-\theta_{p_\alpha})}\prod_{k=m_2+1}^{m}\frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})}\frac{\sinh(\theta_{p_l}-\theta_{p_k})}{\sinh(\theta_{p_l}-\theta_{p_k}+\eta)}\\
&\times\frac{\sinh\eta\,d(u)\,e^{\frac{\theta_{p_l}-\theta_{p_\alpha}}{3}}}{d_{p_l}(\theta_{p_l})\sinh(\theta_{p_\alpha}-\theta_{p_l}-\eta)}\prod_{k=1,k\neq\alpha}^{m_2}\frac{\sinh(\theta_{p_l}-\theta_{p_k})}{\sinh(\theta_{p_l}-\theta_{p_k}+\eta)}\frac{\sinh(\theta_{p_\alpha}-\theta_{p_k}+\eta)}{\sinh(\theta_{p_\alpha}-\theta_{p_k})}\\
&\times\langle\theta_{p_1},\cdots,\underline{\theta_{p_{\alpha-1}},\theta_{p_l},\theta_{p_{\alpha+1}}},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\underline{\theta_{p_m},\theta_{p_\alpha}}|.
\end{aligned}
$$

$$
\begin{aligned}
&\langle\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}|\,D^i_j(\theta_{p_l})=0\,, \quad l=m+1,\cdots,N,\ \text{and } i,j=2,3\,,\\
&\langle\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}|\,B_i(\theta_{p_l})=0\,, \quad l=m+1,\cdots,N,\ \text{and } i=2,3\,,\\
&D^i_j(\theta_{p_l})|\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}\rangle=0\,, \quad l=m+1,\cdots,N,\ \text{and } i=2,3\,,\\
&C^i(\theta_{p_l})|\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}\rangle=0\,, \quad l=m+1,\cdots,N,\ \text{and } i=2,3\,.
\end{aligned}
$$

Some remarks:

1. In the rational limit, the resulting basis serves as the SoV basis for the associated rational spin chain model.

2. We have checked that each basis vector for the su(3) case is an off-shell Bethe state obtained via the nested algebraic Bethe ansatz by replacing the Bethe roots with some sets of the inhomogeneity parameters.

3. This observation provides an efficient way to construct similar nested SoV basis for general high-rank quantum integrable models.

4. From explicit expressions of actions of the monodromy matrix elements on this basis, one can see that in the basis, the operators have no compensating exchange terms on the level of the local operators, which allow us to compute correlation functions for quantum spin chains associated with higher-rank algebras.

A general off-shell Bethe state is

$$|\lambda_1, \cdots, \lambda_m; \lambda_1^{(1)}, \cdots, \lambda_{m-m_2}^{(1)}\rangle = B_{i_1}(\lambda_1)\cdots B_{i_m}(\lambda_m)\, F^{i_1,\cdots,i_m}|0\rangle,$$

where {F} are the vector components of a nested off-shell Bethe state

$$B^{(1)}(\lambda_1^{(1)})\cdots B^{(1)}(\lambda_{m-m_2}^{(1)})|0\rangle^{(1)} = \sum_{i_1,\cdots,i_m=2}^{3} F^{i_1,\cdots,i_m}|i_1,\cdots,i_m\rangle^{(1)}$$

with creation operator and reference state associated with su(2) chain

The above Bethe sate is a linear combination of the vectors

$$|\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}\rangle = B_3(\theta_{p_m})\cdots B_3(\theta_{p_{m_2+1}})\, B_2(\theta_{p_{m_2}})\cdots B_2(\theta_{p_1})|0\rangle,$$

However, if the parameters $\{\lambda_l$, l=1,…, m} are particularly chosen as $\{\lambda_l=\theta_{pl}$, l=1,…, m} and

then the nested parameters $\{\lambda_n^{(1)}$, n=1,…, m-m_2} have to take the values in the chosen set of $\{\lambda_l$, l=1,…, m} (e.g., $\{\lambda_n^{(1)}=\theta_{pn}$, n=m2+1,…, m}),

the corresponding linear combination becomes drastically simple such that only one term such as the right state does remain.

Now, we construct the eigenstates of the su(3) spin torus

Let $|\Psi>$ be a eigenstate of t(u), which does not depend upon u, with an eigenvalue $\Lambda(u)$.

Due to the fact that the left states form a basis of the dual Hilbert space, the eigenstate $|\Psi>$ is completely determined (up to an overall scalar factor) by the following scalar products

$$F_{m_2,m-m_2}(\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}) = \langle\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}|\Psi\rangle\,,$$

$$1\leq p_1<\cdots<p_{m_2}\,,\quad 1\leq p_{m_2+1}<\cdots<p_m\leq N,\quad p_j\neq p_k\,,\quad 0\leq m_2\leq m\leq N\,.$$

Let us consider the quantities

$$\langle\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}|t(\theta_{p_{m+1}})|\Psi\rangle$$

Acting $t(\theta_{p_{m+1}})$ to the right gives rise to the relation

$$\Lambda(\theta_{p_{m+1}})\,F_{m_2,m-m_2}(\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}) = \langle\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}|t(\theta_{p_{m+1}})|\Psi\rangle\,.$$

Acting $t(\theta_{p_{m+1}})$ to the left we readily obtain

$$F_{m_2,m-m_2}(\theta_{p_1},\cdots,\theta_{p_{m_2}};\theta_{p_{m_2+1}},\cdots,\theta_{p_m}) = \left\{\prod_{l=m_2+1}^{m}\Lambda(\theta_{p_l})\right\}F_{m_2}(\theta_{p_1},\cdots,\theta_{p_{m_2}})$$

where the scalar products are given by

$$F_m(\theta_{p_1},\cdots,\theta_{p_m}) = \langle 0|C^2(\theta_{p_1})\cdots C^2(\theta_{p_m})|\Psi\rangle\,,\qquad m=0,\cdots,N\,.$$

The values of scalar products

$$F_m(\theta_{p_1},\cdots,\theta_{p_m}) = \sum_{1\le p'_1<\cdots<p'_m\le N} g_m(\theta_{p_1},\cdots,\theta_{p_m}|\theta_{p'_1},\cdots,\theta_{p'_m}) \times \prod_{\alpha=1}^{m}\prod_{k=m+1}^{N}\sinh(\theta_{p'_\alpha}-\theta_{p_k}+\eta)\frac{\prod_{l=1}^{m}\Lambda(\theta_{p'_l})}{f_m(\theta_{p'_1},\cdots,\theta_{p'_m})}\frac{\prod_{l=1}^{N}a(\theta_k)}{\prod_{k=m+1}^{N}\Lambda(\theta_{p_k})}\langle\bar{0}|\Psi\rangle,$$

where

$$g_m(v_1,\cdots,v_m|u_1,\cdots,u_m) = \frac{\prod_{\alpha=1}^{m}\prod_{k=1}^{m}\sinh(u_\alpha-v_k+\eta)\sinh(u_\alpha-v_k)}{\prod_{k<l}^{m}\sinh(u_l-u_k)\sinh(v_k-v_l)}\det\mathcal{M},$$

$$f_m(\theta_{p_1},\cdots,\theta_{p_m}) = \prod_{l=1}^{m}\sinh\eta\, d_{p_l}(\theta_{p_l})\, a(\theta_{p_l})\prod_{k=1,k\neq l}^{m}\frac{\sinh(\theta_{p_l}-\theta_{p_k}+\eta)}{\sinh(\theta_{p_l}-\theta_{p_k})},$$

M is an m $\times$ m matrix with matrix elements

$$\mathcal{M}_{\alpha,k} = \frac{\sinh\eta\, e^{-\frac{u_\alpha-v_k}{3}}}{\sinh(u_\alpha-v_k+\eta)\,\sinh(u_\alpha-v_k)}, \qquad \alpha,k=1,\cdots,m.$$

$$\langle\bar{0}| = \langle 3,\cdots,3|$$

Therefore, the eigenstate of the transfer matrix corresponding to an eigenvalue $\Lambda(u)$ is

$$
\begin{aligned}
|\Psi\rangle &= \sum_{m=0}^{N} \sum_{m_2=0}^{m} \sum_{P} \frac{\langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m} | \Psi \rangle}{G_m(\theta_{p_1}, \cdots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \cdots, \theta_{p_m})} \\
&\quad \times |\theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}\rangle \\
&= \sum_{m=0}^{N} \sum_{m_2=0}^{m} \sum_{P} \frac{F_{m_2}(\theta_{p_1}, \cdots, \theta_{p_{m_2}}) \prod_{k=m_2+1}^{m} \Lambda(\theta_{p_k})}{G_m(\theta_{p_1}, \cdots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \cdots, \theta_{p_m})} \\
&\quad \times |\theta_{p_1}, \cdots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \cdots, \theta_{p_m}\rangle ,
\end{aligned}
$$

Now, we consider the homogeneous limit.

1. The $T-Q$ relation and the associated BAEs have well-defined homogeneous limits.

2. In the homogeneous limit, the resulting eigenstate becomes the eigenstate of the homogeneous quantum spin chain.

3. We have checked that such a limit of the state does exist for some small N. For an example, here we present the limit of the $N = 2$ case.

$$
\begin{aligned}
\lim_{\theta_1,\theta_2\to 0} |\Psi\rangle \;\propto\;& |0\rangle + \frac{1}{\sinh^3\eta}\left[\Lambda' \, B_3 + \Lambda \, B_3' - 2\coth\eta \, \Lambda \, B_3\right]|0\rangle + \frac{\Lambda^2}{\sinh^8\eta} B_3 \, B_3 |0\rangle \\
&+ \frac{\Lambda^2}{a^2(0)} \left\{ \left[\left(\frac{8}{9} - 2\coth\eta \, \frac{\Lambda'}{\Lambda} + (\frac{\Lambda'}{\Lambda})^2 \right) B_2 + \left(\frac{\Lambda'}{\Lambda} - \coth\eta - \frac{1}{3} \right) B_2' \right] \right. \\
&\qquad + \frac{\Lambda}{\sinh^4\eta} \left[\left(\coth\eta \frac{\Lambda'}{\Lambda} - \frac{\Lambda'}{3\Lambda} - \frac{8}{9} \right) B_3 \, B_2 \right. \\
&\qquad \left. \left. + \left(\frac{\Lambda'}{\Lambda} - \coth\eta - \frac{1}{3} \right) (B_3' \, B_2 - B_3 \, B_2') \right] + \frac{\Lambda^2}{\sinh^8\eta} B_2 \, B_2 \right\} |0\rangle,
\end{aligned}
$$

$$
B_i = B_i(0), \quad B_i' = \frac{\partial}{\partial u} B_i(u)\Big|_{u=0}, \quad i = 2, 3,
$$

$$
\Lambda = \Lambda(0), \quad \Lambda' = \frac{\partial}{\partial u} \Lambda(u)\Big|_{u=0}.
$$

It is conjectured that the eigenstate for generic N has a well-defined homogeneous limit.

Trigonometric su(n) spin torus

R matrix

$$
\begin{aligned}
R(u) &= \sinh(u+\eta)\sum_{k=1}^{n} E^{k,k}\otimes E^{k,k} + \sinh u \sum_{k\neq l}^{n} E^{k,k}\otimes E^{l,l} \\
&\quad + \sinh\eta\left(\sum_{k<l}^{n} e^{\frac{n-2(l-k)}{n}u} + \sum_{k>l}^{n} e^{-\frac{n-2(k-l)}{n}u}\right) E^{k,l}\otimes E^{l,k},
\end{aligned}
$$

twist matrix

$$
g = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}, \quad \text{and } g^n = 1
$$

$$
T(u) = \begin{pmatrix} A(u) & B_2(u) & \cdots & B_n(u) \\ C^2(u) & D_2^2(u) & \cdots & D_n^2(u) \\ \cdots & \cdots & \cdots & \cdots \\ C^n(u) & D_2^n(u) & \cdots & D_n^n(u) \end{pmatrix}
$$

functional relations:

$$\Lambda(\theta_j)\Lambda_m(\theta_j-\eta)=\Lambda_{m+1}(\theta_j), \qquad m=1,\cdots,n-1, \quad j=1,\cdots N,$$

$$\Lambda_m(\theta_j+k\eta)=0, \quad k=1,\cdots,m-1, \quad m=1,\cdots,n-1, \quad j=1,\cdots N,$$

$$\Lambda_n(u)=(-1)^{n-1}\prod_{l=1}^{N}\sinh(u-\theta_l+\eta)\prod_{k=1}^{n-1}\sinh(u-\theta_l-k\eta)\times \mathrm{id},$$

$$\Lambda_m(u+i\pi)=e^{-m(\frac{2}{n})i\pi}((-1)^N)^m\Lambda_m(u), \quad m=1,\cdots,n-1,$$

$$e^{-u+2(\frac{m}{n})u}\Lambda_m(u)\propto e^{\pm(mN-1)u}+\cdots, \quad u\to\pm\infty, \quad m=1,\cdots,n-1.$$

The above relations can completely determine the eigenvalues of the transfer matrix.

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For the su(n) spin torus, let us introduce n−1 non-negative integers $m_2, m_3, \dots, m_n$ such that

$$\sum_{l=2}^{n} m_l \le N$$

Nested SoV basis

$$\boxed{\begin{aligned}\langle\theta_{p_1},\cdots,\theta_{p_{m_2}};\cdots;\theta_{p_{m_2+\cdots m_{n-1}+1}},\cdots,\theta_{p_{m_2+\cdots m_n}}| &= \langle 0|\,C^2(\theta_{p_1})\cdots C^2(\theta_{p_{m_2}})\cdots \\ &\quad\times C^n(\theta_{p_{m_2+\cdots+m_{n-1}+1}})\cdots C^n(\theta_{p_{m_2+\cdots+m_n}}), \\ |\theta_{p_1},\cdots,\theta_{p_{m_2}};\cdots;\theta_{p_{m_2+\cdots m_{n-1}+1}},\cdots,\theta_{p_{m_2+\cdots m_n}}\rangle &= B_n(\theta_{p_{m_2+\cdots+m_n}})\cdots \\ &\quad\times B_n(\theta_{p_{m_2+\cdots+m_{n-1}+1}})\cdots B_2(\theta_{p_{m_2}})\cdots B_2(\theta_{p_1})\,|0\rangle,\end{aligned}}$$

where

$$1\le p_1<\cdots<p_{m_2}\le N,\ \cdots,\ 1\le p_{m_2+\cdots+m_{n-1}+1}<\cdots<p_{m_2+\cdots+m_n}\le N$$

$$p_j \ne p_k$$

Please note that the number of the operators $C^j(u)$ (or $B_j(u)$) in the above expression is m_j.

These states are the eigenstates of operator $D^n_n(u)$

$$\langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \cdots; \theta_{p_{m_2+\cdots m_{n-1}+1}}, \cdots, \theta_{p_{m_2+\cdots m_n}} | D^n_n(u)$$

$$= d(u) \prod_{k=m_2+\cdots m_{n-1}+1}^{m_2+\cdots m_n} \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})}$$

$$\times \langle \theta_{p_1}, \cdots, \theta_{p_{m_2}}; \cdots; \theta_{p_{m_2+\cdots m_{n-1}+1}}, \cdots, \theta_{p_{m_2+\cdots m_n}} |,$$

$$D^n_n(u) |\theta_{p_1}, \cdots, \theta_{p_{m_2}}; \cdots; \theta_{p_{m_2+\cdots m_{n-1}+1}}, \cdots, \theta_{p_{m_2+\cdots m_n}} \rangle$$

$$= d(u) \prod_{k=m_2+\cdots m_{n-1}+1}^{m_2+\cdots m_n} \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})}$$

$$\times |\theta_{p_1}, \cdots, \theta_{p_{m_2}}; \cdots; \theta_{p_{m_2+\cdots m_{n-1}+1}}, \cdots, \theta_{p_{m_2+\cdots m_n}} \rangle.$$

1. For generic values of $\{\theta_j\}$, these right (left) states form an orthogonal right (left) basis of the Hilbert space, and any right (left) state can be decomposed as a unique linear combination of these basis.

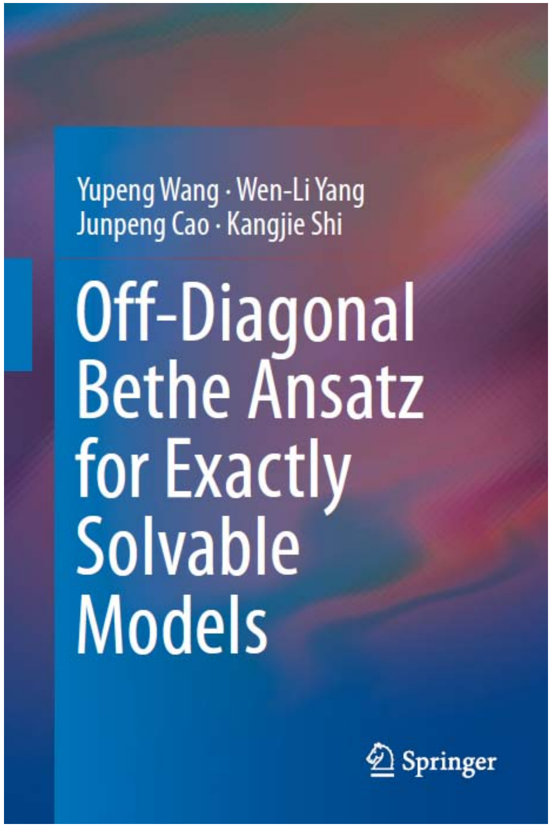
2. Using the similar method, we can obtain the explicit expressions for the operators $\{D^n{}_i(u), D^i{}_n(u) | i = 2, \ldots, n\}$, $B_n(u)$ and $C^n(u)$ in the basis.

3. The operators take some simple forms without compensating exchange terms on the level of the local operators.

4. These resulting simple forms allow one to construct eigenstates of the transfer matrix of the su(n) spin torus via its ODBA solution.

Conclusion & Perspective

ODBA is a universal method to treat the one-dimensional quantum many-body systems.



small polaron: NPB 898, 276 (2015)

AdS/CFT: JHEP 10, 133 (2015)

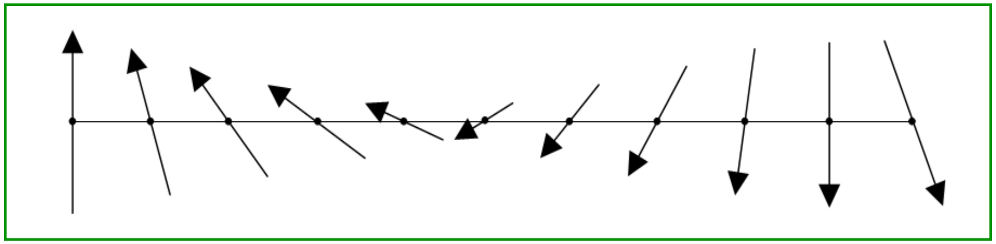
τ_2 model: JHEP 09, 212 (2015)

Thank you for your attention!

Eigenstates of open XXX spin chain

Hamiltonian

$$H=\sum_{j=1}^{N-1}\vec{\sigma}_j\cdot\vec{\sigma}_{j+1}+\frac{\eta}{p}\sigma_1^z+\frac{\eta}{q}(\xi\sigma_N^x+\sigma_N^z),$$



$$H=\eta\frac{\partial\ln t^{(o)}(u)}{\partial u}|_{u=0,\theta_j=0}-N.$$

Transfer matrix

$$\boxed{t^{(o)}(u) = tr_0\left(K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\right)}$$

$$\begin{aligned}
T_0(u) &= R_{0N}(u-\theta_N)\dots R_{01}(u-\theta_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \\
\hat{T}_0(u) &= R_{01}(u+\theta_1)\dots R_{0N}(u+\theta_N) = (-1)^N \begin{pmatrix} D(-u-\eta) & -B(-u-\eta) \\ -C(-u-\eta) & A(-u-\eta) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
K^-(u) &= \begin{pmatrix} p+u & 0 \\ 0 & p-u \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} K_{11}^-(u) & K_{12}^-(u) \\ K_{21}^-(u) & K_{22}^-(u) \end{pmatrix}, \\
K^+(u) &= \begin{pmatrix} q+u+\eta & \xi(u+\eta) \\ \xi(u+\eta) & q-u-\eta \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} K_{11}^+(u) & K_{12}^+(u) \\ K_{21}^+(u) & K_{22}^+(u) \end{pmatrix}
\end{aligned}$$

Functional relations

$\boxed{\text{Analyticity:} \quad \Lambda(u), \text{ as a function of } u, \text{ is a polynomial of degree } 2N+2.}$

$$\text{Crossing symmetry}: \quad \Lambda(-u-1) = \Lambda(u),$$

$$\text{Initial condition}: \Lambda(0) = 2pq\prod_{j=1}^{N}(1-\theta_j)(1+\theta_j) = \Lambda(-1),$$

$$\text{Asymptotic behavior}: \Lambda(u) \sim 2u^{2N+2} + \cdots, \quad u \rightarrow \pm\infty,$$

$$\begin{aligned}\Lambda(\theta_j)\Lambda(\theta_j-1) &= \frac{\Delta_q(\theta_j)}{(1-2\theta_j)(1+2\theta_j)} \\ &= a(\theta_j)d(\theta_j-1), \quad j = 1,\cdots,N,\end{aligned}$$

The inhomogeneous T-Q relation

Each eigenvalue of the transfer matrix can be given in terms of the following inhomogeneous T-Q relation

$$\begin{aligned}\Lambda^{(o)}(u) &= (-1)^N\frac{2u+2\eta}{2u+\eta}(u+p)(\sqrt{1+\xi^2}\,u+q)a(u)d(-u-\eta)\frac{Q(u-\eta)}{Q(u)}\\ &+(-1)^N\frac{2u}{2u+\eta}(u-p+\eta)(\sqrt{1+\xi^2}\,(u+\eta)-q)a(-u-\eta)d(u)\frac{Q(u+\eta)}{Q(u)}\\ &+2(1-\sqrt{1+\xi^2})u(u+\eta)\frac{a(u)a(-u-\eta)d(u)d(-u-\eta)}{Q(u)},\end{aligned}$$

$$a(u) = \prod_{l=1}^{N}(u-\theta_l+\eta), \qquad d(u)=a(u-\eta)=\prod_{l=1}^{N}(u-\theta_l)$$

$$Q(u)=\prod_{j=1}^{N}(u-\lambda_j)(u+\lambda_j+\eta).$$

Bethe ansatz equations

Bethe ansatz equations

$$1+\frac{\lambda_j\,(\lambda_j-p+\eta)\,(\sqrt{1+\xi^2}\,(\lambda_j+\eta)-q)\,a(-\lambda_j-\eta)\,d(\lambda_j)\,Q(\lambda_j+\eta)}{(\lambda_j+\eta)\,(\lambda_j+p)\,(\sqrt{1+\xi^2}\,\lambda_j+q)\,a(\lambda_j)\,d(-\lambda_j-\eta)\,Q(\lambda_j-\eta)}$$
$$=(-1)^N\frac{(\sqrt{1+\xi^2}-1)\,\lambda_j\,(2\lambda_j+\eta)\,a(-\lambda_j-\eta)\,d(\lambda_j)}{(\lambda_j+p)\,(\sqrt{1+\xi^2}\,\lambda_j+q)\,Q(\lambda_j-\eta)},\quad j=1,\ldots,N.$$

All the above results have well-defined homogeneous limit $\{\theta_j\}\rightarrow 0$.

The eigenvalue of the Hamiltonian is

$$E=\left.\frac{\partial\ln\Lambda(u)}{\partial u}\right|_{u=0,\{\theta_j=0\}}-N$$
$$=\sum_{j=1}^{N}\frac{2}{\lambda_j(\lambda_j+1)}+N-1+\frac{1}{p}+\frac{(1+\xi^2)^{\frac{1}{2}}}{q}.$$

Study the eigenstates:

1. Construct the orthogonal basis of Hilbert space of the system.
2. Decompose the eigenstates as the linear combination of the basis and obtain the coefficients from the eigenvalues.
3. Express the eigenstate as the form of Bethe states and obtain the Bethe-like eigenstates.

Gauge transformation

In order to obtain the states, we introduce the **gauge transformation**

$$U = \begin{pmatrix} \xi & \sqrt{1+\xi^2}-1 \\ \xi & -\sqrt{1+\xi^2}-1 \end{pmatrix}$$

K^+-matrix can be diagonalized as

$$\begin{aligned} \bar{K}^+(u) &= UK^+(u)U^{-1} = \begin{pmatrix} q+\sqrt{1+\xi^2}(u+\eta) & 0 \\ 0 & q-\sqrt{1+\xi^2}(u+\eta) \end{pmatrix} \\ &\stackrel{\text{def}}{=} \begin{pmatrix} \bar{K}^+_{11}(u) & 0 \\ 0 & \bar{K}^+_{22}(u) \end{pmatrix}, \end{aligned}$$

Double-row monodromy matrix

$$\begin{aligned} \bar{\mathbb{T}}(u) &= U\,T(u)\,K^-(u)\,\hat{T}(u)U^{-1} = UT(u)U^{-1}\,UK^-(u)U^{-1}\,U\hat{T}(u)U^{-1} \\ &= \bar{T}(u)\,\bar{K}^-(u)\hat{\bar{T}}(u) = \begin{pmatrix} \bar{\mathcal{A}}(u) & \bar{\mathcal{B}}(u) \\ \bar{\mathcal{C}}(u) & \bar{\mathcal{D}}(u) \end{pmatrix}. \end{aligned}$$

Gauge transformation

Local states

$$|1\rangle_n = \frac{\sqrt{1+\xi^2}+1}{2\xi\sqrt{1+\xi^2}}|\uparrow\rangle_n + \frac{1}{2\sqrt{1+\xi^2}}|\downarrow\rangle_n, \quad n=1,\ldots,N,$$

$$|2\rangle_n = \frac{\sqrt{1+\xi^2}-1}{2\xi\sqrt{1+\xi^2}}|\uparrow\rangle_n - \frac{1}{2\sqrt{1+\xi^2}}|\downarrow\rangle_n, \quad n=1,\ldots,N,$$

Dual states

$$\langle 1|_n = \xi\langle\uparrow|_n + (\sqrt{1+\xi^2}-1)\langle\downarrow|_n,$$

$$\langle 2|_n = \xi\langle\uparrow|_n - (\sqrt{1+\xi^2}+1)\langle\downarrow|_n,$$

Orthogonal relations

$$\langle a|_j b\rangle_k = \delta_{a,b}\,\delta_{j,k}, \quad a,b=1,2, \quad j,k=1,\ldots,N.$$

Orthogonal basis of the Hilbert space

Noting that operators C form a commuting family, $[\bar{\mathcal{C}}(u), \bar{\mathcal{C}}(v)] = 0,$

we can use their common (dual) eigenstates to construct the basis of right (left) Hilbert space.

$$\boxed{\begin{aligned} |\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle &= \bar{\mathcal{A}}(\theta_{p_1}) \ldots \bar{\mathcal{A}}(\theta_{p_n})|\Omega\rangle, \quad 1 \leq p_1 < p_2 < \ldots < p_n \leq N, \\ \langle\langle -\theta_{q_1}, \ldots, -\theta_{q_n}| &= \langle\bar{\Omega}|\bar{\mathcal{D}}(-\theta_{q_1}) \ldots \bar{\mathcal{D}}(-\theta_{q_n}), \quad 1 \leq q_1 < q_2 < \ldots < q_n \leq N. \end{aligned}}$$

$$|\Omega\rangle = \otimes_{j=1}^{N}|1\rangle_j, \quad \langle\bar{\Omega}| = \otimes_{j=1}^{N}\langle 2|_j.$$

The above states are exactly the eigenstates of C bar

$$\begin{aligned} \bar{\mathcal{C}}(u)|\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle &= h(u, \{\theta_{p_1}, \ldots, \theta_{p_n}\})|\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle, \\ \langle\langle -\theta_{p_1}, \ldots, -\theta_{p_n}|\bar{\mathcal{C}}(u) &= h'(u, \{-\theta_{p_1}, \ldots, -\theta_{p_n}\})\langle\langle -\theta_{p_1}, \ldots, -\theta_{p_n}|, \end{aligned}$$

Orthogonal basis of the Hilbert space

Orthogonal relations between the left states and the right states

$$\langle\langle -\theta_{q_1}, \ldots, -\theta_{q_m} | \theta_{p_1}, \ldots, \theta_{p_n} \rangle\rangle = f_n^{(o)}(\theta_{p_1}, \ldots, \theta_{p_n})\delta_{m+n,N}\delta_{\{q_1,\ldots,q_m\};\{p_1,\ldots,p_n\}},$$

where $\delta_{\{q_1,\ldots,q_m\};\{p_1,\ldots,p_n\}}$ is defined as

$$\delta_{\{q_1,\ldots,q_m\};\{p_1,\ldots,p_n\}} = \begin{cases} 1 & \text{if } \{q_1, \ldots, q_m, p_1, \ldots, p_n\} = \{1, \ldots, N\}, \\ 0 & \text{otherwise,} \end{cases}$$

Orthogonal basis of the Hilbert space

Total number of the right (or left) states & completeness

$$\sum_{n=0}^{N} \frac{N!}{(N-n)!n!} = 2^N.$$

Thus for generic values $\{\theta_j\}$, these right (or left) states form an orthogonal right (or left) basis of the Hilbert space (or its dual).

And any right (or left) eigenstate can be decomposed as a unique linear combination of these basis.

Retrieve the eigenstates

The eigenstates can thus be expressed as

$$\boxed{\langle\langle\Psi| = \sum_{n=0}^{N}\sum_{p}\frac{F_n(\theta_{p_1},\cdots,\theta_{p_n})}{f_n(\theta_{p_1},\cdots,\theta_{p_n})}\langle\langle\theta_{p_{n+1}},\cdots,\theta_{p_N}|,}$$

with $p_1 < \cdots < p_n$ and $p_{n+1} < \cdots < p_N$.

Now, we should calculate the expansion coefficients F_n。

Retrieve the eigenstates

Let $\langle\langle\Psi|$ be a common eigenstate of the transfer matrix $t^{(o)}(u)$

$$\langle\langle\Psi|\, t^{(o)}(u) = \langle\langle\Psi|\, \Lambda^{(o)}(u),$$

Consider the quantity

$$\langle\langle\Psi|t(\theta_{p_{n+1}})|\theta_{p_1},\ldots,\theta_{p_n}\rangle\rangle$$

Acting to left

$$\langle\langle\Psi|t(\theta_{p_{n+1}})|\theta_{p_1},\ldots,\theta_{p_n}\rangle\rangle = \Lambda^{(o)}(\theta_{p_{n+1}})\bar{F}_n(\theta_{p_1},\ldots,\theta_{p_n})$$

where $\bar{F}_n(\theta_{p_1},\ldots,\theta_{p_n}) = \langle\langle\Psi|\theta_{p_1},\ldots,\theta_{p_n}\rangle\rangle,$

Retrieve the eigenstates

Acting to right

$$
\begin{aligned}
&\langle\langle \Psi | t(\theta_{p_{n+1}}) | \theta_{p_1}, \ldots, \theta_{p_n} \rangle\rangle \\
= \;& \bar{K}^{+}_{11}(\theta_{p_{n+1}})\, \bar{F}_{n+1}(\theta_{p_1}, \ldots, \theta_{p_{n+1}}) \\
&\qquad + \bar{K}^{+}_{22}(\theta_{p_{n+1}})\, \langle\langle \Psi | \bar{\mathcal{D}}(\theta_{p_{n+1}}) \prod_{j=1}^{n} \bar{\mathcal{A}}(\theta_{p_j}) | \Omega \rangle .
\end{aligned}
$$

Using the commutation relations and $\bar{\mathcal{D}}(\theta_{p_j})|\Omega\rangle = \frac{\eta}{2\theta_{p_j} + \eta} \bar{\mathcal{A}}(\theta_{p_j})|\Omega\rangle,$

we obtain the recursive relation

$$
\boxed{\Lambda^{(o)}(\theta_{p_{n+1}}) \bar{F}_n(\theta_{p_1}, \ldots, \theta_{p_n}) = \frac{(2\theta_{p_{n+1}} + \eta) \bar{K}^{+}_{11}(\theta_{p_{n+1}}) + \eta \bar{K}^{+}_{22}(\theta_{p_{n+1}})}{2\theta_{p_{n+1}} + \eta} \bar{F}_{n+1}(\theta_{p_1}, \ldots, \theta_{p_{n+1}})}
$$

Retrieve the eigenstates

The above recursive relation allows us to determine F_n as

$$\bar{F}_n(\theta_{p_1}, \ldots, \theta_{p_n}) = \left\{ \prod_{j=1}^{n} \frac{(2\theta_{p_j} + \eta)\Lambda^{(o)}(\theta_{p_j})}{(2\theta_{p_j} + \eta)\bar{K}_{11}^{+}(\theta_{p_j}) + \eta \bar{K}_{22}^{+}(\theta_{p_j})} \right\} \bar{F}_0,$$

$\bar{F}_0 = \langle\langle\Psi|\Omega\rangle$ is an overall scalar factor.

$$\begin{aligned}
\bar{F}_n(\theta_{p_1}, \ldots, \theta_{p_n}) &= \langle\langle\Psi|\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle \\
&= \left\{ \prod_{j=1}^{n} (-1)^N (\theta_{p_j} + p)\, a(\theta_{p_j}) d(-\theta_{p_j} - \eta) \frac{Q(\theta_{p_j} - \eta)}{Q(\theta_{p_j})} \right\} \bar{F}_0, \\
n &= 0, \ldots, N, \quad 1 \le p_1 < p_2 < \ldots < p_n \le N.
\end{aligned} \qquad (\bigstar)$$

Then the eigenstates are determined completely.

Retrieving the Bethe states

Next we prove that the Bethe states are the eigenstates.

For each solution of the BAEs, the left Bethe states are

$$
{}_B\langle \lambda_1, \ldots, \lambda_N| = \langle 0| \left\{ \prod_{j=1}^{N} \frac{\bar{\mathcal{C}}(\lambda_j)}{(-1)^N \bar{K}_{21}^{-}(\lambda_j) d(\lambda_j) d(-\lambda_j - \eta)} \right\}
$$

Reference states $|0\rangle = \otimes_{j=1}^{N} |\uparrow\rangle_j, \quad \langle 0| = \langle \uparrow|_j \otimes_{j=1}^{N}$

Retrieving the Bethe states

Because we have obtained an orthogonal basis of the Hilbert space, we can decompose the Bethe states as the unique linear combination of these basis. The expansion coefficients are

$$_B\langle\lambda_1, \ldots, \lambda_N|\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle$$

Because

$$\bar{C}(u)|\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle \;\; = \;\; h(u, \{\theta_{p_1}, \ldots, \theta_{p_n}\})|\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle,$$

Then we should calculate the product

$$\langle 0|\theta_{p_1}, \cdots, \theta_{p_n}\rangle\rangle$$

The scalar product $\langle 0|\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle$

From the definition of orthogonal basis and using the commutation relations, we obtain following recursive relation

$$\langle 0|\theta_{p_1}, \ldots, \theta_{p_{n+1}}\rangle\rangle = (-1)^N K_{11}^-(\theta_{p_{n+1}}) a(\theta_{p_{n+1}}) d(-\theta_{p_{n+1}} - \eta) \langle 0|\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle,$$

The solution is

$$\langle 0|\theta_{p_1}, \ldots, \theta_{p_n}\rangle\rangle = \left\{ \prod_{j=1}^{n} (-1)^N (\theta_{p_j} + p)\, a(\theta_{p_j}) d(-\theta_{p_j} - \eta) \right\} \langle 0|\Omega\rangle,$$

$$n = 0, \ldots, N, \quad 1 \leq p_1 < p_2 < \ldots < p_n \leq N.$$

Retrieving the Bethe states

Then we obtain the expansion coefficients of Bethe states as

$$\boxed{{}_B\langle \lambda_1, \ldots, \lambda_N | \theta_{p_1}, \ldots, \theta_{p_n} \rangle\rangle \;=\; \left\{ \prod_{j=1}^{n} (-1)^N (\theta_{p_j} + p)\, a(\theta_{p_j}) d(-\theta_{p_j} - \eta) \frac{Q(\theta_{p_j} - \eta)}{Q(\theta_{p_j})} \right\} \langle 0 | \Omega \rangle,}$$

$$n \;=\; 0, \ldots, N, \quad 1 \leq p_1 < p_2 < \ldots < p_n \leq N.$$

Comparing these expansion coefficients with the expansion coefficients of the eigenstates by the basis of the Hilbert space [equation (★)], we find that, up to a constant, they are the same.

Therefore, the Bethe states are the eigenstates of the transfer matrix provided that the Bethe roots satisfy the Bethe ansatz equations.